Enhancing the entanglement detection power of positive maps

Arvind

Indian Institute of Science Education and Research (IISER) Mohali

Research Interests

- Quantum Entanglement
- Quantum Nonlocality and quantum contextuality.
- Weak quantum measurements.
- Quantum optics and symplectic methods.
- NMR Quantum Information processing.
- Design of experiments for padagogy.

IISER Mohali Group

Professor Kavita Dorai

Ritabrata Sengupta [Graduated]
Shruti Dogra [PhD]
Debmalya Das [PhD]
Harpreet Singh[PhD]
Varinder Singh[PhD]
Chandan Kumar[PhD]
Aakash Sharawat[PhD]
Jaskaran Singh[PhD]

Vikram Sharma[PhD]
Amandeep Singh[PhD]
Mayank Mishra[MS]
Rajinder Bhatti[MS]
Atul Singh Arora[MS]
Kishor Bharti
Akshay[MS]

Acknowledgements

Work Done with Ritabrata Sengupta[PhD] and Mayank Mishra[MS]

Quantum Entanglement

Consider a bipartite system (system composed of two parts) and described by a density operator $\rho \in \mathcal{B}(\mathcal{H}_\infty \otimes \mathcal{H}_\in)$.

Strongly Separable

$$\rho = \rho^1 \otimes \rho^2$$

Weakly Separable

$$\rho = \sum_i \boldsymbol{p_i} \; \rho_i^1 \otimes \rho_i^2 \quad \text{with} \quad \boldsymbol{p_i} > 0.$$

States which are not separable are entangled

Quantum algorithms need entanglement

Quantum non-locality is intimately connected with entanglement

Central Question

To determine whether a given arbitrary (pure or mixed) bipartite state ρ is entangled or separable.

The problem has a simple solution for the case of pure states.

For mixed states such a characterization is not possible and only partial solutions are available.

Maps P anc CP

A map $\varphi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ is said to be positive if it maps the set of positive operators in $\mathcal{B}(\mathcal{H})$ (denoted by $\mathcal{B}(\mathcal{H})_+$) to itself.

A positive map is said to be completely positive if the extension $1_d \otimes \varphi : \mathcal{B}(\mathbb{C}^d \otimes \mathcal{H}) \longrightarrow \mathcal{B}(\mathbb{C}^d \otimes \mathcal{H})$ is a positive map for all $d \geq 1$.

CP maps show a remarkably simple representation due to (Sudarshan, Kraus and Choi), P maps which are not CP are not easily characterizable.

P (NOT CP) maps as entanglement witnesses

- Separable states remain positive when we apply P (NOT CP) to one of the systems.
- Any state which turns into a non-state under the application of a P (NOT CP) to one of the systems, has to be entangled.
- Such maps act as entanglement witnesses.

NPT vs PPT

Transpose is a P but not CP map

- States which are negative under partial transpose are entangled and called NPT entangled states.
- States which are positive under partial transpose are called PPT states.
- PPT states could be separable or entangled.
- Entanglement of PPT entangled states is called bound entanglement.
- No PPT entangled states for $2 \otimes 2$ and $2 \otimes 3$ systems.

Extremal Extensions of Positive Maps

- $\varphi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ to be a positive indecomposable map. (Not CP)
- For any $A \in Gl_n(\mathbb{C})$, we can define a map

$$\begin{aligned} A : \mathcal{B}(\mathcal{H}) &\longrightarrow \mathcal{B}(\mathcal{H}) \\ X &\longmapsto AXA^{\dagger} \quad \text{For } X \in \mathcal{B}(\mathcal{H}) \end{aligned}$$

-

$$\begin{aligned} \varphi \circ A =& \varphi_A \qquad \text{Inner Automorphism} \\ A \circ \varphi =& \varphi^A \qquad \text{Outer Automorphism} \end{aligned}$$

Extremality

The set of positive maps is a convex set described by its ‘extremal points’.

A positive map h is said to be extremal, when for any decomposition $h = h_1 + h_2$, where h_1 and h_2 are positive maps, $h_i = \lambda_i h$, where $\lambda_i \geq 0$ and $\lambda_1 + \lambda_2 = 1$.

For an extremal φ both φ_A and φ^A are extremal.

Theorem

For any positive map $\varphi : \mathcal{B}(\mathcal{H}) \longmapsto \mathcal{B}(\mathcal{H})$, and for any full rank operator A, (such that $AA^\dagger \leq I$) φ_A is a positive map. Moreover, if φ is not completely positive and extremal, so is the map φ_A.

Theorem

1. *For any positive map $\varphi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$, and any invertible operator A, the outer automorphism φ^A is a positive map.*
2. *Any entangled state ρ detected by φ^A is detected by φ and vice versa.*

Thus outer automorphism is useless as far as entanglement detection is concerned.

Positivity under partial transpose is invariant under inner unitary automorphism. In other words, for the transpose map T and any unitary operator U, $(I \otimes T)\rho \geq 0$ implies $(I \otimes T_U)\rho \geq 0$ for any state ρ.

Choi map

- $$\varphi_{C_1} : ((x_{ij})) \longmapsto \frac{1}{2}\begin{pmatrix} x_{11}+x_{22} & -x_{12} & -x_{13} \\ -x_{21} & x_{22}+x_{33} & -x_{23} \\ -x_{31} & -x_{32} & x_{33}+x_{11} \end{pmatrix}$$

- $$\varphi_{C_2} : ((x_{ij})) \longmapsto \frac{1}{2}\begin{pmatrix} x_{11}+x_{33} & -x_{12} & -x_{13} \\ -x_{21} & x_{22}+x_{11} & -x_{23} \\ -x_{31} & -x_{32} & x_{33}+x_{22} \end{pmatrix}$$

Extensions of Choi map

- We are interested in UNITARY INNER AUTOMORPHISMS of the Choi maps.
- $\varphi_{C_{1,2}} \circ U$ where $U \in SU(3)$ is a unitary operator.

TILES Construction

$$
\begin{aligned}
|\psi_0\rangle &= \frac{1}{\sqrt{2}}|0\rangle\left(|0\rangle - |1\rangle\right), \quad |\psi_2\rangle = \frac{1}{\sqrt{2}}|2\rangle\left(|1\rangle - |2\rangle\right), \\
|\psi_1\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)|2\rangle, \quad |\psi_3\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle - |2\rangle\right)|0\rangle, \\
|\psi_4\rangle &= \frac{1}{3}\left(|0\rangle + |1\rangle + |2\rangle\right)\left(|0\rangle + |1\rangle + |2\rangle\right) \qquad (1)
\end{aligned}
$$

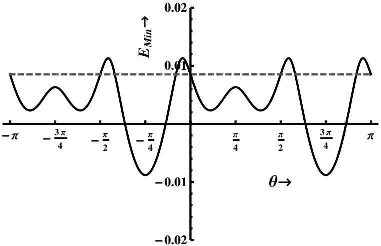
$$
\rho = \frac{1}{4}\left(I_9 - \sum_{i=0}^{4} |\psi_i\rangle\langle\psi_i|\right). \qquad (2)
$$

is PPT entangled.

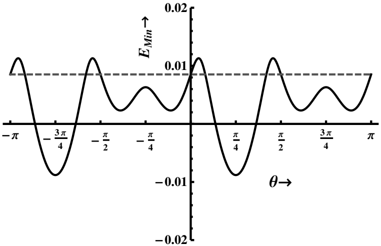
Maps $I \otimes \varphi_{C_{1,2}}$ applied to this state keep it positive.

Consider a one-parameter family of extremal extensions of the Choi maps $\varphi_{C_{1,2}}(\theta) = \varphi_{C_{1,2}} \circ U(\theta)$ with

$$U(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}.$$



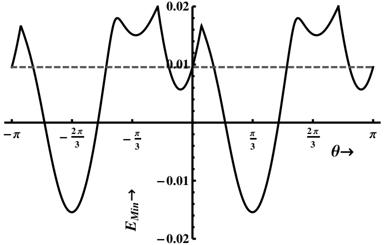
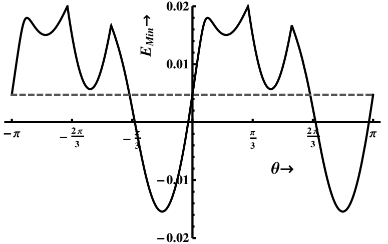
0.02
E_{Min}→
0.01
−0.01
−0.02
−π
−3π/4
−π/2
−π/4
π/4
π/2
3π/4
π
θ→



0.02
E_{Min}→
0.01
−0.01
−0.02
−π
−3π/4
−π/2
−π/4
π/4
π/2
3π/4
π
θ→

The PYRAMID construction

- $$v_i = N\left(\cos\frac{2\pi j}{5}, \sin\frac{2\pi i}{5}, h\right) \quad j = 0, \cdots, 4;$$
  where $h = \frac{1}{2}\sqrt{1+\sqrt{5}}$ and $N = \frac{2}{\sqrt{5+\sqrt{5}}}$.
- $$|\psi_j\rangle = |v_j\rangle \otimes |v_{2j \mod 5}\rangle, \quad j = 0, \cdots, 4.$$
- PPT entangled states can be defined similarly


0.02
0.01
−0.01
−0.02
−π
−2π/3
−π/3
π/3
2π/3
π
θ→
E_Min→

0.02
0.01
−0.01
−0.02
−π
−2π/3
−π/3
π/3
2π/3
π
θ→
E_Min→

Connection between Choi maps

$$\varphi_{C_1} = U\left(\frac{3\pi}{2}\right) \circ \varphi_{C_2} \circ U\left(\frac{\pi}{2}\right).$$

Another connection

We consider an extremal positive in-decomposable map $\varphi : \mathcal{B}(\mathbb{C}^n) \longrightarrow \mathcal{B}(\mathbb{C}^n)$ and the corresponding bi-quadratic $F\begin{pmatrix} X \\ Y \end{pmatrix} = F\begin{pmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{pmatrix} = \langle Y|\varphi(|X\rangle\langle X|)Y\rangle$, where $|X\rangle = (x_1, \cdots, x_n)^t$, and $|Y\rangle = (y_1, \cdots, y_n)^t$, t denotes the transpose, and x_i, y_j are real parameters.

A map φ is positive and extremal if and only if the corresponding real bi-quadratic form is positive and extremal.

Indecomposability of the map implies that the form F can not be written as a sum of square of quadratic forms.

It can be shown that for any set of n non zero real parameters $a_1, \cdots, a_n$; the form

$$G\begin{pmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{pmatrix} = F\begin{pmatrix} a_1 x_1 & \cdots & a_n x_n \\ y_1 & \cdots & y_n \end{pmatrix}$$

is also an extremal positive form. Hence the corresponding map denoted by $\varphi_{(a_1,\cdots,a_n)}$ is an extremal indecomposable positive map.

This extension is useful

$$
(x, t) = \frac{1}{4 + \frac{3}{t} + 4t} \left(\begin{array}{ccc|ccc|ccc}
1+t & 0 & 0 & 0 & x & 0 & 0 & 0 & x \\
0 & t & 0 & x & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{t} & 0 & 0 & 0 & x & 0 & 0 \\
\hline
0 & x & 0 & \frac{1}{t} & 0 & 0 & 0 & 0 & 0 \\
x & 0 & 0 & 0 & 1+t & 0 & 0 & 0 & x \\
0 & 0 & 0 & 0 & 0 & t & 0 & x & 0 \\
\hline
0 & 0 & x & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & x & 0 & \frac{1}{t} & 0 \\
x & 0 & 0 & 0 & x & 0 & 0 & 0 & 1
\end{array} \right)
$$

where $0 < x < 1$ and $0 < t < 1$.

Recasting this extension

$$\varphi_{(a_1,\cdots,a_n)} = \varphi \circ A$$

where A is an operator given by the diagonal matrix

$$A = \mathrm{Diag}(a_1, a_2 \cdots, a_n)$$

This is clearly a non-unitary inner automorphism and connects our earlier result with the present formulation.

Local Filters

Local operators represented by $L \otimes M$ where L,M are local invertible operators.

$$\rho^f = (L \otimes M)\rho(L \otimes M)^\dagger$$

- If L, M are full rank operators. Then the map $\rho \mapsto (L \otimes M)\rho(L \otimes M)^\dagger$ does not change the Schmidt number of the state.
- The PPT or NPT character of a state is invariant under an invertible local filtration operation.

Implementation of filters

- Singular value decomposition given by $L = U_1 D_1 V_1$ and $M = U_2 D_2 V_2$. Here U_1, U_2, V_1, V_2 are unitary operators and D_1, D_2 are diagonal with real positive definite diagonal entries.
- Unitaries can be implemented by Hamiltonians. We therefore need to focus on D_1 and D_2.

Consider the implementation of D_1 on $\mathcal{H}_A$. Consider a set of n orthogonal but un-normalized vectors of the form $\left|u_j\right\rangle = \sqrt{d_j}\,|j\rangle$ in the n dim. system Hilbert space. Extend each of these vectors into a $2n$ dimensional space $\left|\xi_j\right\rangle = \sqrt{d_j}\,|j\rangle + \sqrt{1-d_j}\,|j+n\rangle$.

$$P_j = \left|\xi_j\right\rangle\left\langle\xi_j\right| = \left(\begin{array}{c|c} \eta_j & \delta_j \\ \hline \delta_j & \eta'_j \end{array}\right)_{2n\times 2n}$$

$$\eta_j = d_j\,|j\rangle\langle j|\,, \quad \eta'_j = \left(1-d_j\right)|j\rangle\langle j|\,, \quad \delta_j = \sqrt{d_j\left(1-d_j\right)}\,|j\rangle\langle j|$$

$$P = \sum_{j=1}^{n} P_j = \left(\begin{array}{c|c} D_1 & \Delta \\ \hline \Delta & D'_1 \end{array}\right)_{2n\times 2n}$$

where $D_1 = \eta_1 + \cdots + \eta_n$ original operator that we wanted to implement, $D'_1 = \eta'_1 + \cdots + \eta'_n$ is a complementary operator obtained from D_1 and $\Delta = \delta_1 + \cdots + \delta_n$ represents the cross

Now consider the system to be in an arbitrary state ρ_A and the one-qubit ancilla to be in the state $|0\rangle \langle 0|$. Consider a measurement of P on this composite system. If the outcome of the measurement is positive, we retain the state. The state after such a selection is given by the action of the projection operator P on the composite state:

$$
\begin{aligned}
& P \left(|0\rangle \langle 0| \otimes \rho_A \right) P \\
= & \left(\begin{array}{c|c} D_1 \rho_A D_1 & D_1 \rho_A \Delta \\ \hline \Delta \rho_A D_1 & \Delta \rho_A \Delta \end{array} \right)
\end{aligned}
$$

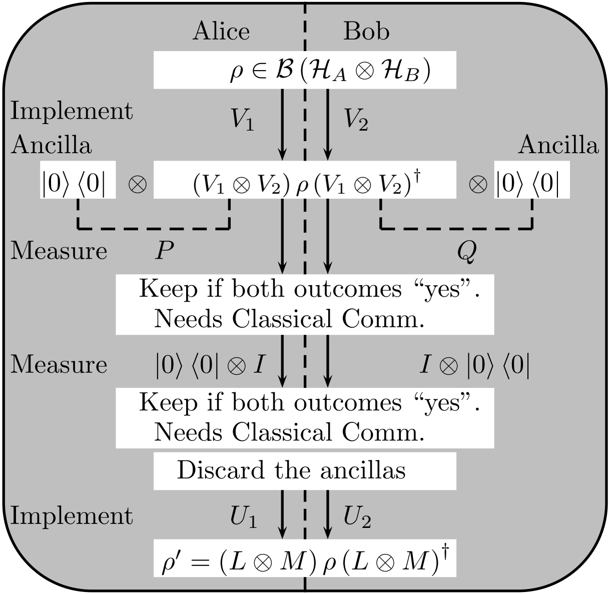


Figure : Schematic diagram for performing the local filtration via

Cryptography with Bound Entanglement

- Can bound entangled states be used?
- Teleportation, cryptography
- Examples of Key distillable bound entangled states have been given.
- We have shown that by using appropriate filtration schemes one can turn a non-key distillable bound entangled state into a state with distillable key.

Conclusions

- Considered a method for extremal extension of entanglement witnesses.
- Used the method to produces useful extensions.
- Connection with UPBS.
- Found a new class of Bound Entangled states.
- Local filters as as dual to automorphisms of maps.
- Physical implementation of local filters.
- Enhancement of distillable key from bound entangled states.