Strong monogamy of multi-party quantum entanglement

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Entanglement

- Non-local Nature of Quantum State
- Useful Applications
  - Quantum Teleportation
  - Dense Coding
  - Quantum Cryptography (QKD)
  - Etc.

- Quantification and Qualification

Entanglement of Formation (EoF)

- For bipartite pure state $|\psi\rangle_{AB} \in \mathbb{C}^d \otimes \mathbb{C}^{d'}$

$$E_f\left(|\psi\rangle\right) = S\left(\rho_A\right) \quad \left(= S\left(\rho_B\right)\right)$$

$$\rho_A = \mathrm{tr}_B\left(|\psi\rangle_{AB}\langle\psi|\right), \quad S\left(\rho\right) = -\mathrm{tr}\rho\log\rho$$

- Mixed state $\rho_{AB} \in \mathcal{B}\left(\mathbb{C}^d \otimes \mathbb{C}^{d'}\right)$

$$E_f\left(\rho_{AB}\right) = \min \sum_i p_i E_f\left(|\psi_i\rangle\right)$$

min: *over all possible pure state decompositions*

$$\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB}\langle\psi_i|$$

Tangle (Linear entropy)

- Pure state $|\psi\rangle_{AB} \in \mathbb{C}^d \otimes \mathbb{C}^{d'}$

$$\tau\left(|\psi\rangle\right) = 2\left(1 - \mathrm{tr}\rho_A^2\right) = S_l\left(\rho_A\right)$$

- Mixed state $\rho_{AB} \in \mathcal{B}\left(\mathbb{C}^d \otimes \mathbb{C}^{d'}\right)$

$$\tau\left(\rho_{AB}\right) = \left[\min \sum_i p_i \sqrt{\tau\left(|\psi_i\rangle_{AB}\right)}\right]^2$$

min: *over all possible pure state decompositions*

$$\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB} \langle\psi_i|$$

Tangle

- Analytic formula for two-qubit system
  - For a two-qubit state $\rho_{AB} \in \mathcal{B}\left(\mathbb{C}^2 \otimes \mathbb{C}^2\right)$

$$\tilde{\rho}_{AB} = \left(\sigma_y \otimes \sigma_y\right) \rho_{AB}^{*} \left(\sigma_y \otimes \sigma_y\right)$$

$$\lambda_i : \text{ the singularvalues of } \rho_{AB}\tilde{\rho}_{AB} \text{ in decreasing order}$$

$$C(\rho_{AB}) = \max\{0,\ \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} : \quad \text{concurrence}$$

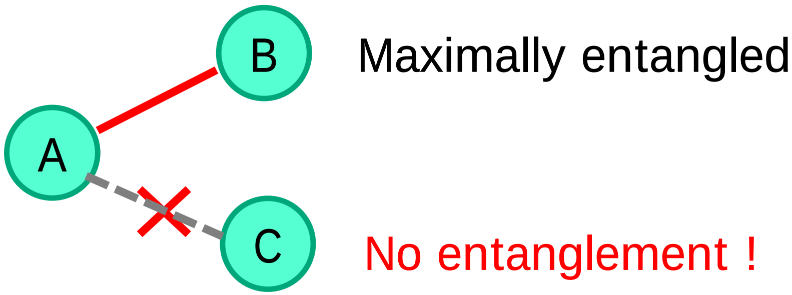
$$\Rightarrow \quad \tau\left(\rho_{AB}\right) = C\left(\rho_{AB}\right)^2$$

[W. K. Wootters, PRL 80 2245 (1998)]

Multi-party quantum entanglement

Monogamy of entanglement (MoE)

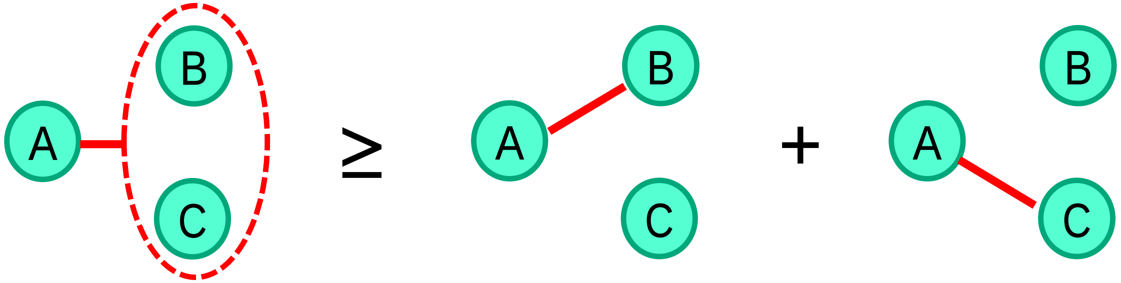
- Restricted shareability of multi-party entanglement
  - Three-qubit systems: $|\psi\rangle_{ABC} = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B\right) \otimes |\varphi\rangle_C$



- Unique characteristic of quantum correlation with no classical counterpart: classical correlations can be shared freely among different parties
- Applications in quantum information processing
  - Bound on the amount of information to eveasdropper: security proof of quantum cryptography
  - Characterization of multi-party entanglement

Characterization of MoE

- Upper bound on a sum of bipartite entanglement measures showing that bipartite sharing of entanglement is bounded.
  - Three-qubit systems: Coffman-Kundu-Wootters inequality



$$\tau\left(|\psi\rangle_{A|BC}\right) \geq \tau\left(\rho_{A|B}\right) + \tau\left(\rho_{A|C}\right)$$

[V. Coffman, J. Kundu and W. K. Wootters PRA 61. 052306 (2000)]

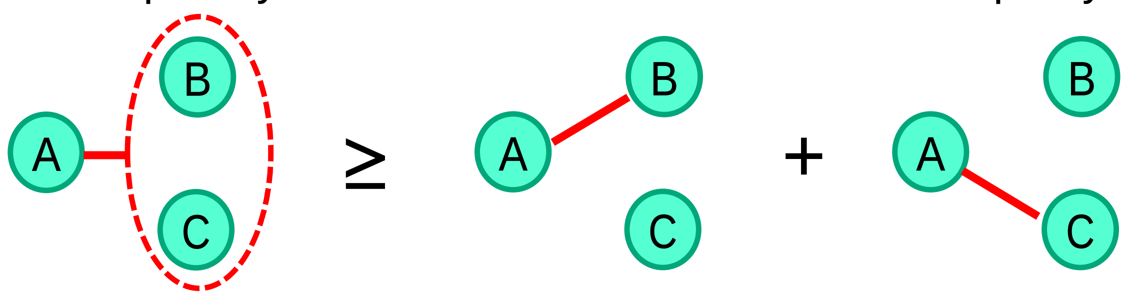
- Tangle

$$\tau\left(|\psi\rangle_{AB}\right) = 4\det\rho_A = C^2\left(|\psi\rangle_{AB}\right)$$

$$\tau\left(\rho_{AB}\right) = \min\sum_i p_i \tau\left(|\psi_i\rangle_{AB}\right), \qquad \rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB}\langle\psi_i|$$

Characterization of MoE

- Upper bound on a sum of bipartite entanglement measures showing that bipartite sharing of entanglement is bounded.
  - Three-qubit systems: Coffman-Kundu-Wootters inequality



$\tau\left(|\psi\rangle_{A|BC}\right)$ $\tau\left(\rho_{A|B}\right)$ $\tau\left(\rho_{A|C}\right)$

[V. Coffman, J. Kundu and W. K. Wootters PRA 61. 052306 (2000)]

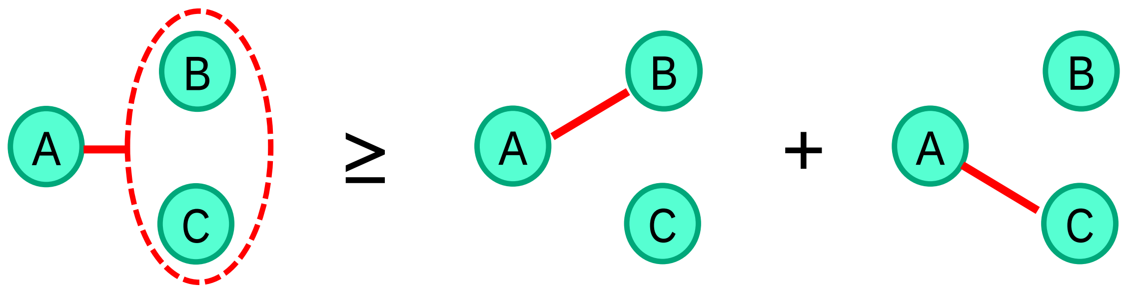
- 3-Tangle

$$\tau\left(|\psi\rangle_{A|B|C}\right)=\tau\left(|\psi\rangle_{A|BC}\right)-\tau\left(\rho_{A|B}\right)-\tau\left(\rho_{A|C}\right)$$

  - Genuine three-party entanglement

Characterization of MoE

- Upper bound on a sum of bipartite entanglement measures showing that bipartite sharing of entanglement is bounded.
  - Three-qubit systems: Coffman-Kundu-Wootters inequality



$\tau\left(|\psi\rangle_{A|BC}\right) \quad \tau\left(\rho_{A|B}\right) \quad \tau\left(\rho_{A|C}\right)$

[V. Coffman, J. Kundu and W. K. Wootters PRA 61. 052306 (2000)]

- Generalization of CKW inequality into multi-qubit systems

$$\tau\left(|\psi\rangle_{A_1|A_2\cdots A_n}\right) \geq \tau\left(\rho_{A_1|A_2}\right) + \cdots + \tau\left(\rho_{A_1|A_n}\right)$$

[T. J. Osborne and F. Verstraete PRL 96. 220503 (2006)]

W-class state

- n-qubit generalized W-class state

$$|W\rangle_{A_1\ldots A_n} = a_1|10\ldots0\rangle + a_2|01\ldots0\rangle + \cdots + a_n|00\ldots1\rangle \quad \text{with} \quad \sum_{i=1}^{n}|a_i|^2 = 1$$

  - Generalization of W state
    - Three-qubit W state: $|W\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right)$

- Saturation of CKW inequality

$$\tau_{A_1|A_2\ldots A_n} = \tau_{A_1|A_2} + \tau_{A_1|A_3} + \cdots + \tau_{A_1|A_n}$$

[JSK and B. C. Sanders, J. Phys. A 41. 495301 (2008)]

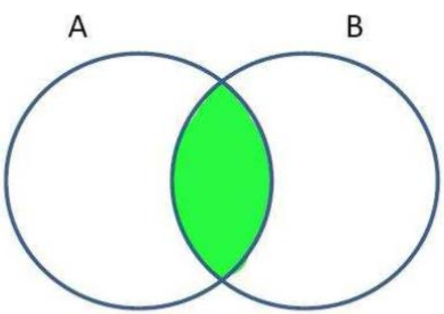
General Monogamy Inequalities

- Squashed entanglement
  - For ρ_{AB}, $\mathrm{ext}\rho_{AB} := \{\rho_{ABE} \mid \mathrm{tr}_E \rho_{ABE} = \rho_{AB}\}$

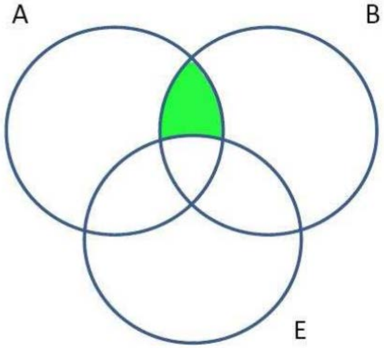
$$E_{sq}(\rho_{AB}) := \frac{1}{2} \inf \{ I(A;B|E) := S(\rho_{AE}) + S(\rho_{BE}) - S(\rho_{ABE}) - S(\rho_E) \}$$

inf: over all $\rho_{\mathrm{ABE}} \in \mathrm{ext}\rho_{AB}$

[M. Christandl and A. Winter, J. Math. Phys. 45, p. 829-840 (2004)]



$I(A;B)$



$I(A;B|E)$

General Monogamy Inequalities

- Squashed entanglement
  - For ρ_{AB}, $\mathrm{ext}\rho_{AB} := \{\rho_{ABE} \mid \mathrm{tr}_E \rho_{ABE} = \rho_{AB}\}$

$$E_{sq}(\rho_{AB}) := \frac{1}{2}\inf\{I(A;B|E) := S(\rho_{AE}) + S(\rho_{BE}) - S(\rho_{ABE}) - S(\rho_E)\}$$

  - Entanglement monotone
  - Lower bound of $E_f(\rho_{AB})$, upper bound of $E_D(\rho_{AB})$
  - For $\rho_{AB} = |\psi\rangle_{AB}\langle\psi|$, $\rho_{\mathrm{ABE}} \in \mathrm{ext}\rho_{AB} \Rightarrow \rho_{\mathrm{ABE}} = |\psi\rangle_{AB}\langle\psi| \otimes \rho_{\mathrm{E}}$

$$E_{sq}(|\psi\rangle_{AB}) = S(\rho_A), \quad \rho_A = \mathrm{tr}_B(|\psi\rangle_{AB}\langle\psi|)$$

General Monogamy Inequalities

- Squashed entanglement
  - For ρ_{AB}, $\operatorname{ext}\rho_{AB} := \left\{ \rho_{ABE} \mid \operatorname{tr}_E \rho_{ABE} = \rho_{AB} \right\}$

$$E_{sq}\left(\rho_{AB}\right) := \frac{1}{2}\inf\left\{ I\left(A;B\mid E\right) := S\left(\rho_{AE}\right) + S\left(\rho_{BE}\right) - S\left(\rho_{ABE}\right) - S\left(\rho_E\right) \right\}$$

  - Monogamy inequality
  - ρ_{ABCE}, $\quad I\left(A;BC\mid E\right) = I\left(A;B\mid E\right) + I\left(A;C\mid BE\right) \quad$ (chain rule)

$$\Rightarrow \quad S_{sq}\left(\rho_{A(BC)}\right) \geq S_{sq}\left(\rho_{AB}\right) + S_{sq}\left(\rho_{AC}\right)$$

$$\left(\text{by minimizing } E \text{ for } I\left(A;BC\mid E\right)\right)$$

[M. Koashi and A. Winter, Phys. Rev. A 69, 022309 (2004)]

General Monogamy Inequalities

- Squashed entanglement
  - For ρ_{AB}, $\mathrm{ext}\rho_{AB} := \{\rho_{ABE} \mid \mathrm{tr}_E \rho_{ABE} = \rho_{AB}\}$

$$E_{sq}(\rho_{AB}) := \frac{1}{2}\inf\{I(A;B|E) := S(\rho_{AE}) + S(\rho_{BE}) - S(\rho_{ABE}) - S(\rho_E)\}$$

  - Monogamy inequality
  - ρ_{ABCE}, $I(A;BC|E) = I(A;B|E) + I(A;C|BE)$ (chain rule)

$$\Rightarrow \quad S_{sq}(\rho_{A(BC)}) \geq S_{sq}(\rho_{AB}) + S_{sq}(\rho_{AC})$$

[M. Koashi and A. Winter, Phys. Rev. A 69, 022309 (2004)]

  - $E_{sq}(\rho_{AB}) = 0$ iff ρ_{AB} : separable

[F.G.S.L. Brandao, M. Christandl and Jon Yard, Commun. Math. Phys. 306, 805 (2011)]

Polygamy Inequality

- Dual monogamy inequality
  - For three-qubit pure state $\left|\psi\right\rangle_{ABC} \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

$$\tau\left(\left|\psi\right\rangle_{A(BC)}\right) \le \tau_a\left(\rho_{AB}\right) + \tau_a\left(\rho_{AB}\right)$$

[G. Gour, D. Meyer and B. C. Sanders PRA 72 042329 (2005)]

$\tau_a\left(\rho_{AB}\right)$: tangle of assistance

$$\tau_a\left(\rho_{AB}\right) = \max \sum_i p_i \tau\left(\left|\psi_i\right\rangle_{AB}\right)$$

max: *over all possible pure state decompositions*

$$\rho_{AB} = \sum_i p_i \left|\psi_i\right\rangle_{AB} \left\langle\psi_i\right|$$

General Polygamy Inequality

- Entanglement of Assistance

$$\rho_{AB} \in \mathcal{B}\left(\mathbb{C}^d \otimes \mathbb{C}^{d'}\right) \quad E_a\left(\rho_{AB}\right) = \max \sum_i p_i E_f\left(\left|\psi_i\right\rangle\right)$$

max: *over all possible pure state decompositions*

$$\rho_{AB} = \sum_i p_i \left|\psi_i\right\rangle_{AB}\left\langle\psi_i\right|$$

For any $\rho_{A_1A_2\cdots A_n} \in \mathcal{B}\left(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_n}\right)$

$$E_a\left(\rho_{A_1(A_2\cdots A_n)}\right) \leq E_a\left(\rho_{A_1A_2}\right) + E_a\left(\rho_{A_1A_3}\right) + \cdots + E_a\left(\rho_{A_1A_n}\right)$$

[JSK, PRA 85, 062302 (2012)]

Mono-poly inequality

- For any $|\psi\rangle_{A_1A_2\cdots A_n} \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_n}$

$$E_{sq}\left(|\psi\rangle_{A_1(A_2\cdots A_n)}\right) = S\left(\rho_A\right) = E_a\left(|\psi\rangle_{A_1(A_2\cdots A_n)}\right)$$

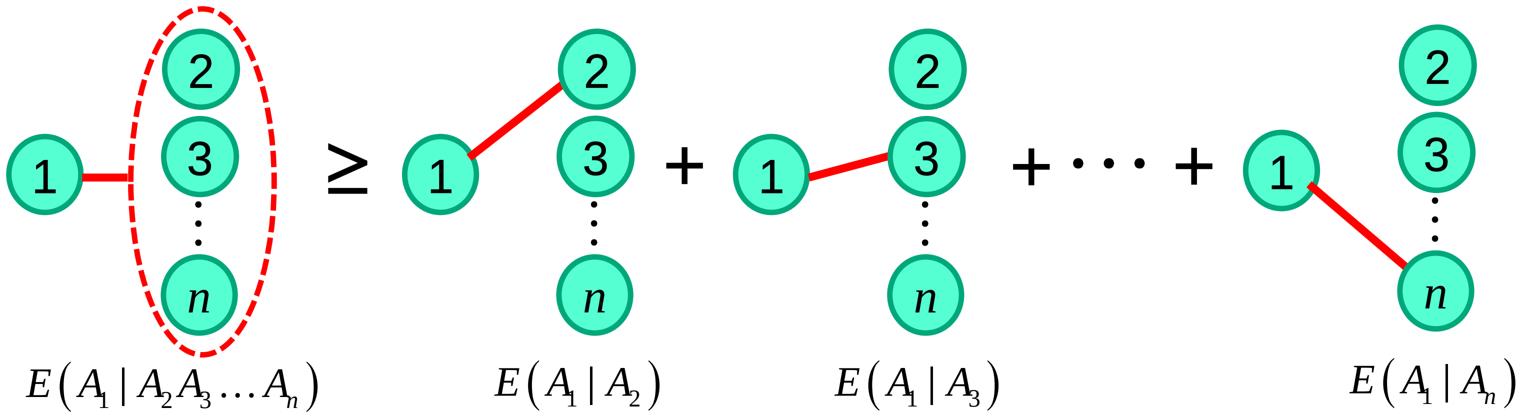
$$E_{sq}\left(\rho_{A_1A_2}\right) + E_{sq}\left(\rho_{A_1A_3}\right) + \cdots + E_{sq}\left(\rho_{A_1A_n}\right) \leq S\left(\rho_A\right)$$

$$\leq E_a\left(\rho_{A_1A_2}\right) + E_a\left(\rho_{A_1A_3}\right) + \cdots + E_a\left(\rho_{A_1A_n}\right)$$

Strong monogamy of entanglement

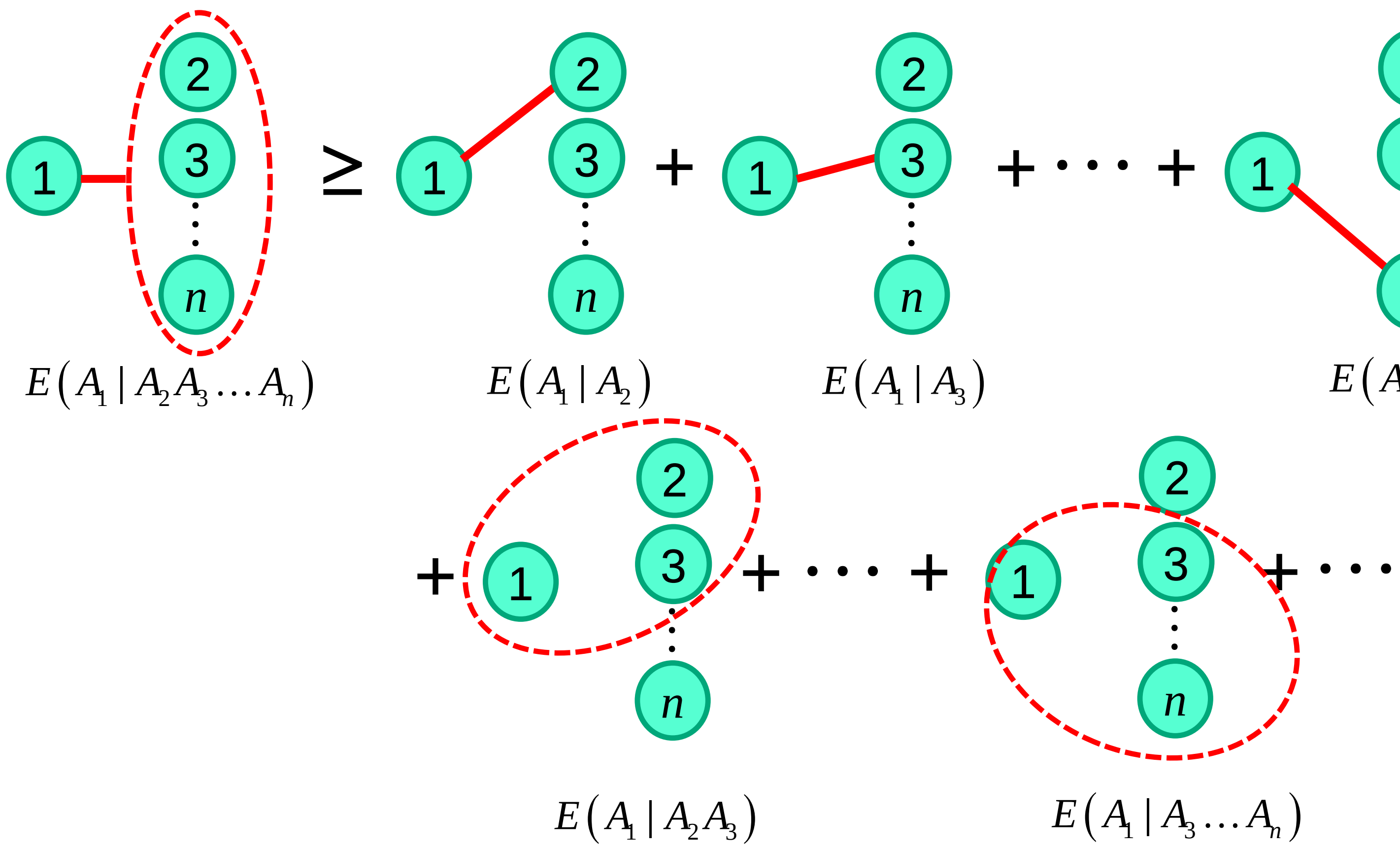
Strong monogamy of entanglement

- CKW-type monogamy inequality



Strong monogamy of entanglement

- Stronger (or finer) monogamy inequality ?



n-tangle

- 3-Tangle
  - For three-qubit pure state $\left|\psi\right\rangle_{ABC}$

$$\tau\left(\left|\psi\right\rangle_{A|B|C}\right)=\tau\left(\left|\psi\right\rangle_{A|BC}\right)-\tau\left(\rho_{A|B}\right)-\tau\left(\rho_{A|C}\right)$$

- n-tangle
  - For n-qubit pure state $\left|\psi\right\rangle_{A_1A_2\ldots A_n}$

$$\tau\left(\left|\psi\right\rangle_{A_1|A_2|\ldots|A_n}\right)=\tau\left(\left|\psi\right\rangle_{A_1|A_2\ldots A_n}\right)-\sum_{m=2}^{n-1}\sum_{\vec{j}^m}\tau\left(\rho_{A_1|A_{j_1^m}|\ldots|A_{j_{m-1}^m}}\right)^{m/2}$$

  - $\vec{j}^m=\left(j_1^m,\ldots,j_{m-1}^m\right)$: index vector spans over all (m-1)-ordered subsets of $\{2,3,\ldots,n\}$

$$\tau\left(\rho_{A_1|A_{j_1^m}|\ldots|A_{j_{m-1}^m}}\right)=\left[\min_{\{p_h,|\psi_h\rangle\}}\sum_h p_h\sqrt{\tau\left(\left|\psi_h\right\rangle_{A_1|A_{j_1^m}|\ldots|A_{j_{m-1}^m}}\right)}\right]^2,\quad \rho_{A_1A_{j_1^m}\ldots A_{j_{m-1}^m}}=\sum_h p_h\left|\psi_h\right\rangle\left\langle\psi_h\right|$$

n-tangle

- 3-Tangle
  - For three-qubit pure state $|\psi\rangle_{ABC}$

$$\tau\left(|\psi\rangle_{A|B|C}\right)=\tau\left(|\psi\rangle_{A|BC}\right)-\tau\left(\rho_{A|B}\right)-\tau\left(\rho_{A|C}\right)$$

- n-tangle
  - For n=2

$$\tau\left(\rho_{A_1|A_{j_1^m}|\ldots|A_{j_{m-1}^m}}\right)=\left[\min_{\{p_h,|\psi_h\rangle\}}\sum_h p_h\sqrt{\tau\left(|\psi_h\rangle_{A_1|A_{j_1^m}|\ldots|A_{j_{m-1}^m}}\right)}\right]^2$$

$$\Rightarrow\quad \tau\left(\rho_{A|B}\right)=\left[\min_{\{p_h,|\psi_h\rangle\}}\sum_h p_h\sqrt{\tau\left(|\psi_h\rangle_{A|B}\right)}\right]^2 \quad : \quad \text{two-tangle}$$

  - ...bsets of $\{2,3,\ldots,n\}$

$$\tau\left(\rho_{A_1|A_{j_1^m}|\ldots|A_{j_{m-1}^m}}\right)=\left[\min_{\{p_h,|\psi_h\rangle\}}\sum_h p_h\sqrt{\tau\left(|\psi_h\rangle_{A_1|A_{j_1^m}|\ldots|A_{j_{m-1}^m}}\right)}\right]^2,\quad \rho_{A_1A_{j_1^m}\ldots A_{j_{m-1}^m}}=\sum_h p_h|\psi_h\rangle\langle\psi_h|$$

n-tangle

- 4-tangle
  - For four-qubit pure state $|\psi\rangle_{ABCD}$
    - $\tau\left(|\psi\rangle_{A|B|C|D}\right)=\tau\left(|\psi\rangle_{A|BCD}\right)-\tau\left(\rho_{A|B|C}\right)^{3/2}-\tau\left(\rho_{A|B|D}\right)^{3/2}-\tau\left(\rho_{A|C|D}\right)^{3/2}$
      $-\tau\left(\rho_{A|B}\right)-\tau\left(\rho_{A|C}\right)-\tau\left(\rho_{A|D}\right)$
    - $\rho_{ABC}=\mathrm{tr}_D\,|\psi\rangle_{ABCD}\langle\psi|$

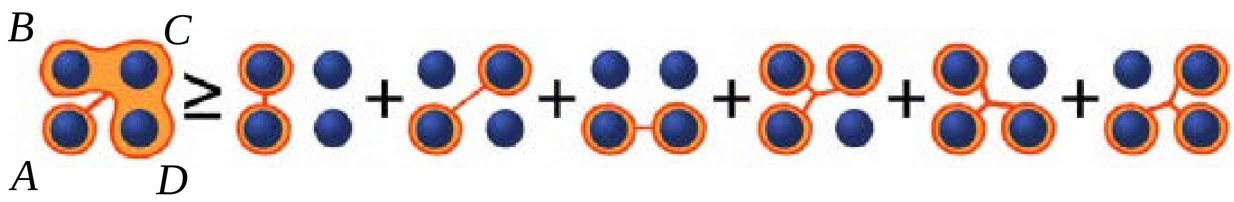
$$\tau\left(\rho_{A|B|C}\right)=\left[\min_{\{p_h,|\psi_h\rangle\}}\sum_h p_h\sqrt{\tau\left(|\psi_h\rangle_{A|B|C}\right)}\right]^2,\quad \rho_{ABC}=\sum_h p_h|\psi_h\rangle_{ABC}\langle\psi_h|$$

$$\tau\left(|\psi_h\rangle_{A|B|C}\right)=\tau\left(|\psi\rangle_{A|BC}\right)-\tau\left(\rho_{A|B}\right)-\tau\left(\rho_{A|C}\right)$$

Strong monogamy conjecture

- 4-tangle
  - Assuming non-negativity of 4-tangle
    - $\tau\left(\left|\psi\right\rangle_{A|B|C|D}\right) \geq 0 \quad \Leftrightarrow$

$$\tau\left(\left|\psi\right\rangle_{A|BCD}\right) \geq \tau\left(\rho_{A|B|C}\right)^{3/2} + \tau\left(\rho_{A|B|D}\right)^{3/2} + \tau\left(\rho_{A|C|D}\right)^{3/2}$$
$$+ \tau\left(\rho_{A|B}\right) + \tau\left(\rho_{A|C}\right) + \tau\left(\rho_{A|D}\right)$$
$$\geq \tau\left(\rho_{A|B}\right) + \tau\left(\rho_{A|C}\right) + \tau\left(\rho_{A|D}\right)$$



[B. Regula, et. al., PRL **113** 110501 (2014)]

Strong monogamy conjecture

- n-tangle

$$\tau\left(|\psi\rangle_{A_1|A_2|\ldots|A_n}\right)=\tau\left(|\psi\rangle_{A_1|A_2\ldots A_n}\right)-\sum_{m=2}^{n-1}\sum_{\vec{j}^m}\tau\left(\rho_{A_1|A_{j_1^m}|\ldots|A_{j_{m-1}^m}}\right)^{m/2}$$

- Assuming non-negativity of n-tangle

$$\Rightarrow \quad \tau\left(|\psi\rangle_{A_1|A_2\ldots A_n}\right)\geq\sum_{j=2}^{n}\tau\left(\rho_{A_1|A_j}\right)+\sum_{m=3}^{n-1}\sum_{\vec{j}^m}\tau\left(\rho_{A_1|A_{j_1^m}|\ldots|A_{j_{m-1}^m}}\right)^{m/2}$$

$$\geq\sum_{j=2}^{n}\tau\left(\rho_{A_1|A_j}\right)$$

- Strong monogamy inequality of multi-qubit entanglement

[B. Regula, et. al., PRL **113** 110501 (2014)]

Strong monogamy conjecture

- Proving strong monogamy conjecture?

$$\tau\left(\left|\psi\right\rangle_{A_1|A_2|\cdots|A_n}\right)=\tau\left(\left|\psi\right\rangle_{A_1|A_2\cdots A_n}\right)-\sum_{m=2}^{n-1}\sum_{\vec{j}^m}\tau\left(\rho_{A_1|A_{j_1^m}|\cdots|A_{j_{m-1}^m}}\right)^{m/2}$$

$$\tau\left(\rho_{A_1|A_{j_1^m}|\cdots|A_{j_{m-1}^m}}\right)=\left[\min_{\{p_h,|\psi_h\rangle\}}\sum_h p_h\sqrt{\tau\left(\left|\psi_h\right\rangle_{A_1|A_{j_1^m}|\cdots|A_{j_{m-1}^m}}\right)}\right]^2,$$

  - Exponentially many optimization processes w.r.t. n

- Numerical test for 4-qubit systems
  - 8×10^6 random 4-qubit pure states

[B. Regula, et. al., PRL **113** 110501 (2014)]

Saturation of multi-qubit strong monogamy inequality

Saturation of CKW inequality

- n-qubit generalized W-class state

$$|W\rangle_{A_1\ldots A_n} = a_1|10\ldots0\rangle + a_2|01\ldots0\rangle + \cdots + a_n|00\ldots1\rangle \quad \text{with} \quad \sum_{i=1}^{n}|a_i|^2 = 1$$

- Saturation of CKW inequality

$$\tau\left(|W\rangle_{A_1|A_2\ldots A_n}\right) = \tau\left(\rho_{A_1|A_2}\right) + \tau\left(\rho_{A_1|A_3}\right) + \cdots + \tau\left(\rho_{A_1|A_n}\right)$$

[JSK and B. C. Sanders, J. Phys. A 41. 495301 (2008)]

  - Good candidate of possible counterexample for strong monogamy inequality

W-class state and strong monogamy inequality

- Strong monogamy conjecture

$$\tau\left(|\psi\rangle_{A_1|A_2\ldots A_n}\right) \geq \sum_{j=2}^{n} \tau\left(\rho_{A_1|A_j}\right) + \sum_{m=3}^{n-1} \sum_{\vec{j}^m} \tau\left(\rho_{A_1|A_{j_1^m}|\ldots|A_{j_{m-1}^m}}\right)^{m/2}$$

- W-class state

$$\tau\left(|W\rangle_{A_1|A_2\ldots A_n}\right) = \tau\left(\rho_{A_1|A_2}\right) + \cdots + \tau\left(\rho_{A_1|A_n}\right) = \sum_{j=2}^{n} \tau\left(\rho_{A_1|A_j}\right)$$

$$|W\rangle_{A_1\ldots A_n} = a_1|10\ldots0\rangle + a_2|01\ldots0\rangle + \cdots + a_n|00\ldots1\rangle$$

- Strong monogamy conjecture for W-class states

$$\Leftrightarrow \sum_{m=3}^{n-1} \sum_{\vec{j}^m} \tau\left(\rho_{A_1|A_{j_1^m}|\ldots|A_{j_{m-1}^m}}\right)^{m/2} = 0 \quad \text{for W-class states}$$

W-class state and strong monogamy inequality

- Strong monogamy conjecture

Lemma

$$\tau\left(\rho_{A_1|A_{j_1^m}|\dots|A_{j_{m-1}^m}}\right)=0$$

for all the index vectors $\vec{j}^{\,m}=\left(j_1^m,\dots,j_{m\text{-}1}^m\right)$ with $3\le m\le n\text{-}1$

for generalized W-class states

[JSK, PRA 90, 062306 (2014)]

$\Leftrightarrow$ $\displaystyle\sum_{m=3}^{n-1}\sum_{\vec{j}^m}\tau\left(\rho_{A_1|A_{j_1^m}|\dots|A_{j_{m-1}^m}}\right)^{m/2}=0$ for W-class states

W-class state and strong monogamy inequality

- Saturation of strong monogamy inequality
  - For any generalized W-class state

$$|W\rangle_{A_1\ldots A_n} = a_1|10\ldots0\rangle + a_2|01\ldots0\rangle + \cdots + a_n|00\ldots1\rangle$$

$$\tau\left(|W\rangle_{A_1|A_2\ldots A_n}\right) = \sum_{j=2}^{n}\tau\left(\rho_{A_1|A_j}\right) + \sum_{m=3}^{n-1}\sum_{\vec{j}^m}\tau\left(\rho_{A_1|A_{j_1^m}|\ldots|A_{j_{m-1}^m}}\right)^{m/2}$$

  - Moreover, the saturation strong monogamy inequality is also true for

$$|\psi\rangle_{A_1\ldots A_n} = a|00\ldots0\rangle + b_1|10\ldots0\rangle + b_2|01\ldots0\rangle + \cdots + b_n|00\ldots1\rangle$$

[JSK, PRA 90, 062306 (2014)]

Negativity and SM inequality in higher-dimensional systems

Counterexamples in higher dimension

- Multi-qubit SM inequality

$$\tau\left(|\psi\rangle_{A_1|A_2\ldots A_n}\right) \geq \sum_{j=2}^{n} \tau\left(\rho_{A_1|A_j}\right) + \sum_{m=3}^{n-1} \sum_{\vec{j}^m} \tau\left(\rho_{A_1|A_{j_1^m}|\ldots|A_{j_{m-1}^m}}\right)^{m/2}$$

n-qubit systems

$$\Rightarrow \; \tau\left(|\psi\rangle_{A|BC}\right) \geq \tau\left(\rho_{A|B}\right) + \tau\left(\rho_{A|C}\right)$$

3-qubit systems

- Counterexamples

$$|\psi\rangle_{ABC} = \frac{1}{\sqrt{6}}\left(|012\rangle - |021\rangle + |120\rangle - |102\rangle + |201\rangle - |210\rangle\right)$$

$\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$

[Y. Ou, PRA 75. 034305 (2007)]

$$|\psi\rangle_{ABC} = \frac{\sqrt{2}}{\sqrt{6}}\left(|010\rangle + |101\rangle\right) + \frac{1}{\sqrt{6}}\left(|200\rangle + |211\rangle\right)$$

$\mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

[JSK and B. C. Sanders, J. Phys. A 41. 495301 (2008)]

Counterexamples in higher dimension

- Multi-qubit SM inequality

$$\tau\left(|\psi\rangle_{A_1|A_2\ldots A_n}\right) \geq \sum_{j=2}^{n}\tau\left(\rho_{A_1|A_j}\right)+\sum_{m=3}^{n-1}\sum_{\vec{j}^m}\tau\left(\rho_{A_1|A_{j_1^m}|\ldots|A_{j_{m-1}^m}}\right)^{m/2}$$

n-qubit systems

$$\Rightarrow\ \tau\left(|\psi\rangle_{A|BC}\right) \geq \tau\left(\rho_{A|B}\right)+\tau\left(\rho_{A|C}\right)$$

3-qubit systems

- Counterexamples

$$|\psi\rangle_{ABC}=\frac{1}{\sqrt{6}}\left(|012\rangle-|021\rangle+|120\rangle-|102\rangle+|201\rangle-|210\rangle\right)$$

$\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$

034305 (2007)]

$$\tau\left(|\psi\rangle_{A|BC}\right) < \tau\left(\rho_{A|B}\right)+\tau\left(\rho_{A|C}\right)$$

$\Rightarrow$ violation of SM inequality in terms of tangle

$|\psi\rangle_{ABC} = \ldots$ $\otimes\mathbb{C}^2$

[JSK and B. C. Sanders, J. Phys. A 41. 495301 (2008)]

Square of convex-roof extended negativity (SCREN)

- Negativity
  - Bipartite pure state with Schmidt decomposition $|\psi\rangle_{AB} = \sum_i \sqrt{\lambda_i} |ii\rangle_{AB}$

$$\mathcal{N}\left(|\psi\rangle_{A|B}\right) := \left\| \left(|\psi\rangle_{AB} \langle\psi|\right)^{\Gamma} \right\|_1 - 1 \qquad \|\cdot\|_1 : \text{ Trace norm,}$$

$$= 2\sum_{i<j} \sqrt{\lambda_i \lambda_j} \qquad \Gamma : \text{ Partial transposition}$$

  - For bipartite pure state with Schmidt-rank 2

$$|\psi\rangle_{AB} = \sqrt{\lambda_0} |e_0 f_0\rangle_{AB} + \sqrt{\lambda_1} |e_1 f_1\rangle_{AB}$$

Negativity: $\left(\mathcal{N}\left(|\psi\rangle_{A|B}\right)\right)^2 = 4\lambda_0\lambda_1$

two-tangle: $\tau\left(|\psi\rangle_{A|B}\right) = 2\left(1 - tr\rho_A^2\right) = 4\lambda_0\lambda_1$

Square of convex-roof extended negativity (SCREN)

- Negativity vs. Tangle
  - For bipartite pure state with Schmidt-rank 2

$$|\psi\rangle_{AB} = \sqrt{\lambda_0}\,|e_0 f_0\rangle_{AB} + \sqrt{\lambda_1}\,|e_1 f_1\rangle_{AB} \qquad \left(\mathcal{N}\left(|\psi\rangle_{A|B}\right)\right)^2 = 4\lambda_0\lambda_1 = \tau\left(|\psi\rangle_{A|B}\right)$$

  - For two-qubit state $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB}\langle\psi_i|$

$$\tau\left(\rho_{A|B}\right) = \left[\min \sum_i p_i \sqrt{\tau\left(|\psi_i\rangle_{A|B}\right)}\right]^2$$

$$= \left[\min \sum_i p_i \mathcal{N}\left(|\psi\rangle_{A|B}\right)\right]^2 \qquad = \mathcal{N}_{SC}\left(\rho_{A|B}\right) \qquad \text{2-SCREN}$$

Square of convex-roof extended negativity (SCREN)

- For n-qudit pure state $\left|\psi\right\rangle_{A_1 A_2 \ldots A_n} \in \mathbb{C}^d \otimes \mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d$
  - n-SCREN

$$\mathcal{N}_{SC}\left(\left|\psi\right\rangle_{A_1|A_2|\ldots|A_n}\right) = \mathcal{N}_{SC}\left(\left|\psi\right\rangle_{A_1|A_2\ldots A_n}\right) - \sum_{m=2}^{n-1}\sum_{\vec{j}^m} \mathcal{N}_{SC}\left(\rho_{A_1|A_{j_1^m}|\ldots|A_{j_{m-1}^m}}\right)^{m/2}$$

$\vec{j}^m = \left(j_1^m, \ldots, j_{m-1}^m\right)$: index vector spans over all $(m-1)$-subsets of $\{1,2, \ldots ,n\}$

  - Mixed state

$$\mathcal{N}_{SC}\left(\rho_{A_1|A_{j_1^m}|\ldots|A_{j_{m-1}^m}}\right) = \left[\min_{\{p_h, |\psi_h\rangle\}} \sum_h p_h \sqrt{\mathcal{N}_{SC}\left(\left|\psi_h\right\rangle_{A_1|A_{j_1^m}|\ldots|A_{j_{m-1}^m}}\right)}\right]^2,$$

$$\rho_{A_1 A_{j_1^m}\ldots A_{j_{m-1}^m}} = \sum_h p_h \left|\psi_h\right\rangle\left\langle\psi_h\right|$$

[JSK PRA 92 042307 (2015)]

n-SCREN vs. n-tangle

- For n-qubit states $|\psi\rangle_{A_1A_2\ldots A_n} \in \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$

$$\mathcal{N}_{SC}\left(|\psi\rangle_{A_1|A_2|\ldots|A_n}\right) = \tau\left(|\psi\rangle_{A_1|A_2|\ldots|A_n}\right)$$

- n-qubit SM inequality

$$\tau\left(|\psi\rangle_{A_1|A_2\ldots A_n}\right) \geq \sum_{j=2}^{n} \tau\left(\rho_{A_1|A_j}\right) + \sum_{m=3}^{n-1} \sum_{\vec{j}^m} \tau\left(\rho_{A_1|A_{j_1^m}|\ldots|A_{j_{m-1}^m}}\right)^{m/2}$$

$$\Leftrightarrow \quad \mathcal{N}_{SC}\left(|\psi\rangle_{A_1|A_2\ldots A_n}\right) \geq \sum_{j=2}^{n} \mathcal{N}_{SC}\left(\rho_{A_1|A_j}\right) + \sum_{m=3}^{n-1} \sum_{\vec{j}^m} \mathcal{N}_{SC}\left(\rho_{A_1|A_{j_1^m}|\ldots|A_{j_{m-1}^m}}\right)^{m/2}$$

n-SCREN vs. n-tangle

- Saturation of SCREN SM inequality
  - For multi-qubit generalized W-class state

$$|W\rangle_{A_1\ldots A_n} = a_1|10\ldots0\rangle + a_2|01\ldots0\rangle + \cdots + a_n|00\ldots1\rangle$$

$$\mathscr{N}_{SC}\left(|W\rangle_{A_1|A_2\ldots A_n}\right) = \sum_{j=2}^{n} \mathscr{N}_{SC}\left(\rho_{A_1|A_j}\right) + \sum_{m=3}^{n-1}\sum_{\vec{j}^m} \mathscr{N}_{SC}\left(\rho_{A_1|A_{j_1^m}|\ldots|A_{j_{m-1}^m}}\right)^{m/2}$$

  - Moreover, the saturation SCREN SM inequality is also true for

$$|\psi\rangle_{A_1\ldots A_n} = a|00\ldots0\rangle + b_1|10\ldots0\rangle + b_2|01\ldots0\rangle + \cdots + b_n|00\ldots1\rangle$$

[JSK PRA 92 042307 (2015)]

n-SCREN vs. n-tangle

- Counterexamples of tangle SM inequality
  - $$|\psi\rangle_{ABC} = \frac{1}{\sqrt{6}}\left(|012\rangle - |021\rangle + |120\rangle - |102\rangle + |201\rangle - |210\rangle\right) \qquad \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$$

    $$|\psi\rangle_{ABC} = \frac{\sqrt{2}}{\sqrt{6}}\left(|010\rangle + |101\rangle\right) + \frac{1}{\sqrt{6}}\left(|200\rangle + |211\rangle\right) \qquad \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$$

$$\mathcal{N}_{SC}\left(|\psi\rangle_{A|BC}\right) \geq \mathcal{N}_{SC}\left(\rho_{A|B}\right) + \mathcal{N}_{SC}\left(\rho_{A|C}\right)$$

$\Rightarrow$ Validity of SCREN SM inequality

Beyond multi-qubit systems

- Multi-qudit generalized W-class states

$$\left|W^d\right\rangle_{A_1\ldots A_n} = \sum_{i=1}^{d-1}\left(a_{1i}\left|i0\ldots0\right\rangle + a_{2i}\left|0i\ldots0\right\rangle + \cdots + a_{ni}\left|00\ldots i\right\rangle\right) \quad \text{with} \quad \sum_{s=1}^{n}\sum_{i=1}^{d-1}\left|a_{si}\right|^2 = 1$$

[JSK and B. C. Sanders, J. Phys. A 41. 495301 (2008)]

  - For d=2

$$\left|W\right\rangle_{A_1\ldots A_n} = a_1\left|10\ldots0\right\rangle + a_2\left|01\ldots0\right\rangle + \cdots + a_n\left|00\ldots1\right\rangle$$

n-qubit generalized W-class state

- Saturation of SM inequality

$$\mathcal{N}_{SC}\left(\left|W\right\rangle_{A_1|A_2\ldots A_n}\right) = \sum_{j=2}^{n}\mathcal{N}_{SC}\left(\rho_{A_1|A_j}\right) + \sum_{m=3}^{n-1}\sum_{\vec{j}^m}\mathcal{N}_{SC}\left(\rho_{A_1|A_{j_1^m}|\ldots|A_{j_{m-1}^m}}\right)^{m/2}$$

[JSK PRA 92 042307 (2015)]

Beyond multi-qubit systems

- Partially coherent superposition of $\left|W^d\right\rangle_{A_1\ldots A_n}$ with vacuum

$$\rho^{(p,\lambda)}_{A_1\ldots A_n} = p\left|W^d\right\rangle_{A_1\ldots A_n}\left\langle W^d\right| + (1-p)\left|0\ldots0\right\rangle_{A_1\ldots A_n}\left\langle 0\ldots0\right|$$
$$+\lambda\sqrt{p(1-p)}\left(\left|W^d\right\rangle_{A_1\ldots A_n}\left\langle 0\ldots0\right| + \left|0\ldots0\right\rangle_{A_1\ldots A_n}\left\langle W^d\right|\right) \quad \text{for } 0 \le p,\lambda \le 1$$

- $\lambda=1: \quad \rho_{A_1\ldots A_n} = \sqrt{p}\left|W_n^d\right\rangle + \sqrt{1-p}\left|0\right\rangle^{\otimes n} \qquad \text{(coherent superposition)}$

$\lambda=0: \quad \rho_{A_1\ldots A_n} = p\left|W_n^d\right\rangle\left\langle W_n^d\right| + (1-p)\left|0\right\rangle^{\otimes n}\left\langle 0\right|^{\otimes n} \text{ (incoherent superposition)}$

Beyond multi-qubit systems

- Partially coherent superposition of $\left|W^d\right\rangle_{A_1\ldots A_n}$ with vacuum

$$\rho^{(p,\lambda)}_{A_1\ldots A_n} = p\left|W^d\right\rangle_{A_1\ldots A_n}\left\langle W^d\right| + (1-p)\left|0\ldots0\right\rangle_{A_1\ldots A_n}\left\langle 0\ldots0\right|$$
$$+\lambda\sqrt{p(1-p)}\left(\left|W^d\right\rangle_{A_1\ldots A_n}\left\langle 0\ldots0\right| + \left|0\ldots0\right\rangle_{A_1\ldots A_n}\left\langle W^d\right|\right) \quad \text{for } 0\le p,\lambda\le 1$$

- In terms of decoherence

For $\left|\psi\right\rangle_{A_1\ldots A_n} = \sqrt{p}\left|W_n^d\right\rangle + \sqrt{1-p}\left|0\right\rangle^{\otimes n}$

$$\rho^{(p,\lambda)}_{A_1\ldots A_n} = \Lambda\left(\left|\psi\right\rangle\left\langle\psi\right|\right) = E_0\left|\psi\right\rangle\left\langle\psi\right|E_0^{+} + E_1\left|\psi\right\rangle\left\langle\psi\right|E_1^{+} + E_2\left|\psi\right\rangle\left\langle\psi\right|E_2^{+}$$

where $E_0 = \sqrt{\lambda}I,\ E_1 = \sqrt{1-\lambda}\left(I - \left|0\right\rangle\left\langle 0\right|\right)$ and $E_2 = \sqrt{1-\lambda}\left|0\right\rangle\left\langle 0\right|$

$\rho^{(p,\lambda)}_{A_1\cdots A_n}$: resulting state from a coherent state $\left|\psi\right\rangle$ by the decoherence process Λ.

Beyond multi-qubit systems

- Partially coherent superposition of $\left|W^d\right\rangle_{A_1\ldots A_n}$ with vacuum

$$\rho^{(p,\lambda)}_{A_1\ldots A_n} = p\left|W^d\right\rangle_{A_1\ldots A_n}\left\langle W^d\right| + (1-p)\left|0\ldots0\right\rangle_{A_1\ldots A_n}\left\langle 0\ldots0\right|$$

$$+\lambda\sqrt{p(1-p)}\left(\left|W^d\right\rangle_{A_1\ldots A_n}\left\langle 0\ldots0\right| + \left|0\ldots0\right\rangle_{A_1\ldots A_n}\left\langle W^d\right|\right) \quad \text{for } 0 \le p,\lambda \le 1$$

- Saturation of SCREN inequalities

$$\mathscr{N}_{SC}\left(\rho^{(p,\lambda)}_{A_1|A_2\ldots A_n}\right) = \sum_{i=2}^{n}\mathscr{N}_{SC}\left(\rho_{A_1|A_i}\right)$$

$$\mathscr{N}_{SC}\left(\rho^{(p,\lambda)}_{A_1|A_2|\ldots|A_n}\right) = \left[\min_{\{p_h,|\psi_h\rangle\}}\sum_h p_h\sqrt{\mathscr{N}_{SC}\left(\left|\psi_h\right\rangle_{A_1|A_2|\ldots|A_n}\right)}\right]^2 = 0$$

$$\rho_{A_1A_{j_1^m}\ldots A_{j_{m-1}^m}} = \sum_h p_h\left|\psi_h\right\rangle\left\langle\psi_h\right|$$

[JSK in preparation]

Summary

- Monogamy of multi-party quantum entanglement
  - Mathematical characterization: CKW-type inequality
  - Squashed entanglement
  - General polygamy inequality
- Strong monogamy conjecture in multi-qudit systems
  - Non-negativity of n-tangle : strong monogamy inequality
  - SCREN SM inequality for qudits
  - Saturation of SCREN SM monogamy inequality

- Future works
  - Analytic proof of strong monogamy inequality ?
  - SM inequality of entanglement and other correlations