Quantum error correction and the Connes embedding problem

Joint work with Uffe Haagerup, and part in collaboration with

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Outline

Theorem (Birkhoff, 1946)

Every doubly stochastic matrix is a convex combination of permutation matrices.

Let $D = l_\infty(\{1, 2, \ldots, n\})$ with trace $\tau(\chi_{\{i\}}) = 1/n\,,\ 1 \leq i \leq n\,.$
The unital positive trace-preserving maps on D are precisely the linear operators on D given by doubly stochastic $n \times n$ matrices.
An automorphism of D is given by a permutation of $\{1\,, 2\,, \ldots\,, n\}\,.$

$\rightsquigarrow$ **Quantum Birkhoff Conjecture**:

- **Kümmerer** (1983): $\mathrm{UCPT}(2) = \mathrm{conv}(\mathrm{Aut}(M_2(\mathbb{C})))$.
- For $n \geq 3$: $\mathrm{UCPT}(n) \supsetneq \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))$

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The asymptotic quantum Birkhoff conjecture

- **Gregoratti-Werner** (2003): Channels which are convex combinations of unitarily implemented ones allow for complete error correction, given suitable feedback of classical information from the environment.

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Let $T \in UCPT(n)$, $n \geq 3$. Then T satisfies the following asymptotic quantum Birkhoff property *(***AQBP***):*

$$\lim_{k \to \infty} d_{cb}\Big(\bigotimes_{i=1}^{k} T \,, conv(Aut(\bigotimes_{i=1}^{k} M_n(\mathbb{C})))\Big) = 0 \,.$$

- **Mendl-Wolf** (2009): $\exists\, T \in \mathrm{UCPT}(3)$ such that

$$T \notin \mathrm{conv}(\mathrm{Aut}(M_3(\mathbb{C}))), \text{ but } T \otimes T \in \mathrm{conv}\big(\mathrm{Aut}\big(M_9(\mathbb{C})\big)\big)\,.$$

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Remark: No such examples can occur among Schur channels:

Theorem (Haagerup-M)

Let T be a unital Schur channel on $M_n(\mathbb{C})$ and S be a unital Schur channel on $M_k(\mathbb{C})$, $k, n \geq 2$. Then

$$d_{cb}\Big(T \otimes S, conv(Aut(M_{nk}(\mathbb{C}))\Big) \geq \frac{1}{2}\, d_{cb}\Big(T, conv(Aut(M_n(\mathbb{C}))\Big)$$

In particular, if $T \notin conv(Aut(M_n(\mathbb{C})))$, then T fails the AQBP.

Example: Let $\beta = 1/\sqrt{5}$ and set

$$B := \begin{pmatrix} 1 & \beta & \beta & \beta & \beta & \beta \\ \beta & 1 & \beta & -\beta & -\beta & \beta \\ \beta & \beta & 1 & \beta & -\beta & -\beta \\ \beta & -\beta & \beta & 1 & \beta & -\beta \\ \beta & -\beta & -\beta & \beta & 1 & \beta \\ \beta & \beta & -\beta & -\beta & \beta & 1 \end{pmatrix}$$

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Factorizable maps

Definition (Anantharaman-Delaroche, 2005)

A UCPT(n) map $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is called *factorizable* if $\exists$ vN algebra N with n.f. tracial state ϕ and injective unital $*$-homs $\alpha, \beta\colon M_n(\mathbb{C}) \to M_n(\mathbb{C}) \otimes N$ such that $T = \beta^* \circ \alpha$.

$$\begin{array}{ccc} M_n(\mathbb{C}) & \xrightarrow{\quad T \quad} & M_n(\mathbb{C}) \\ & \searrow^{\alpha} \qquad \nearrow_{\beta^* = \beta^{-1} \circ \mathbb{E}_{\beta(M_n(\mathbb{C}))}} & \\ & M_n(\mathbb{C}) \otimes N & \end{array}$$

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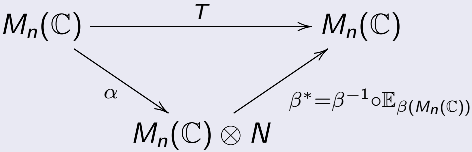

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Let $T \in UCPT(n)$, $n \geq 3$, written in Choi canonical form

$$Tx = \sum_{i=1}^{d} a_i^* x a_i \,, \; x \in M_n(\mathbb{C}) \,.$$

Then T is factorizable iff one of the following conditions hold:

1) *$\exists$ vN algebra N with nf tracial state τ_N and a unitary $u \in M_n(\mathbb{C}) \otimes N$ st*

$$Tx = (id_{M_n(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes 1_N)u) \,, \quad x \in M_n(\mathbb{C}) \,.$$

We say that T has an exact factorization through $M_n(\mathbb{C}) \otimes N$.

2) *$\exists$ vN algebra N with nf tracial state τ_N and $v_1, \ldots, v_d \in N$ st*

$$u = \sum a_i \otimes v_i \in \mathcal{U}(M_n(\mathbb{C}) \otimes N), \quad \tau_N(v_i^* v_j) = \delta_{ij} \,, 1 \leq i,j \leq d$$

Interpretation in Quantum Information Theory (R. Werner):
Factorizable maps are obtained by coupling the input system to a maximally mixed ancillary one, executing a unitary rotation on the combined system, and tracing out the ancilla.

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- Any $\phi \in \mathrm{Aut}(M_n(\mathbb{C}))$ exactly factors through $M_n(\mathbb{C}) \otimes \mathbb{C}$.
- The set $\mathcal{F}(M_n(\mathbb{C}))$ of factorizable UCPT(n) maps is convex:

If $T, S \in \mathcal{F}(M_n(\mathbb{C}))$, with unitaries u, resp., $v \in M_n(\mathbb{C}) \otimes N$, then

$$tT + (1-t)S \in \mathcal{F}(M_n(\mathbb{C})), \quad 0 < t < 1,$$

with unitary $u \otimes p + v \otimes (1-p) \in M_n(\mathbb{C}) \otimes N \otimes L^\infty([0,1])$, where p is a projection of trace t in $L^\infty([0,1])$.

Hence $\mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C}))) \subseteq \mathcal{F}(M_n(\mathbb{C}))$.

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Corollary

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Choi (1975): $T \in \partial_e(\mathrm{UCP}(n))$ if and only if $\{a_i^* a_j : 1 \leq i, j \leq d\}$ is a linearly independent set.

By the corollary, if

$$T \in \partial_e(\mathrm{UCP}(n)) \cap \mathrm{UCPT}(n) \quad \text{and} \quad \text{Choi-rank}(T) \geq 2$$

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Example

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With $a_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$, $a_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, $a_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ we obtained the first example of a non-factorizable map. It is the Holevo-Werner channel W_3^-.

Theorem (Haagerup-M-Ruskai)

Let $n \geq 3$ and let $S : \mathbb{C}^n \to \mathbb{C}^n$ be the cyclic shift.

(1) *Let $U_1, \ldots, U_n \in \mathcal{U}(n-1)$ and set*

$$a_i = \frac{1}{\sqrt{n-1}} S^i \begin{pmatrix} U_i & 0 \\ 0 & 0 \end{pmatrix} S^{-i}, \quad 1 \leq i \leq n\,.$$

Set $Tx = \sum_{i=1}^n a_i^* x a_i$, $x \in M_n(\mathbb{C})$. Then $T \in UCPT(n)$ and with probability one (w.r.t. Haar measure on $\prod_{i=1}^n \mathcal{U}(n-1)$)

$$T \in \partial_e(UCP(n)) \cap \partial_e(CPT(n))\,.$$

In particular, T is not factorizable.

(2) *Same conclusion holds for*

$$a_i = \frac{1}{\sqrt{n-1+t^2}} S^i \begin{pmatrix} U_i & 0 \\ 0 & t \end{pmatrix} S^{-i}, \quad 1 \leq i \leq n\,,$$

where $t > 0$, $t \neq 1$ (fixed).

Extreme points in UCPT(n)

Landau and Streater (1993): $T \in \partial_e(\mathrm{UCPT}(n))$ if and only if

$$\{a_i^* a_j \oplus a_j a_i^* : 1 \le i, j \le d\}$$

is a linearly independent set.
Hence

$$\partial_e(\mathrm{UCPT}(n)) \supseteq \big(\partial_e(\mathrm{UCP}(n)) \cup \partial_e(\mathrm{CPT}(n))\big) \cap \mathrm{UCPT}(n).$$

Mendl-Wolf (2009): above inclusion is strict for $n = 3$.

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Haagerup-M.-Ruskai (motivated by a question of Farenick):

For $t \in [0,1]$, let $T_t(x) = \frac{1}{2}\sum_{i=1}^{4} a_i(t)^* x a_i(t)$, $x \in M_3(\mathbb{C})$, where

$$a_1(t) = \begin{pmatrix} \sqrt{t} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, a_2(t) = \begin{pmatrix} 0 & 0 & \sqrt{1-t} \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$a_3(t) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \sqrt{t} \end{pmatrix}, a_4(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \sqrt{1-t} & 0 & 0 \end{pmatrix}.$$

Then, for $t \notin \{0, 1/2, 1\}$,

$$T_t \in \partial_e(\mathrm{UCPT}_3) \setminus \left(\partial_e(\mathrm{UCP}_3) \cup \partial_e(\mathrm{CPT}_3)\right).$$

Moreover, T_t is factorizable, $0 \le t \le 1$, (through $M_3(\mathbb{C}) \otimes M_2(\mathbb{C})$).

Outline

Let $n \geq 2$. Consider the *Holevo-Werner channels* in dimension n:

$$W_n^-(x) = \frac{1}{n-1}(\,\mathrm{Tr}(x)1_n - x^t\,)\,, \quad x \in M_n(\mathbb{C})\,.$$

$$W_n^+(x) = \frac{1}{n+1}(\,\mathrm{Tr}(x)1_n + x^t\,)\,, \quad x \in M_n(\mathbb{C})\,.$$

- $W_n^-, W_n^+ \in \mathrm{UCPT}(n)$.
- $S_n = \tau_n(\cdot)1_n \in \mathrm{conv}\{W_n^-, W_n^+\}$, since

$$S_n(x) = \frac{1}{n}\mathrm{Tr}(x)1_n = \frac{n-1}{2n}W_n^-(x) + \frac{n+1}{2n}W_n^+(x)\,,\; x \in M_n(\mathbb{C})\,.$$

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Theorem (Mendl-Wolf, 2009)

(1) $W_n^+ \in conv(Aut(M_n(\mathbb{C})))$, *for all* $n \geq 2$.

(2) $W_n^- \in conv(Aut(M_n(\mathbb{C})))$, *for all* n *even*.

(3) *For* n *odd and* $0 \leq \lambda \leq 1$,

$$\lambda W_n^+ + (1-\lambda) W_n^- \in conv\big(Aut(M_n(\mathbb{C}))\big) \iff \lambda \geq 1/n.$$

In particular, $W_n^- \notin conv\big(Aut(M_n(\mathbb{C}))\big)$.

Theorem (Haagerup-M)

(1) $d_{cb}(W_3^-, \mathcal{F}(M_3(\mathbb{C}))) = 4/27$.

(2) *For* n *odd,* $n \neq 3$,

$$W_n^- \text{ exactly factors through } M_n(\mathbb{C}) \otimes M_4(\mathbb{C}) \otimes L^\infty([0,1]).$$

(3) $\lambda W_3^+ + (1-\lambda) W_3^- \in \mathcal{F}(M_3(\mathbb{C})) \iff 2/27 \leq \lambda \leq 1$.

In each such case, $\lambda W_3^+ + (1-\lambda) W_3^-$ *has an exact factorization through* $M_3(\mathbb{C}) \otimes M_3(\mathbb{C})$.

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For $0 \leq \lambda \leq 1$, set $T_\lambda := \lambda W_3^+ + (1-\lambda) W_3^-$. By **Mendl-Wolf**:

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There exists $\lambda_0 \in (0, 1/3)$ such that for all $\lambda \geq \lambda_0$,

$$T_\lambda \otimes T_\lambda \in conv\big(Aut(M_9(\mathbb{C})\big).$$

Theorem (Haagerup-M)

For every $\lambda \in [1/4, 1]$ and every integer $k \geq 2$,

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An averaging technique (twirl)

For $T \in \mathcal{B}(M_n(\mathbb{C}))$ set

$$F(T) := \int_{\mathcal{U}(n)} \mathrm{ad}(u)\, T\, \mathrm{ad}(u^t)\, du\,,$$

where du is the Haar measure on $\mathcal{U}(n)$.

Properties:

- If $T \in \mathrm{UCPT}(n)$ then $F(T) \in \mathrm{UCPT}(n)$. Moreover,

$$\|F(T)\|_{\mathrm{cb}} \leq \|T\|_{\mathrm{cb}}\,.$$

- $F(\mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))) \subseteq \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))$.
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Choi (1975): $T \in \mathcal{B}(M_n(\mathbb{C}))$ is CP $\iff$ $\hat{T}$ is positive, where

$$\hat{T}: = \frac{1}{n}\sum_{i,j=1}^{n} T(e_{ij}) \otimes e_{ij} \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C}).$$

Vollbrecht-Werner (2001): $\widehat{F(T)} = E(\hat{T})$, where

$$E(x) = \int_{\mathcal{U}(n)} (u \otimes u)x(u^* \otimes u^*)du, \quad x \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C}),$$

is the trace-preserving cond. expectation of $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ onto span$\{P^+, P^-\}$, where P^+, P^- are the orthogonal projections onto $(\mathbb{C}^n \otimes \mathbb{C}^n)_{\text{sym}}$ and $(\mathbb{C}^n \otimes \mathbb{C}^n)_{\text{antisym}}$, respectively.

- $F(W_n^+) = W_n^+$ and $F(W_n^-) = W_n^-$.

Theorem (Haagerup-M)

If $T \in UCPT(n)$, then

$$F(T) \in conv\{W_n^+, W_n^-\}.$$

More precisely, if $T = \sum_{i=1}^{d} a_i x a_i^*$ (Choi canonical form), then

$$F(T) = c^+(T) W_n^+ + c^-(T) W_n^-,$$

where $c^+(T) = \frac{1}{4} \sum_{i=1}^{d} \|a_i + a_i^t\|_2^2$, $c^-(T) = \frac{1}{4} \sum_{i=1}^{d} \|a_i - a_i^t\|_2^2$.

Outline

Theorem (Haagerup-M)

Let $T \in UCPT(n)$ be factorizable. The following are equivalent:

(1) *T has an exact factorization through a finite vN algebra which embeds into $\mathcal{R}^\omega$, i.e., $\exists (N, \tau_N) \hookrightarrow \mathcal{R}^\omega$, $\exists\ u \in \mathcal{U}(M_n(N))$ st*

$$Tx = (id_{M_n(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes 1_N)u), \quad x \in M_n(\mathbb{C}).$$

(2) *T admits an approximate factorization through matrix alg.*

(3) $\lim_{k\to\infty} d_{cb}\Big(T \otimes S_k\, , conv\big(Aut(M_n(\mathbb{C}) \otimes M_k(\mathbb{C}))\big)\Big) = 0,$

where S_k is the completely depolarizing channel:

$$S_k(y) = \tau_k(y)1_k, \quad y \in M_k(\mathbb{C}).$$

- $T \otimes S_k \in \text{conv}(\text{Aut}(M_n(\mathbb{C}) \otimes M_k(\mathbb{C}))$ iff T has a exact factorization through $M_n(\mathbb{C}) \otimes M_k(\mathbb{C}) \otimes L^\infty([0,1])$.

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Theorem (Haagerup-M)

The Connes embedding problem has a positive answer if and only if every factorizable $UCPT_n$ map satisfies one of the equivalent conditions in above theorem, for all $n \geq 3$.

Idea of proof: ($\Leftarrow$) **Dykema-Jushenko** (2009):

$$\mathcal{F}_n := \overline{\bigcup_{k \geq 1} \left\{ B = (b_{ij}) \in M_n(\mathbb{C}) : b_{ij} = \tau_k(u_i u_j^*)\,, u_j \in \mathcal{U}(M_k(\mathbb{C})) \right\}}$$

$$\mathcal{G}_n := \Big\{ B = (b_{ij}) \in M_n(\mathbb{C}) : b_{ij} = \tau_M(u_i u_j^*)\,, u_j \in \mathcal{U}(M)\,, \text{ for some vN algebra } (M, \tau_M) \text{ with n.f. tracial state} \Big\}$$

Kirchberg (1993): The Connes embedding problem has a positive answer iff $\mathcal{F}_n = \mathcal{G}_n$, for all $n \geq 1$.

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Idea of proof — continued:

Assume that the Connes embedding problem has a negative answer. Then $\mathcal{G}_n \setminus \mathcal{F}_n \neq \emptyset$, for some $n \geq 1$. Choose

$$B = (b_{ij})_{i,j=1}^n \in \mathcal{G}_n \setminus \mathcal{F}_n \,.$$

Then the Schur multiplier T_B has an exact factorization through a finite vN algebra embeddable into $\mathcal{R}^\omega$, so $\exists u_1, \ldots, u_n \in \mathcal{U}(\mathcal{R}^\omega)$ s.t.

$$b_{ij} = \tau_{\mathcal{R}^\omega}(u_i^* u_j), \quad 1 \leq i, j \leq n \,.$$

Approximate each b_{ij} by $\tau_{\mathcal{R}}(v_i^* v_j)$, where $v_i \in \mathcal{U}(\mathcal{R})$, and further by unitary matrices (via Kaplansky). Hence B can be approximated by a sequence B_k s.t. the Schur multiplier T_{B_k} admits an exact factorization through a matrix algebra. This implies $B \in \mathcal{F}_n$ $\lightning$.

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Problem

Let $T \in UCPT(n)$ be factorizable satisfying condition (3) above. Is it then true that

$$\min\Big\{d_{cb}\Big(T \otimes S_k\,, conv\big(Aut(M_n(\mathbb{C}) \otimes M_k(\mathbb{C}))\big)\Big) : k \in \mathbb{N}\Big\} = 0\,?$$

Theorem (Haagerup-M)

Let $T \in UCPT(n)$ be factorizable. The following are equivalent:

(1) *T has an exact factorization through a finite vN algebra which embeds into $\mathcal{R}^\omega$, i.e., $\exists (N, \tau_N) \hookrightarrow \mathcal{R}^\omega$, $\exists\ u \in \mathcal{U}(M_n(N))$ st*

$$Tx = (id_{M_n(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes 1_N)u), \quad x \in M_n(\mathbb{C}).$$

(2) *T admits an approximate factorization through matrix alg.*

(3) $\lim_{k \to \infty} d_{cb}\Big(T \otimes S_k\, , conv\big(Aut(M_n(\mathbb{C}) \otimes M_k(\mathbb{C}))\big)\Big) = 0,$
where S_k is the completely depolarizing channel.

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