Recent progress on the distillability problem

Lin Chen

School of Mathematics and Systems Science,
Beihang University, Beijing, China

Email: linchen@buaa.edu.cn

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The talk is based on two papers:

1. J. Phys. A. 44, 285303 (2011),

2. quant-ph/1602.04416 (2016).

Collaborator:

Dragomir Z Djokovic

Department of Pure Mathematics and Institute for Quantum Computing, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada

Outlines

- The distillability problem and entanglement distillation

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- $M \times N$ NPT states of rank $\max\{M, N\}$

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- $M \times N$ NPT states of rank $\max\{M, N\}$
- Two-qutrit NPT states of rank four and five
- Open problems

Entanglement distillation

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$$\to \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}} \right)^{\otimes m}$$

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If no pure entangled states can be obtained, then ρ is not distillable, or equivalently ρ is undistillable.

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- Attempts for the proof: Yes, there is something...

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PPT and NPT

Definition

The partial transpose of a bipartite quantum state ρ acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ is computed in an orthonormal (o .n.) basis $\{|a_i\rangle\}$ of system A, is defined by $\rho^\Gamma := \sum_{ij} |a_i\rangle\langle a_j| \otimes \langle a_j|\rho|a_i\rangle$.

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- For example, all separable states are PPT. All pure entangled states are NPT.

PPT and NPT

- Example. If

$$\rho = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}$$

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(1) ρ is n-distillable if the bipartite state $\rho^{\otimes n}$ is 1-distillable.
(2) ρ is distillable if it is n-distillable for some $n \geq 1$, i.e.,

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- For example, PPT states are not distillable.

The math/mess of many-copy states

- $\rho^{\otimes n} = \rho_{A_1 B_1} \otimes \cdots \otimes \rho_{A_n B_n} := \rho_{A_1 \cdots A_n : B_1 \cdots B_n}$.

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- Example. Consider the "critical" Werner state

$$\rho_{A_1 B_1} = \sum_{i,j} (|i,j\rangle\langle i,j| - \frac{1}{2}|i,j\rangle\langle j,i|)_{A_1 B_1}$$

$$\rho_{A_2 B_2} = \sum_{m,n} (|m,n\rangle\langle m,n| - \frac{1}{2}|m,n\rangle\langle n,m|)_{A_2 B_2}$$

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- Then

$$\rho^{\otimes 2} = \rho_{A_1 B_1} \otimes \rho_{A_2 B_2}$$

$$= \sum_{i,j,m,n} \Big(|im,jn\rangle\langle im,jn| - \frac{1}{2}|im,jn\rangle\langle jm,in|$$

$$-\frac{1}{2}|im,jn\rangle\langle in,jm| + \frac{1}{4}|im,jn\rangle\langle jn,im| \Big)_{A_1 A_2, B_1 B_2}$$

List of 1-distillable NPT states

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- Two-qutrit states of rank four
  LC and DZ, 2016

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$$(P \otimes I_B)\rho(P \otimes I_B) = (|11\rangle + |22\rangle)(\langle 11| + \langle 22|)$$

is a Bell state.

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- then

$$(P \otimes I_B)\rho(P \otimes I_B)$$

is a two-qubit mixed entangled state. So ρ is also distillable.

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- When is $(P \otimes I_B)\rho(P \otimes I_B)$ entangled?
- What if $(P \otimes I_B)\rho(P \otimes I_B)$ is PPT?
- A popular trick: let $(P \otimes I_B)\rho(P \otimes I_B)$ be a $2 \times N$ state then it has to be PPT, or some entries have to be zero.

Outlines

- The distillability problem and entanglement distillation

  $M \times N$ NPT states of rank $\max\{M, N\}$

  Two-qutrit NPT states of rank four and five

  Open problems

Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

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Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

- LFRP. Let ρ_{AB} be an $M \times N$ NPT states of rank $\max\{M, N\}$. We say ρ_{AB} has left full-rank property (LFRP) if there is some state $|x\rangle$ such that $\langle x|_B \rho_{AB} |x\rangle_B$ is invertible.

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- then

$$\max_x \left(\operatorname{rank}(\langle x|_B \rho_{AB} |x\rangle_B) \right) = 2 < \operatorname{rank} \rho_A = 3.$$

$$\max_x \left(\operatorname{rank}(\langle x|_B \sigma_{AB} |x\rangle_B) \right) = 3 = \operatorname{rank} \rho_A = 3.$$

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- So ρ_{AB} has no LFRP, and σ_{AB} has LFRP.
- The right full-rank property (RFRP) can be similarly defined.
- Strategy of proof. Prove that ρ_{AB} is 1-distillable when (1) ρ_{AB} has no LFRP or RFRP, and (2) ρ_{AB} has LFRP and RFRP.

Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

- (1) ρ_{AB} has no LFRP or RFRP.
  Using the matrix decomposition of semidefinite positive matrix $\rho = C^\dagger C$, where

$$C = (C_1, \ldots, C_i, \ldots, C_M)$$

and each matrix C_i is of size $(\operatorname{rank} \rho) \times N$.

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- Project ρ to the following state by using the projector $P = |1\rangle\langle 1| + |i\rangle\langle i|$

$$\rho_{1,i} = (P \otimes I_B)\rho(P \otimes I_B)$$

$$= (C_1, C_i)^\dagger \cdot (C_1, C_i) = \begin{pmatrix} C_1^\dagger C_1 & C_1^\dagger C_i \\ C_i^\dagger C_1 & C_i^\dagger C_i \end{pmatrix}$$

Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

- We split each C_i into four blocks $C_i = \begin{pmatrix} C_{i1} & C_{i2} \\ C_{i3} & C_{i4} \end{pmatrix}$ with C_{i1} square of size r_1, where $C_1 = I_{r_1} \oplus 0$ because of ρ has no LFRP or RFRP. We have

$$\rho_{1,i} = \begin{pmatrix} I_{r_1} & 0 & \vdots & C_{i1} & C_{i2} \\ 0 & 0 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdot & \cdots & \cdots \\ C_{i1}^\dagger & 0 & \vdots & * & * \\ C_{i2}^\dagger & 0 & \vdots & * & * \end{pmatrix}$$

, where $i > 1$ and the asterisk stands for an unspecified block.

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- If some $C_{i2} \neq 0$, then ρ is 1-distillable. Thus we may assume that all $C_{i2} = 0$.

Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

- Now $\rho = C^\dagger C$ where

$$C = \left[\begin{pmatrix} I_{r_1} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} C_{21} & 0 \\ C_{23} & {\color{red}C_{24}} \end{pmatrix}, \cdots, \begin{pmatrix} C_{M1} & 0 \\ C_{M3} & {\color{red}C_{M4}} \end{pmatrix} \right]$$

Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

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$$C = \left[\begin{pmatrix} I_{r_1} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} C_{21} & 0 \\ C_{23} & \color{red}{C_{24}} \end{pmatrix}, \cdots, \begin{pmatrix} C_{M1} & 0 \\ C_{M3} & \color{red}{C_{M4}} \end{pmatrix} \right]$$

- Since ρ has no LFRP or RFRP, the linear combination of $C_{21}, \cdots, C_{N1}$ is of deficient rank. We may assume

$$\color{red}{C_{24}} = \begin{pmatrix} I_{r_2} & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\color{red}{C_{i4}} = \begin{pmatrix} C_{i41} & C_{i42} \\ C_{i43} & C_{i44} \end{pmatrix}$$

Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

- Project ρ to the state $(C')^\dagger C'$ where

$$C' = \left[\begin{pmatrix} I_{r_2} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} C_{341} & C_{342} \\ C_{343} & C_{344} \end{pmatrix}, \cdots, \begin{pmatrix} C_{M41} & C_{M42} \\ C_{M43} & C_{M44} \end{pmatrix} \right]$$

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$$\left. , \cdots, \begin{pmatrix} C_{M1} & 0 & 0 \\ C_{M21} & C_{M41} & 0 \\ C_{M23} & C_{M43} & C_{M44} \end{pmatrix} \right]$$

Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

- The process continues and the facts $C_{i2} = C_{i42} = \cdots = 0$ implies that ρ has RFRP. It is a contradiction and we obtain that the process must terminate.

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- So ρ is distillable when it has no LFRP or RFRP.

Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

- (2) ρ_{AB} has LFRP and RFRP.
  Using the matrix decomposition of semidefinite positive matrix $\rho = C^\dagger C$, we have

$$\rho = (C_1, \ldots, C_{M-1}, I_N)^\dagger \cdot (C_1, \ldots, C_{M-1}, I_N)$$

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- Since ρ is NPT, there exist i, j such that $[C_i, C_j] \neq 0$.
- One can show that $(xC_i + C_j, I_N)^\dagger \cdot (xC_i + C_j, I_N)$ is distillable for some complex number x.

Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

- Conclusion 1: the LFRP (RFRP) is a key property for the distillation.

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Corollary

The bipartite state of rank four is separable if and only if it is PPT and its range contains at least one product state.

Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

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For a tripartite pure state $\rho = |\psi\rangle\langle\psi|$, the bipartite reduced density operators ρ_{AB} and ρ_{AC} are PPT if and only if $|\psi\rangle = \sum_i |a_i\rangle|ii\rangle$ up to local unitary operations.

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$$\rho_{AB} = \rho_{AC} = \sum_i |a_i, i\rangle\langle a_i, i|$$

are both separable states.

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  (5) Majorization criterion: $\rho_A \succ \rho_{AB}$ and $\rho_B \succ \rho_{AB}$
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  (6) Conditional entropy criterion:
  $H_\rho(B|A) = H(\rho_{AB}) - H(\rho_A) \geq 0$ and
  $H_\rho(A|B) = H(\rho_{AB}) - H(\rho_B) \geq 0$, where H is the von Neumann entropy.

Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

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Theorem

For a tripartite state $|\Psi\rangle_{ABC}$ with a non-distillable reduced state ρ_{BC} namely condition **(3)**, *then conditions* **(1)**-**(6)** *are equivalent for ρ_{AB}.*

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For a tripartite state $|\Psi\rangle_{ABC}$ with a non-distillable reduced state ρ_{BC} namely condition **(3)**, *then conditions* **(1)**-**(6)** *are equivalent for ρ_{AB}.*

- It is a way of unifying the six well-known conditions.

Outlines

- The distillability problem and entanglement distillation

$M \times N$ NPT states of rank $\max\{M, N\}$

Two-qutrit NPT states of rank four and five

Open problems

Distilling two-qutrit NPT states of rank four

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- Facts: $2 \times N$ NPT states are distillable, and $M \times N$ NPT states of rank $\max\{M, N\}$ are distillable.

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- Entanglement distillation of $M \times N$ states ρ of rank bigger than $\max\{M, N\}$ turns out to be much harder.
- For example ρ can be the Werner state.
- Facts: $2 \times N$ NPT states are distillable, and $M \times N$ NPT states of rank $\max\{M, N\}$ are distillable.
- Hence, the first unsolved problem is to distill 3×3 NPT states of rank four.

Distilling two-qutrit NPT states of rank four

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Theorem

If ρ is a two-qutrit NPT state and ρ^Γ has at least two non-positive eigenvalues counting multiplicities, then ρ is 1-distillable.

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Theorem

If ρ is a two-qutrit NPT state and ρ^Γ has at least two non-positive eigenvalues counting multiplicities, then ρ is 1-distillable.

Proof.

By the hypothesis, there exist two eigenvectors of ρ^Γ, say $|\alpha\rangle$ and $|\beta\rangle$ with matrices A and B, such that $\rho^\Gamma|\alpha\rangle = \lambda|\alpha\rangle$, $\lambda < 0$, $\rho^\Gamma|\beta\rangle = \mu|\beta\rangle$, $\mu \le 0$, and $\langle\alpha|\beta\rangle = 0$. If A is not invertible, then its rank is 2 and so ρ is 1-distillable.
If $N := A^{-1}B$ is not nilpotent, then $\det(I_3 + tN)$ is a nonconstant polynomial in t and we can choose t so that this determinant is 0. Thus $A + tB$ is singular, and $|\phi\rangle := |\alpha\rangle + t|\beta\rangle$ satisfies $\langle\phi|\rho^\Gamma|\phi\rangle = \lambda\|\alpha\|^2 + \mu|t|^2\|\beta\|^2 < 0$. Hence ρ is 1-distillable. The case that N is nilpotent is similar. □

Distilling two-qutrit NPT states of rank four

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Proof.

We can assume that $|0,0\rangle \in \ker \rho$. Consequently, the first diagonal entry of ρ is 0, and the same is true for ρ^Γ. If the first column of ρ^Γ is not 0, then ρ is 1-distillable by projecting to a 2×3 NPT state. Otherwise $|0,0\rangle \in \ker \rho^\Gamma$ and ρ is 1-distillable by last Theorem. □

Distilling two-qutrit NPT states of rank four

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If ρ is a 1-undistillable two-qutrit NPT state, then $\ker \rho$ *is a completely entangled space, and ρ^Γ has exactly one negative and eight positive eigenvalues. Consequently,* $\operatorname{rank} \rho > 4$ *and* $\det \rho^\Gamma \neq 0$.

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- So the minimum rank of 1-undistillable NPT states is at least five.

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- So the minimum rank of 1-undistillable NPT states is at least five.
- We construct an example below.

Distilling two-qutrit NPT states of rank four

- The following state σ is an edge PPT entangled state of birank $(5, 8)$ constructed by Kye and Osaka, 2012.

$$\frac{1}{N}\begin{bmatrix} 2\cos\theta & 0 & 0 & 0 & -\cos\theta & 0 & 0 & 0 & -\cos\theta \\ 0 & \frac{1}{b} & 0 & -e^{-i\theta} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & -e^{i\theta} & 0 & 0 \\ 0 & -e^{i\theta} & 0 & b & 0 & 0 & 0 & 0 & 0 \\ -\cos\theta & 0 & 0 & 0 & 2\cos\theta & 0 & 0 & 0 & -\cos\theta \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{b} & 0 & -e^{-i\theta} & 0 \\ 0 & 0 & -e^{-i\theta} & 0 & 0 & 0 & \frac{1}{b} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -e^{i\theta} & 0 & b & 0 \\ -\cos\theta & 0 & 0 & 0 & -\cos\theta & 0 & 0 & 0 & 2\cos\theta \end{bmatrix},$$

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,

- where

$$N = 3(2\cos\theta + b + 1/b),$$

and the two parameters $b > 0$ and $0 < |\theta| < \pi/3$.

Distilling two-qutrit NPT states of rank four

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- For any pure state $|\psi\rangle$ of Schmidt rank two, we have

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Distilling two-qutrit NPT states of rank four

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Lemma

For any integer n, and sufficiently small $\epsilon = \epsilon(n) > 0$, the two-qutrit NPT state $\rho = \frac{1}{1-\epsilon}(\sigma - \epsilon|f,g\rangle\langle f,g|)$ is n-undistillable.

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Proof.

For any pure state $|\psi\rangle$ of Schmidt rank two, we have

$$(1-\epsilon)^n \langle\psi|(\rho^\Gamma)^{\otimes n}|\psi\rangle := \langle\psi|(\sigma^\Gamma)^{\otimes n}|\psi\rangle + \sum_{k=1}^{n} c_k \epsilon^k$$

$$\geq p^n \langle\psi|(I_9 - |\Psi\rangle\langle\Psi|)^{\otimes n}|\psi\rangle + \sum_{k=1}^{n} c_k \epsilon^k$$

where c_k are complex numbers and p is the minimum positive eigenvalue of σ^Γ. Since the first summand is positive and has nothing to do with ϵ, the assertion holds. □

Distilling two-qutrit NPT states of rank four

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Distilling two-qutrit NPT states of rank four

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Lemma

$$\min_{\psi \in \mathrm{sr}_2} \langle\psi|(I_9 - |\Psi\rangle\langle\Psi|)^{\otimes n}|\psi\rangle \geq \frac{1}{3^n},$$

where sr_2 *is the set of bipartite pure states of Schmidt rnak two, and* $|\Psi\rangle$ *is the two-qutrit maximally entangled state.*

Comparing with Werner states

- The comparison between our suspicious two-qutrit NPT states ρ of rank five and the "critical" NPT Werner states $\rho_w = \frac{2}{15}(I_9 - \frac{1}{2}\sum_{i,j=1}^{3} |ij\rangle\langle ji|)$.

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rank of partial transpose	9	9
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- Whether there is a “critical” ρ is unknown.
- The condition of rank nine prevents the further investigation in both cases.

Open problems

- Can we distill more NPT states satisfying LFRP and RFRP?

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- Distill $3 \times N$ NPT states of rank $N + 1$ for $N \geq 4$.

Open problems

- Can we distill more NPT states satisfying LFRP and RFRP?
- Distill $3 \times N$ NPT states of rank $N + 1$ for $N \geq 4$.
- Is there an undistillable suspicious two-qutrit NPT state $\rho = \frac{1}{1-\epsilon}(\sigma - \epsilon|f, g\rangle\langle f, g|)$ by a constant $\epsilon > 0$?

End

- Thanks for your attention!