A class of exposed positive maps

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Mathematical Aspects in Current Quantum Information Theory
Daejeon, Korea
February 16, 2016

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- ϕ is *completely positive* (or CP) if it is k-positive for any $k \in \mathbb{N}$.
- ϕ is *decomposable* if $\phi(X) = \phi_1(X) + \phi_2(X)^{\mathrm{t}}$, $X \in B(K)$, where ϕ_1, ϕ_2 are CP maps.

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Assume one of the following conditions holds:

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Another examples of non-decomposable maps were given by Woronowicz, Tang, Ha, Osaka, Robertson, Kye and others.

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A map ϕ is said to be *optimal* if

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where $Z = \sum_i X_i \otimes Y_i$, $X_i \in \mathfrak{B}(\mathscr{K})$, $Y_i \in \mathfrak{B}(\mathscr{H})$.

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- Consider dual cone $\mathfrak{P}^\circ \subset B(K)\hat{\otimes}T^1(H)$
$$\mathfrak{P}^\circ = \{Z \in B(K)\hat{\otimes}T^1(H) : \langle Z, \phi\rangle_{\mathrm{d}} \geq 0 \text{ for all } \phi \in \mathfrak{P}\}.$$
$\mathfrak{P}^\circ$ consist of block-positive matrices, i.e.
$$Z \in \mathfrak{P}^\circ \quad \Leftrightarrow \quad \langle \xi \otimes \eta, Z\xi \otimes \eta\rangle \geq 0,\;\; \xi \in K, \eta \in H.$$

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- A map $\phi \in \mathscr{P}$ is said to be exposed if there is a block-positive matrix Z_0 such that

$$\{\psi \in \mathfrak{P} : \langle Z_0, \psi\rangle_{\mathrm{d}} = 0\} = \mathbb{R}_+\phi$$

Straszewicz's theorem

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Assume that V is a real locally compact topological vector space with a topology which induced bo some strictly convex norm. If $K \subset V$ is compact and convex then $\operatorname{cl}(\operatorname{Exp} K) = \operatorname{Ext} K$.

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It follows from the above theorems that the problem of the description of positive maps can be reduced to the problem of characterization of exposed positive maps.

Examples

- (MM'2011) For finite dimensional dimensional K and H and any $A : K \to H$, the maps

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- Choi map is an extremal **non**exposed positive map.
- Other examples are due to Chruściński and Sarbicki, Ha and Kye, and others..

Strong spanning property

A positive map $\phi : B(K) \to B(H)$ is said to be irreducible, if

$$\forall\, Y \in B(H) : \Big([\phi(X), Y] = 0,\ \forall X \in B(K) \Big) \quad \Rightarrow \quad Y \in \mathbb{C}\mathbb{I}_H$$

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Theorem (Chruściński, Sarbicki)

Assume K and H are finite dimensional. Let $\phi : B(K) \to B(H)$ be a positive map irreducible on its image. If the subspace $N_\phi \subset B(K) \otimes H$ satisfies

$$\dim N_\phi = (\dim_K)^2 \dim_H - \operatorname{rank} \phi(\mathbb{I}_K)$$

then it is exposed.

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Theorem (Miller, Olkiewicz)

S is a bistochastic, exposed and nondecomposable (even atomic) map.

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$$\Lambda_d(X) = \frac{1}{d} \begin{pmatrix} \sum_{i=1}^{d} x_{ii} & \cdots & 0 & 0 & \sqrt{d}x_{1,d+1} \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \cdots & \sum_{i=1}^{d} x_{ii} & 0 & \sqrt{d}x_{d-1,d+1} \\ 0 & \cdots & 0 & \sum_{i=1}^{d} x_{ii} & \sqrt{d}x_{d+1,d} \\ \sqrt{d}x_{d+1,1} & \cdots & \sqrt{d}x_{d+1,d-1} & \sqrt{d}x_{d,d+1} & dx_{d+1,d+1} \end{pmatrix}$$

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Theorem (Rutkowski et al.)

Λ_d is a bistochastic positive, nondecomposable and optimal map.

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Construction contd. – structure of $\tilde{S}_{\text{ess}}$

- identity

$$\tilde{S}_{\text{ess}} : \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \mapsto \begin{pmatrix} x_{11} & 0 & x_{13} \\ 0 & x_{22} & x_{32} \\ x_{31} & x_{23} & x_{33} \end{pmatrix}$$

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- merging of identity and transposition

$$\tilde{S}_{\text{ess}} : \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \mapsto \begin{pmatrix} x_{11} & 0 & x_{13} \\ 0 & x_{22} & x_{32} \\ x_{31} & x_{23} & x_{33} \end{pmatrix}$$

Construction, contd.

Let $k_1, k_2 \in \mathbb{N}$, $k = k_1 + k_2 + 1$. Every matrix $X \in B(\mathbb{C}^k)$ can be represented in the block form

$$\left(\begin{array}{ccc:ccc:c}
x_{1,1} & \cdots & x_{1,k_1} & x_{1,k_1+1} & \cdots & x_{1,k_1+k_2} & x_{1,k} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots \\
x_{k_1,1} & \cdots & x_{k_1,k_1} & x_{k_1,k_1+1} & \cdots & x_{k_1,k_1+k_2} & x_{k_1,k} \\
\hdashline
x_{k_1+1,1} & \cdots & x_{k_1+1,k_1} & x_{k_1+1,k_1+1} & \cdots & x_{k_1+1,k_1+k_2} & x_{k_1+1,k} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots \\
x_{k_1+k_2,1} & \cdots & x_{k_1+k_2,k_1} & x_{k_1+k_2,k_1+1} & \cdots & x_{k_1+k_2,k_1+k_2} & x_{k_1+k_2,k} \\
\hdashline
x_{k,1} & \cdots & x_{k,k_1} & x_{k,k_1+1} & \cdots & x_{k,k_1+k_2} & x_{k,k}
\end{array}\right)$$

Construction, contd.

Let $k_1, k_2 \in \mathbb{N}$, $k = k_1 + k_2 + 1$. Every matrix $X \in B(\mathbb{C}^k)$ can be represented in the block form

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\hline
x_{k_1+1,1} & \cdots & x_{k_1+1,k_1} & x_{k_1+1,k_1+1} & \cdots & x_{k_1+1,k_1+k_2} & x_{k_1+1,k} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots \\
x_{k_1+k_2,1} & \cdots & x_{k_1+k_2,k_1} & x_{k_1+k_2,k_1+1} & \cdots & x_{k_1+k_2,k_1+k_2} & x_{k_1+k_2,k} \\
\hline
x_{k,1} & \cdots & x_{k,k_1} & x_{k,k_1+1} & \cdots & x_{k,k_1+k_2} & x_{k,k}
\end{array}\right)$$

$$\left(\begin{array}{c|c|c}
X_{11} & X_{12} & X_{13} \\
\hline
X_{21} & X_{22} & X_{23} \\
\hline
X_{31} & X_{32} & X_{33}
\end{array}\right), \qquad X_{ij} \in B(\mathbb{C}^{k_j}, \mathbb{C}^{k_i}),\ i,j = 1,2,3$$

$k_3 = 1$

Construction, contd.

$$\tilde{S}_{\text{ess}} : \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \mapsto \begin{pmatrix} x_{11} & 0 & x_{13} \\ 0 & x_{22} & x_{32} \\ x_{31} & x_{23} & x_{33} \end{pmatrix}$$

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is generalized to

$$\phi_{\text{ess}} : B(\mathbb{C}^{k_1+k_2+1}) \to B(\mathbb{C}^{k_1+k_2+1})$$

Construction, contd.

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$$X_{i3} \in B(\mathbb{C}, \mathbb{C}^{k_i}) = \mathbb{C}^{k_i}, \qquad X_{3j} \in B(\mathbb{C}^{k_j}, \mathbb{C}) = (\mathbb{C}^{k_j})^*$$

In particular X_{33} is a scalar. Hence $X_{33}^{t} = X_{33}$.

Construction contd.

The diagonal part

$$\tilde{S}_{\mathrm{diag}} : \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \longmapsto \begin{pmatrix} x_{22} & 0 & 0 \\ 0 & x_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Construction contd.

The diagonal part

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is generalized to

$$\phi_{\text{diag}} : \left(\begin{array}{c:c:c} X_{11} & X_{22} & X_{33} \\ \hdashline X_{11} & X_{22} & X_{33} \\ \hdashline X_{11} & X_{22} & X_{33} \end{array}\right) \longmapsto \left(\begin{array}{c:c:c} \mathrm{Tr}(X_{22})\mathbb{I}_{k_1} & 0 & 0 \\ \hdashline 0 & \mathrm{Tr}(X_{11})\mathbb{I}_{k_2} & 0 \\ \hdashline 0 & 0 & 0 \end{array}\right)$$

Construction

Therefore, we get

$$\phi : B(\mathbb{C}^{k_1+k_2+1}) \to B(\mathbb{C}^{k_1+k_2+1})$$

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Further,

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- replace $\mathbb{C}^{k_i}$ by some (not necessarily finite dimensional) Hilbert space K_i, $i = 1, 2$,
- consider some Hilbert-Shmidt operators $A_i : K_i \to H_i$ ($\mathrm{Tr}(A_i^* A_i) < \infty$), where K_i are some another Hilbert spaces

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- consider some Hilbert-Shmidt operators $A_i : K_i \to H_i$ ($\mathrm{Tr}(A_i^* A_i) < \infty$), where K_i are some another Hilbert spaces
- replace identity and transposition parts by

$$B(K_1) \ni X_{11} \mapsto A_1 X_{11} A_1^* \in B(H_1)$$

$$B(K_2) \ni X_{22} \to A_2 X_{22}^{t} A_2^* \in B(H_2)$$

Construction

Finally, we arrive at the following generalization

$$\phi : B(K_1 \oplus K_2 \oplus \mathbb{C}) \to B(H_1 \oplus H_2 \oplus \mathbb{C})$$

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$$\left(\begin{array}{c|c|c} A_1 X_{11} A_1^* + \operatorname{Tr}(A_2 X_{22}^{t} A_2^*) E_1 & 0 & A_1 X_{13} \\ \hline 0 & A_2 X_{22}^{t} A_2^* + \operatorname{Tr}(A_1 X_{11} A_1^*) E_2 & A_2 X_{32}^{t} \\ \hline X_{31} A_1^* & X_{23}^{t} A_2^* & X_{33} \end{array}\right)$$

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$$\begin{pmatrix} A_1 X_{11} A_1^* + \mathrm{Tr}(A_2 X_{22}^{t} A_2^*) E_1 & 0 & A_1 X_{13} \\ 0 & A_2 X_{22}^{t} A_2^* + \mathrm{Tr}(A_1 X_{11} A_1^*) E_2 & A_2 X_{32}^{t} \\ X_{31} A_1^* & X_{23}^{t} A_2^* & X_{33} \end{pmatrix}$$

where

- $A_i : K_i \to H_i$ are Hilbert-Schmidt operators, $i = 1, 2$.
- E_i is the projection in $B(H_i)$ onto the range of A_i for $i = 1, 2$.

Properties of ϕ

Theorem (MM,Rutkowski)

ϕ is a positive map. Moreover, it is exposed in the cone of positive maps.

Proposition

Assume $\dim K_i < \infty$, $\dim H_i < \infty$. *The map* ϕ *does not satisfy the strong spanning property, unless one of the following conditions is satisfied:*

1. $K_2 = H_2 = \{0\}$ *and* $\mathrm{rank} A_1 = \dim K_1$,
2. $K_1 = H_1 = \{0\}$ *and* $\mathrm{rank} A_2 = \dim K_2$,

Main idea of the proof

- $K = K_1 \oplus K_2 \oplus \mathbb{C}$, $H = H_1 \oplus H_2 \oplus \mathbb{C}$
- $\mathscr{Z} = \{(\xi, \eta) \in K \times H : \langle \eta, \phi(\xi\xi^*)\eta \rangle = 0$

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- By Kye's characterization of exposed faces, $\phi : B(K) \to B(H)$ is exposed iff
  $\forall\, \psi \in \mathfrak{P} : \;\; (\forall\, (\xi,\eta) \in \mathscr{Z} : \langle \eta, \psi(\xi\xi^*)\eta\rangle = 0) \quad \Rightarrow \quad \psi \in \mathbb{R}^+\phi.$

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- $\langle \eta, \phi(\xi\xi^*)\eta\rangle$ is equal to

$$\|A_1\xi_1\|^2\|E_2\eta_2\|^2 + \|A_2\overline{\xi_2}\|^2\|E_1\eta_1\|^2 + |\langle \eta_1, A_1\xi_1\rangle|^2 + |\langle \eta_2, A_2\overline{\xi_2}\rangle|^2$$

if $\alpha = 0$, and

$$|\alpha|^{-2}\left(\left||\alpha|^2\overline{\beta} + \overline{\alpha}\langle \eta_1, A_1\xi_1\rangle + \alpha\langle \eta_2, A_2\overline{\xi_2}\rangle\right|^2 + \left\|\alpha E_1\eta_1 \otimes A_2\overline{\xi_2} - \overline{\alpha} A_1\xi_1 \otimes E_2\eta_2\right\|^2\right),$$

if $\alpha \neq 0$.

Sketch of the proof

- Thus $(\xi, \eta) \in \mathscr{Z}$ iff one of the following conditions holds

$$
\begin{array}{lll}
\alpha = 0,\ A_1\xi_1 = 0,\ A_2\overline{\xi_2} = 0 & & \\
\alpha = 0,\ A_1\xi_1 \neq 0,\ A_2\overline{\xi_2} = 0 & \text{and} & \eta_1 \perp A_1\xi_1,\ E_2\eta_2 = 0 \\
\alpha = 0,\ A_1\xi_1 = 0,\ A_2\overline{\xi_2} \neq 0 & \text{and} & E_1\eta_1 = 0,\ \eta_2 \perp A_2\overline{\xi_2} \\
\alpha = 0,\ A_1\xi_1 \neq 0,\ A_2\overline{\xi_2} \neq 0 & \text{and} & E_1\eta_1 = 0,\ E_2\eta_2 = 0 \\
\alpha \neq 0,\ A_1\xi_1 = 0,\ A_2\overline{\xi_2} = 0 & \text{and} & \beta = 0 \\
\alpha \neq 0,\ A_1\xi_1 \neq 0,\ A_2\overline{\xi_2} = 0 & \text{and} & \langle A_1\xi_1, \eta_1\rangle = -\overline{\alpha}\beta,\ E_2\eta_2 = 0 \\
\alpha \neq 0,\ A_1\xi_1 = 0,\ A_2\overline{\xi_2} \neq 0 & \text{and} & E_1\eta_1 = 0,\ \langle A_2\overline{\xi_2}, \eta_2\rangle = -\alpha\beta \\
\alpha \neq 0,\ A_1\xi_1 \neq 0,\ A_2\overline{\xi_2} \neq 0 & \text{and} & \begin{cases} E\eta_1 = -\dfrac{\overline{\alpha}\beta}{\|A_1\xi_1\|^2 + \|A_2\overline{\xi_2}\|^2} A_1\xi_1, \\ E\eta_2 = -\dfrac{\alpha\beta}{\|A_1\xi_1\|^2 + \|A_2\overline{\xi_2}\|^2} A_2\overline{\xi_2} \end{cases}
\end{array}
$$

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- Now, assume $\langle \eta, \psi(\xi\xi^*)\eta \rangle = 0$ for all $(\xi, \eta) \in \mathcal{Z}$.

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- One shows that there are sesquilinear vector valued forms

$$\Psi_{kl} : (K_1 \oplus K_2) \times (K_1 \oplus K_2) \to B(H_l, H_k), \quad k,l = 1,2$$

and linear maps $R_k, Q_k : K_1 \oplus K_2 \to H_k$ for $k = 1,2$ such that $\psi(\xi\xi^*)$ is equal to

$$\begin{pmatrix} \Psi_{11}(\xi_0,\xi_0) & \Psi_{12}(\xi_0,\xi_0) & \overline{\alpha}R_1\xi_0 + \alpha Q_1\overline{\xi_0} \\ \Psi_{21}(\xi_0,\xi_0) & \Psi_{22}(\xi_0,\xi_0) & \overline{\alpha}R_2\xi_0 + \alpha Q_2\overline{\xi_0} \\ \alpha(R_1\xi_0)^* + \overline{\alpha}(Q_1\overline{\xi_0})^* & \alpha(R_2\xi_0)^* + \overline{\alpha}(Q_2\overline{\xi_0})^* & \lambda|\alpha|^2 \end{pmatrix}$$

for any $\xi \in K$ where $\xi = \xi_0 + \alpha e$ for a unique $\xi_0 = \xi_1 + \xi_2 \in K_1 \oplus K_2$ and $\alpha \in \mathbb{C}$.

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for any $\xi \in K$ where $\xi = \xi_0 + \alpha e$ for a unique $\xi_0 = \xi_1 + \xi_2 \in K_1 \oplus K_2$ and $\alpha \in \mathbb{C}$.
- Finally, by a sequence of reasonings using linearity-antilinearity interplay, one that all ingredients are multiples by λ of respective terms of ϕ.

Further properties of maps

- Obviously, ϕ is nondecomposable and even atomic.
- ϕ is not locally completely positive.
- For $K_i = H_i = \mathbb{C}^{k_i}$ (finite dimensional) and $A_i = \mathrm{id}$ one can normalize map ϕ to obtain a unital map

$$X \mapsto \left(\begin{array}{c:c:c} \frac{1}{1+k_2}\left(X_{11} + \mathrm{Tr}(X_{22})\mathbb{I}_{k_1}\right) & 0 & \frac{1}{\sqrt{1+k_2}} X_{13} \\ \hdashline 0 & \frac{1}{1+k_1}\left(X_{22}^{t} + \mathrm{Tr}(X_{11})\mathbb{I}_{k_2}\right) & \frac{1}{\sqrt{1+k_1}} X_{32}^{t} \\ \hdashline \frac{1}{\sqrt{1+k_2}} X_{31} & \frac{1}{\sqrt{1+k_1}} X_{23}^{t} & X_{33} \end{array}\right)$$

which becomes bistochastic if $k_1 = k_2$.

Further generalizations

- Consider two positive maps $\phi_i : B(K_i) \to B(H_i)$, $i = 1, 2$.

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- Consider two positive maps $\phi_i : B(K_i) \to B(H_i)$, $i = 1,2$.
- Let $C_k, D_k : K_k \to H_k$, $k = 1,2$, be linear maps.
- Define merging of the maps ϕ_1 and ϕ_2 by operators C_k, D_k as a linear map

$$\phi : B(K_1 \oplus K_2 \oplus \mathbb{C}) \to B(H_1 \oplus H_2 \oplus \mathbb{C})$$

which to X assigns

$$\left(\begin{array}{c|c|c} \phi_1(X_{11}) + \mathrm{Tr}(\phi_2(X_{22}))E_1 & 0 & C_1 X_{13} + D_1 X_{31}^{\mathrm{t}} \\ \hline 0 & \phi_2(X_{22}) + \mathrm{Tr}(\phi_1(X_{11}))E_2 & C_2 X_{23} + D_2 X_{32}^{\mathrm{t}} \\ \hline X_{31} C_1^* + X_{13}^{\mathrm{t}} D_1^* & X_{32} C_2^* + X_{23}^{\mathrm{t}} D_2^* & X_{33} \end{array}\right)$$

where E_k is support projection of $\phi_k(\mathbb{I}_k)$.

Further generalizations

Theorem

If ϕ_1 is 2-positive and ϕ_2 is 2-copositive, then there are operators C_k, D_k such that merging of ϕ_1 and ϕ_2 by C_k, D_k is a positive nondecomposable map.

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