A decomposition theorem for k-positive linear maps on matrix algebras*

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A Corollary for Generalized Choi Maps in Three Dimensional Matrix Algebra

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For nonnegative real numbers a,b and c, the generalized Choi map $\Phi[a, b, c]$ is defined by

$$\Phi[a,b,c](X) = \begin{pmatrix} ax_{11} + bx_{22} + cx_{33} & -x_{12} & -x_{13} \\ -x_{21} & cx_{11} + ax_{22} + bx_{33} & -x_{23} \\ -x_{31} & -x_{32} & bx_{11} + cx_{22} + ax_{33} \end{pmatrix}$$

for $X = [x_{ij}] \in M_3(\mathbb{C})$.

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It has been completely characterized that the generalized Choi map $\Phi[a, b, c]$ is

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A Corollary

If the linear map $\Phi[a, b, c]$ is 2-positive or 2-copositive, then it is decomposable.

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"Is every 2-positive/2-copostive map in $B(M_3(\mathbb{C}), M_3(\mathbb{C}))$ decomposable?"

Strong Evidence that All PPTES in $\mathbb{C}^3 \otimes \mathbb{C}^3$ (Two Qutrits) have Schmidt Number 2.

Let ρ be the density matrix for a quantum state in a bipartite system $\mathcal{H}_A \otimes \mathcal{H}_B$. The *Schmidt number* of the density matrix (or the state) ρ is defined by

$$SN(\rho) = \min \left\{ \max_k SR(z_k) \right\},$$

where the minimum is taken over all possible decompositions

$$\rho = \sum_k p_k \cdot z_k z_k^*$$

with z_k being vectors in $\mathcal{H}_A \otimes \mathcal{H}_B$ and $p_k > 0$, $\sum_k p_k = 1$.

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A Conjecture

All bound entangled states with positive partial transpose in $\mathbf{C}^3 \otimes \mathbf{C}^3$ have Schmidt number 2.

Dual Cone Relations

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Let us consider the duality between the space $M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$ and the space $B(M_m(\mathbb{C}), M_n(\mathbb{C}))$. Let E_{ij} be the canonical matrix units in $M_m(\mathbb{C})$. For $A = \sum_{i,j=1}^{m} E_{ij} \otimes A_{ij} \in M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$ and a linear map $\phi \in B(M_m(\mathbb{C}), M_n(\mathbb{C}))$, define a bilinear form:

$$\langle A, \phi \rangle = \sum_{i,j=1}^{m} Tr(\phi(E_{ij}) A_{ij}^t).$$

Dual Cone Relations

Denote by $\mathbb{P}_k[m, n]$ and $\mathbb{P}^k[m, n]$ the set of all k-positive maps and the set of all k-copositive maps in $B(M_m(\mathbb{C}), M_n(\mathbb{C}))$, respectively.

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Define convex cones $\mathbb{V}_k[m, n]$ and $\mathbb{V}^k[m, n]$ in $M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$ as

$$\mathbb{V}_k[m, n] = \{zz^* : \ SR(z) \leq k, \ z \text{ in } \mathbb{C}^m \otimes \mathbb{C}^n\}^{\circ\circ},$$
$$\mathbb{V}^k[m, n] = \{(zz^*)^\tau : \ SR(z) \leq k, \ z \text{ in } \mathbb{C}^m \otimes \mathbb{C}^n\}^{\circ\circ}.$$

Here τ is partial transposition that acts as transposition only on the first part of a tensor product.

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$$\begin{array}{ccccccccc} \mathbb{V}_1 & \subsetneqq & \cdots & \mathbb{V}_k & \subsetneqq & \mathbb{V}_{m\wedge n} & = & (M_m(\mathbb{C}) \otimes M_n(\mathbb{C}))^+ \\ \updownarrow & & & \updownarrow & & \updownarrow & & \\ \mathbb{P}_1 & \supsetneqq & \cdots & \mathbb{P}_k & \supsetneqq & \mathbb{P}_{m\wedge n} & \cong & (M_m(\mathbb{C}) \otimes M_n(\mathbb{C}))^+ \end{array}$$

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where $m \wedge n = \min\{m, n\}$, and a similar diagram holds in case of copositivity.

Dual Cone Relations when $m = n = 3$

Denote by $\mathbb{D}$ the cone of all decomposable maps and $\mathbb{T}$ the cone of all positive partial transpose states.

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Dual Cone Relations when $m = n = 3$

$$\begin{array}{lccccccc} \mathit{Conj2}: & \mathbb{V}_1 & \subsetneq & \mathbb{T}(?) & \subsetneq & \mathbb{V}_2 & \subsetneq & \mathbb{V}_3 = (M_3(\mathbb{C}) \otimes M_3(\mathbb{C}))^+ \\ & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ \mathit{Conj1}: & \mathbb{P}_1 & \supsetneq & \mathbb{D}(?) & \supsetneq & \mathbb{P}_2 & \supsetneq & \mathbb{P}_3 \cong (M_3(\mathbb{C}) \otimes M_3(\mathbb{C}))^+ \end{array}$$

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Peel-off Theorem (Marciniak)

If ϕ is a non-zero 2-positive map, then there exists a non-zero completely positive map ψ such that $\phi \geq \psi$.

Trivial Lifting

We will present a slightly stronger version (Choi Decomposition) of the peel-off result by block-matrix approach, which was shown by Choi for the case of 2-positive maps.

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Definition of Trivial Lifting

Given a linear map $\chi \in B(M_s(\mathbb{C}), M_n(\mathbb{C}))$, fix the canonical matrix unit basis E_{ij}, $i, j = 1, .., s$, in $M_s(\mathbb{C})$, under which the Choi matrix is $C_\chi = [\chi(E_{ij})]_{i,j=1}^s \in M_s(M_n(\mathbb{C}))$. Given $L = \{n_1, ..., n_p\} \subset \{1, ..., s+p\}$, where $n_1 < \cdots < n_p$, extend the matrix C_χ to a $(s+p) \times (s+p)$ block matrix $C_L^{lift} \in M_{s+p}(M_n(\mathbb{C}))$ by adding one row and one column of $n \times n$ zero matrices at the n_k^{th} level for each $k = 1, ..., p$ as follows:

Trivial Lifting

Definition of Trivial Lifting

$$C_L^{lift} \triangleq \begin{array}{c} \\ 1^{st} \\ \vdots \\ n_k^{th} \\ \vdots \\ (s+p)^{th} \end{array} \begin{array}{c} \begin{array}{ccccc} 1^{st} & \cdots & n_k^{th} & \cdots & (s+p)^{th} \end{array} \\ \begin{pmatrix} \chi(E_{11}) & \cdots & 0 & \cdots & \chi(E_{1,s}) \\ \vdots & \ddots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & 0 & \ddots & \vdots \\ \chi(E_{s,1}) & \cdots & 0 & \cdots & \chi(E_{s,s}) \end{pmatrix} \end{array}.$$

Denote by $\tilde{\chi}_L$ the map in $B(M_{s+p}(\mathbb{C}), M_n(\mathbb{C}))$ associated with the Choi matrix $C_{\tilde{\chi}_L} = [\tilde{\chi}_L(E_{ij})]_{i,j=1}^{s+p} = C_L^{lift}$. Then the map $\tilde{\chi}_L$ is called a L-trivial lifting of the original map χ. If $L = \{q\}$ is a singleton, simply denote by $\tilde{\chi}_q$ the q-trivial lifting of χ.

Trivial Lifting

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Remark 1 for trivial lifting

A map χ is k-positive or k-copositive, if and only if its trivial lifting $\tilde{\chi}_L$ is k-positive or k-copositive, respectively.

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A map χ is k-positive or k-copositive, if and only if its trivial lifting $\tilde{\chi}_L$ is k-positive or k-copositive, respectively.

Remark 2 for trivial lifting

A map χ is decomposable, if and only if its trivial lifting $\tilde{\chi}_L$ is decomposable.

Main Result

Motivated by Choi's block matrix approach regarding the peel-off theorem, we come up with the following straightforward result:

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Choi Decomposition Theorem

Let ϕ be a non-zero k-positive ($2 \leq k < \min\{m, n\}$) map in $B(M_m(\mathbb{C}), M_n(\mathbb{C}))$. Then there exists a decomposition $\phi = \psi + \gamma$, where ψ is a non-zero completely positive map and γ is a p-trivial lifting of a $(k-1)$-positive map in $B(M_{m-1}(\mathbb{C}), M_n(\mathbb{C}))$, for some $p \in \{1, ..., m\}$.

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Notice that the dimension of the space where the remaining map γ resides is reduced.

Sketch of the Proof: Useful Lemmas

Lemma 1: Positivity in terms of Block Matrix

Suppose a hermitian matrix M is partitioned as

$$M = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix},$$

where A and C are square matrices. TFAE:

1. $M \geq 0$,
2. $A \geq 0,\ M/A = C - B^* A^\dagger B \geq 0,\ range(B) \subset range(A)$,
3. $C \geq 0,\ M/C = A - BC^\dagger B^* \geq 0,\ range(B^*) \subset range(C)$.

Here $A^\dagger$ and $C^\dagger$ refer to the Moore-Penrose pseudo inverses of A and C, respectively.

Sketch of the Proof: Useful Lemmas

Lemma 2: Properties of the Moore-Penrose Pseudo Inverse

1. $AA^{\dagger}A = A$, $A^{\dagger}AA^{\dagger} = A^{\dagger}$.
2. $(AA^{\dagger})^* = AA^{\dagger}$, $(A^{\dagger}A)^* = A^{\dagger}A$.
3. $AA^{\dagger}$ is the orthogonal projector onto the range of A, $A^{\dagger}A$ is the orthogonal projector onto the range of A^*.
4. If A is invertible, then $A^{\dagger} = A^{-1}$.
5. If $A \geq 0$, then $A^{\dagger} \geq 0$.

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Choi Decomposition: Original Part

$$C_\phi = \begin{pmatrix} A_{11} & \cdots & A_{1j} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{i1} & \cdots & A_{ij} & \cdots & A_{im} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mj} & \cdots & A_{mm} \end{pmatrix}$$

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Observation 1: WLOG, assume that $\phi(E_{mm}) \neq 0$.

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Choi Decomposition: Peel-off Part

$$U = \begin{pmatrix} A_{1m}A_{mm}^{\dagger}A_{m1} & \cdots & A_{1m}A_{mm}^{\dagger}A_{mj} & \cdots & A_{1m}A_{mm}^{\dagger}A_{mm} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{im}A_{mm}^{\dagger}A_{m1} & \cdots & A_{im}A_{mm}^{\dagger}A_{mj} & \cdots & A_{im}A_{mm}^{\dagger}A_{mm} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{mm}A_{mm}^{\dagger}A_{m1} & \ddots & A_{mm}A_{mm}^{\dagger}A_{mj} & \cdots & A_{mm}A_{mm}^{\dagger}A_{mm} \end{pmatrix}$$

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Observation 2: $U \geq 0$, and U is non-zero.

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The remaining part is a matrix with $R_{ij} = A_{ij} - A_{im}A_{mm}^{\dagger}A_{mj}$ in its (i,j)-entry.

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$$R = \begin{pmatrix} A_{11} - A_{1m}A_{mm}^{\dagger}A_{m1} & \cdots & A_{1j} - A_{1m}A_{mm}^{\dagger}A_{mj} & \cdots & A_{1m} - A_{1m}A_{mm}^{\dagger}A_{mm} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{i1} - A_{im}A_{mm}^{\dagger}A_{m1} & \cdots & A_{ij} - A_{im}A_{mm}^{\dagger}A_{mj} & \cdots & A_{im} - A_{im}A_{mm}^{\dagger}A_{mm} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{m1} - A_{mm}A_{mm}^{\dagger}A_{m1} & \ddots & A_{mj} - A_{mm}A_{mm}^{\dagger}A_{mj} & \cdots & A_{mm} - A_{mm}A_{mm}^{\dagger}A_{mm} \end{pmatrix}$$

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Choi Decomposition: Remaining Part

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Observation 3: entries in last row and last column of R are zero matrices

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Question: what will γ be?

Good News: k-positivity of ϕ guarantees the $(k-1)$-positivity of γ.

Choi Decomposition: Employ k-positivity of ϕ for $\xi\xi^*$

$$\xi\xi^* = \begin{pmatrix} w^1(w^1)^* & \cdots & w^1(w^j)^* & \cdots & w^1 e_m^* \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ w^i(w^1)^* & \cdots & w^i(w^j)^* & \cdots & w^i e_m^* \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ e_m(w_1)^* & \cdots & e_m(w^j)^* & \cdots & e_m e_m^* \end{pmatrix} \geq 0$$

Here $\xi = (w^1, ..., w^{k-1}, e_m)^T$ where $w^1, w^2, .., w^{k-1} \in \mathbb{C}^m$ are arbitrary column vectors, and $e_m = (0, ..., 0, 1)^T \in \mathbb{C}^m$.

Sketch of the Proof: Block Matrix Approach

Choi Decomposition: Employ k-positivity of ϕ for $\xi\xi^*$

$$(id_k \otimes \phi)(\xi\xi^*) = \begin{pmatrix} \phi(w^1(w^1)^*) & \cdots & \phi(w^1(w^j)^*) & \cdots & \phi(w^1 e_m^*) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \phi(w^i(w^1)^*) & \cdots & \phi(w^i(w^j)^*) & \cdots & \phi(w^i e_m^*) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \phi(e_m(w_1)^*) & \cdots & \phi(e_m(w^j)^*) & \cdots & \phi(e_m e_m^*) \end{pmatrix} \geq 0.$$

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Observation 4: Recall Lemma 1.

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By equivalence of Condition 1 and Condition 3 in Lemma 1, the condition $(id_k \otimes \phi)(\xi\xi^*) \geq 0$ expands to:

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$$\begin{pmatrix} \phi(w^1 e_m^*) \\ \vdots \\ \phi(w^{k-1} e_m^*) \end{pmatrix} \phi(e_m e_m^*)^\dagger \begin{pmatrix} \phi(e_m(w^1)^*) & \cdots & \phi(e_m(w^{k-1})^*) \end{pmatrix}.$$

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$$\begin{pmatrix} \phi(w^1 e_m^*) \\ \vdots \\ \phi(w^{k-1} e_m^*) \end{pmatrix} \phi(e_m e_m^*)^\dagger \begin{pmatrix} \phi(e_m(w^1)^*) & \cdots & \phi(e_m(w^{k-1})^*) \end{pmatrix}.$$

Observation 5: the (s,t) entry amounts to $\gamma(w^s(w^t)^*)$, How?

Sketch of the Proof: Block Matrix Approach

Choi Decomposition: Employ k-positivity of ϕ for $\xi\xi^*$

$$\begin{aligned}
&\phi(w^s(w^t)^*) - \phi(w^s e_m^*)\phi(e_m e_m^*)^\dagger \phi(e_m (w^t)^*) \\
&= \phi(w^s(w^t)^*) - \phi\left(\sum_{i=1}^m w_i^s e_i e_m^*\right)\phi(e_m e_m^*)^\dagger \phi\left(\sum_{j=1}^m \overline{w_j^t} e_m e_j^*\right) \because \textit{Linearity} \\
&= \phi(w^s(w^t)^*) - \left(\sum_{i=1}^m w_i^s \phi(E_{im})\right)\phi(E_{mm})^\dagger \left(\sum_{j=1}^m \overline{w_j^t}\phi(E_{mj})\right) \because \textit{Linearity} \\
&= \phi(w^s(w^t)^*) - \sum_{i=1}^m \sum_{j=1}^m w_i^s \overline{w_j^t}\left(\phi(E_{im})\phi(E_{mm})^\dagger \phi(E_{mj})\right) \because \textit{Linearity} \\
&= \phi(w^s(w^t)^*) - \sum_{i=1}^m \sum_{j=1}^m w_i^s \overline{w_j^t}(A_{im}A_{mm}^\dagger A_{mj}) \\
&= \phi(w^s(w^t)^*) - \sum_{i=1}^m \sum_{j=1}^m w_i^s \overline{w_j^t} U_{ij} \\
&= \phi(w^s(w^t)^*) - \sum_{i=1}^m \sum_{j=1}^m w_i^s \overline{w_j^t}\psi(e_i e_j^*) \because U = C_\psi \\
&= \phi(w^s(w^t)^*) - \psi(w^s(w^t)^*) \\
&= \gamma(w^s(w^t)^*) \because \gamma = \phi - \psi.
\end{aligned}$$

Sketch of the Proof: Block Matrix Approach

The following proves that γ is $(k-1)$-positive.

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Choi Decomposition: Employ k-positivity of ϕ for $\xi\xi^*$

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Combining Observation 3 and the above fact, we know how the remaining map γ looks like.

Sketch of the Proof: Block Matrix Approach

Denote the matrix $R = C_\gamma$ by:

$$R = \begin{pmatrix} \text{\Huge K} & & 0 \\ & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} \text{\Huge C_K} & & 0 \\ & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

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$$R = \begin{pmatrix} K & & 0 \\ & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} C_\kappa & & 0 \\ & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

Choi Decomposition: γ is a trivial-lifting of κ

The map $\kappa \in B(M_{m-1}(\mathbb{C}), M_n(\mathbb{C}))$ is defined by the Choi matrix $K \in M_{(m-1)n}(\mathbb{C})$ through $\kappa(E_{st}) = K_{st}, s, t = 1, .., m-1$. It is obvious that $\gamma \in B(M_m(\mathbb{C}), M_n(\mathbb{C}))$ is the m-trivial lifting of $\kappa \in B(M_{m-1}(\mathbb{C}), M_n(\mathbb{C}))$.

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A similar result holds for k-copositive maps.

Positivity for Dimension

Theorem

Let $2 \leq k < \min\{m, n\}$. Any non-zero k-positive (respectively k-copositive) map in $B(M_m(\mathbb{C}), M_n(\mathbb{C}))$ is the sum of at most $(k-1)$ many non-zero completely positive (respectively completely copositive) maps and a positive map which is the trivial lifting of a positive map in $B(M_{m-k+1}(\mathbb{C}), M_n(\mathbb{C}))$.

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Remark: The Choi decomposition may no longer be valid for a general positive map ϕ even when ϕ is in $B(M_2(\mathbb{C}), M_2(\mathbb{C}))$.

An Affirmative Answer to the Conjecture

Theorem

Every 2-positive or 2-copositive map ϕ in $B(M_3(\mathbb{C}), M_3(\mathbb{C}))$ is decomposable.

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Proof: WLOG, we assume the 2-positive(respectively 2-copositive) map ϕ is not zero. In this concrete case of $B(M_3(\mathbb{C}), M_3(\mathbb{C}))$, the peel-off process yields that:

$$\phi = \psi + \tilde{\kappa}_p \textit{ for some } p \in \{1, ..., m\}$$

where ψ is completely positive (respectively completely copositive) and $\tilde{\kappa}_p$ is a p-trivial lifting of a positive map $\kappa \in B(M_2(\mathbb{C}), M_3(\mathbb{C}))$.

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Hence, $\phi = \psi + \tilde{\kappa}_p$ is also decomposable.

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There exist different methods to decompose a 2-positive map $\Phi[a, b, c]$ into a sum of a completely positive map and a completely copositive map.

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There exist different methods to decompose a 2-positive map $\Phi[a, b, c]$ into a sum of a completely positive map and a completely copositive map.

An Example

$$\Phi[a,b,c](X) = \begin{pmatrix} ax_{11} + bx_{22} + cx_{33} & -x_{12} & -x_{13} \\ -x_{21} & cx_{11} + ax_{22} + bx_{33} & -x_{23} \\ -x_{31} & -x_{32} & bx_{11} + cx_{22} + ax_{33} \end{pmatrix}$$

for $X = [x_{ij}] \in M_3(\mathbb{C})$. Here $a \in [1, 2)$ and $bc \geq (2-a)(b+c)$.

A Corollary & An Example

An Example: Decomposition 1

$\Phi[a,b,c] = \Phi_1 + \Phi_2$, where

$$\Phi_1 \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} ax_{11} + bx_{22} + cx_{33} & -x_{12} & -x_{13} \\ -x_{21} & cx_{11} + ax_{22} & (\frac{2}{a} - a)x_{23} \\ -x_{31} & (\frac{2}{a} - a)x_{32} & bx_{11} + ax_{33} \end{bmatrix}, (CP)$$

$$\Phi_2 \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & bx_{33} & (a - 1 - \frac{2}{a})x_{23} \\ 0 & (a - 1 - \frac{2}{a})x_{32} & cx_{22} \end{bmatrix} (CcoP).$$

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Another decomposition given by Cho, Kye and Lee is:

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Another decomposition given by Cho, Kye and Lee is:

An Example: Decomposition 2

$$\Phi[a,b,c] = (1-\sqrt{bc})\Phi\left[\frac{a-\sqrt{bc}}{1-\sqrt{bc}}, 0, 0\right](CP) + \sqrt{bc}\Phi\left[1, \sqrt{\frac{b}{c}}, \sqrt{\frac{c}{b}}\right](CcoP).$$

An Algorithm to Decompose

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"Is such an algorithm possible even in $B(M_2(\mathbb{C}), M_2(\mathbb{C}))$?"

An Example in Higher Dimensions

"Does there exist a 2-positive but indecomposable map in $B(M_3(\mathbb{C}), M_4(\mathbb{C}))$?"

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Thank You!