

Recent progress on the distillability problem

Lin Chen

School of Mathematics and Systems Science,
Beihang University, Beijing, China

Email: linchen@buaa.edu.cn

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The talk is based on two papers:

1. [J. Phys. A. 44, 285303 \(2011\)](#),
2. [quant-ph/1602.04416 \(2016\)](#).

Collaborator:

[Dragomir Z Djokovic](#)

Department of Pure Mathematics and Institute for Quantum Computing, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada

Outlines

- The distillability problem and entanglement distillation

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- $M \times N$ NPT states of rank $\max\{M, N\}$

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- $M \times N$ NPT states of rank $\max\{M, N\}$
- Two-qutrit NPT states of rank four and five
- Open problems

Entanglement distillation

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$$\rightarrow \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}} \right)^{\otimes m}$$

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If no pure entangled states can be obtained, then ρ is not distillable, or equivalently ρ is undistillable.

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- Proof of the existence 2-undistillable NPT Werner states: **Not found yet.**

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- Proof of the existence 2-undistillable NPT Werner states: **Not found yet.**
- Attempts for the proof: **Yes, there is something...**

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PPT and NPT

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The **partial transpose** of a bipartite quantum state ρ acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ is computed in an orthonormal (o.n.) basis $\{|a_i\rangle\}$ of system A, is defined by $\rho^\Gamma := \sum_{ij} |a_i\rangle\langle a_j| \otimes \langle a_j|\rho|a_i\rangle$.

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- For example, all separable states are PPT. All pure entangled states are NPT.

PPT and NPT

- **Example.** If

$$\rho = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}$$

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- (2) ρ is **distillable** if it is n -distillable for some $n \geq 1$, i.e.,

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- For example, PPT states are not distillable.

The math/mess of many-copy states

- $\rho^{\otimes n} = \rho_{A_1 B_1} \otimes \cdots \otimes \rho_{A_n B_n} := \rho_{A_1 \cdots A_n : B_1 \cdots B_n}$.

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- **Example.** Consider the “critical” Werner state

$$\rho_{A_1 B_1} = \sum_{i,j} (|i,j\rangle\langle i,j| - \frac{1}{2}|i,j\rangle\langle j,i|)_{A_1 B_1}$$

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- Then

$$\begin{aligned} \rho^{\otimes 2} &= \rho_{A_1 B_1} \otimes \rho_{A_2 B_2} \\ &= \sum_{i,j,m,n} \left(|im,jn\rangle\langle im,jn| - \frac{1}{2}|im,jn\rangle\langle jm,in| \right. \\ &\quad \left. - \frac{1}{2}|im,jn\rangle\langle in,jm| + \frac{1}{4}|im,jn\rangle\langle jn,im| \right)_{A_1 A_2, B_1 B_2} \end{aligned}$$

List of 1-distillable NPT states

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- **Example 1.** If

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is a Bell state.

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- So ρ is 1-distillable.

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- **Example 2.** If

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- then

$$(P \otimes I_B)\rho(P \otimes I_B)$$

is a two-qubit mixed entangled state. So ρ is also distillable.

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- Finding a good P is hard, although P belongs to LOCC.
- When is $(P \otimes I_B)\rho(P \otimes I_B)$ entangled?
- What if $(P \otimes I_B)\rho(P \otimes I_B)$ is PPT?
- A popular trick: let $(P \otimes I_B)\rho(P \otimes I_B)$ be a $2 \times N$ state then it has to be PPT, or some entries have to be zero.

Outlines

- The distillability problem and entanglement distillation

$M \times N$ NPT states of rank $\max\{M, N\}$

Two-qutrit NPT states of rank four and five

Open problems

Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

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Lemma

$M \times N$ NPT states of rank *smaller than* $\max\{M, N\}$ is 1-distillable.

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Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

- **LFRP**. Let ρ_{AB} be an $M \times N$ NPT states of rank $\max\{M, N\}$. We say ρ_{AB} has left full-rank property (LFRP) if there is some state $|x\rangle$ such that $\langle x|_B \rho_{AB} |x\rangle_B$ is invertible.

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- then

$$\max_x \left(\text{rank}(\langle x|_B \rho_{AB} |x\rangle_B) \right) = 2 < \text{rank } \rho_A = 3.$$

$$\max_x \left(\text{rank}(\langle x|_B \sigma_{AB} |x\rangle_B) \right) = 3 = \text{rank } \rho_A = 3.$$

Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

- So ρ_{AB} has no LFRP, and σ_{AB} has LFRP.

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- So ρ_{AB} has no LFRP, and σ_{AB} has LFRP.
- The right full-rank property (RFRP) can be similarly defined.
- **Strategy of proof.** Prove that ρ_{AB} is 1-distillable when
 - (1) ρ_{AB} has no LFRP or RFRP, and
 - (2) ρ_{AB} has LFRP and RFRP.

Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

- (1) ρ_{AB} has no LFRP or RFRP.

Using the matrix decomposition of semidefinite positive matrix $\rho = C^\dagger C$, where

$$C = (C_1, \dots, C_i, \dots, C_M)$$

and each matrix C_i is of size $(\text{rank } \rho) \times N$.

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- Project ρ to the following state by using the projector $P = |1\rangle\langle 1| + |i\rangle\langle i|$

$$\begin{aligned} \rho_{1,i} &= (P \otimes I_B) \rho (P \otimes I_B) \\ &= (C_1, C_i)^\dagger \cdot (C_1, C_i) = \begin{pmatrix} C_1^\dagger C_1 & C_1^\dagger C_i \\ C_i^\dagger C_1 & C_i^\dagger C_i \end{pmatrix} \end{aligned}$$

Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

- We split each C_i into four blocks $C_i = \begin{pmatrix} C_{i1} & C_{i2} \\ C_{i3} & C_{i4} \end{pmatrix}$ with C_{i1} square of size r_1 , where $C_1 = I_{r_1} \oplus 0$ because of ρ has no LFRP or RFRP. We have

$$\rho_{1,i} = \begin{pmatrix} I_{r_1} & 0 & \vdots & C_{i1} & C_{i2} \\ 0 & 0 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdot & \cdots & \cdots \\ C_{i1}^\dagger & 0 & \vdots & * & * \\ C_{i2}^\dagger & 0 & \vdots & * & * \end{pmatrix}$$

, where $i > 1$ and the asterisk stands for an unspecified block.

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- If some $C_{i2} \neq 0$, then ρ is 1-distillable. Thus we may assume that all $C_{i2} = 0$.

Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

- Now $\rho = C^\dagger C$ where

$$C = \left[\left(\begin{array}{cc} I_{r_1} & 0 \\ 0 & 0 \end{array} \right), \left(\begin{array}{cc} C_{21} & 0 \\ C_{23} & C_{24} \end{array} \right), \dots, \left(\begin{array}{cc} C_{M1} & 0 \\ C_{M3} & C_{M4} \end{array} \right) \right]$$

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- Since ρ has no LFRP or RFRP, the linear combination of C_{21}, \dots, C_{M1} is of deficient rank. We may assume

$$C_{24} = \begin{pmatrix} I_{r_2} & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$C_{i4} = \begin{pmatrix} C_{i41} & C_{i42} \\ C_{i43} & C_{i44} \end{pmatrix}$$

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- Project ρ to the state $(C')^\dagger C'$ where

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Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

- The process continues and the facts $C_{i2} = C_{i42} = \dots = 0$ implies that ρ has RFRP. It is a contradiction and we obtain that the process must terminate.

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- So ρ is distillable when it has no LFRP or RFRP.

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- (2) ρ_{AB} has LFRP and RFRP.

Using the matrix decomposition of semidefinite positive matrix $\rho = C^\dagger C$, we have

$$\rho = (C_1, \dots, C_{M-1}, I_N)^\dagger \cdot (C_1, \dots, C_{M-1}, I_N)$$

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- Since ρ is NPT, there exist i, j such that $[C_i, C_j] \neq 0$.
- One can show that $(xC_i + C_j, I_N)^\dagger \cdot (xC_i + C_j, I_N)$ is distillable for some complex number x .

Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

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Corollary

The bipartite state of rank four is separable if and only if it is PPT and its range contains at least one product state.

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For a tripartite pure state $\rho = |\psi\rangle\langle\psi|$, the bipartite reduced density operators ρ_{AB} and ρ_{AC} are PPT if and only if $|\psi\rangle = \sum_i |a_i\rangle|ii\rangle$ up to local unitary operations.

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$$\rho_{AB} = \rho_{AC} = \sum_i |a_i, i\rangle\langle a_i, i|$$

are both separable states.

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 - (6) Conditional entropy criterion:
 $H_\rho(B|A) = H(\rho_{AB}) - H(\rho_A) \geq 0$ and
 $H_\rho(A|B) = H(\rho_{AB}) - H(\rho_B) \geq 0$, where H is the von Neumann entropy.

Distilling $M \times N$ NPT states of rank $\max\{M, N\}$

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Theorem

*For a tripartite state $|\Psi\rangle_{ABC}$ with a non-distillable reduced state ρ_{BC} namely condition **(3)**, then conditions **(1)**-**(6)** are equivalent for ρ_{AB} .*

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- It is a way of unifying the six well-known conditions.

Outlines

- The distillability problem and entanglement distillation

$M \times N$ NPT states of rank $\max\{M, N\}$

Two-qutrit NPT states of rank four and five

Open problems

Distilling two-qutrit NPT states of rank four

- Entanglement distillation of $M \times N$ states ρ of rank **bigger than** $\max\{M, N\}$ turns out to be much harder.

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- **Facts:** $2 \times N$ NPT states are distillable, and $M \times N$ NPT states of rank $\max\{M, N\}$ are distillable.
- Hence, the first unsolved problem is to distill **3×3 NPT states of rank four.**

Distilling two-qutrit NPT states of rank four

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If ρ is a two-qutrit NPT state and ρ^Γ has at least two non-positive eigenvalues counting multiplicities, then ρ is 1-distillable.

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Theorem

If ρ is a two-qutrit NPT state and ρ^Γ has at least two non-positive eigenvalues counting multiplicities, then ρ is 1-distillable.

Proof.

By the hypothesis, there exist two eigenvectors of ρ^Γ , say $|\alpha\rangle$ and $|\beta\rangle$ with matrices A and B , such that $\rho^\Gamma|\alpha\rangle = \lambda|\alpha\rangle$, $\lambda < 0$, $\rho^\Gamma|\beta\rangle = \mu|\beta\rangle$, $\mu \leq 0$, and $\langle\alpha|\beta\rangle = 0$. If A is not invertible, then its rank is 2 and so ρ is 1-distillable.

If $N := A^{-1}B$ is not nilpotent, then $\det(I_3 + tN)$ is a nonconstant polynomial in t and we can choose t so that this determinant is 0. Thus $A + tB$ is singular, and $|\phi\rangle := |\alpha\rangle + t|\beta\rangle$ satisfies $\langle\phi|\rho^\Gamma|\phi\rangle = \lambda\|\alpha\|^2 + \mu|t|^2\|\beta\|^2 < 0$. Hence ρ is 1-distillable. The case that N is nilpotent is similar. \square

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Proof.

We can assume that $|0, 0\rangle \in \ker \rho$. Consequently, the first diagonal entry of ρ is 0, and the same is true for ρ^Γ . If the first column of ρ^Γ is not 0, then ρ is 1-distillable by projecting to a 2×3 NPT state. Otherwise $|0, 0\rangle \in \ker \rho^\Gamma$ and ρ is 1-distillable by last Theorem. \square

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- So the minimum rank of 1-undistillable NPT states is at least five.

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- We construct an example below.

Distilling two-qutrit NPT states of rank four

- The following state σ is an **edge PPT entangled state** of birank (5, 8) constructed by Kye and Osaka, 2012.

$$\frac{1}{N} \begin{bmatrix} 2 \cos \theta & 0 & 0 & 0 & -\cos \theta & 0 & 0 & 0 & -\cos \theta \\ 0 & \frac{1}{b} & 0 & -e^{-i\theta} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & -e^{i\theta} & 0 & 0 \\ 0 & -e^{i\theta} & 0 & b & 0 & 0 & 0 & 0 & 0 \\ -\cos \theta & 0 & 0 & 0 & 2 \cos \theta & 0 & 0 & 0 & -\cos \theta \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{b} & 0 & -e^{-i\theta} & 0 \\ 0 & 0 & -e^{-i\theta} & 0 & 0 & 0 & \frac{1}{b} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -e^{i\theta} & 0 & b & 0 \\ -\cos \theta & 0 & 0 & 0 & -\cos \theta & 0 & 0 & 0 & 2 \cos \theta \end{bmatrix},$$

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- where

$$N = 3(2 \cos \theta + b + 1/b),$$

and the two parameters $b > 0$ and $0 < |\theta| < \pi/3$.

Distilling two-qutrit NPT states of rank four

- Since $\text{rank } \sigma = 5$ and σ is an edge state, $\mathcal{R}(\sigma)$ contains a product state $|f, g\rangle$ such that $|f^*, g\rangle \notin \mathcal{R}(\sigma^\Gamma)$.

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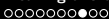
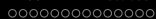
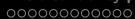
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Distilling two-qutrit NPT states of rank four

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Lemma

For any integer n , and sufficiently small $\epsilon = \epsilon(n) > 0$, the two-qutrit NPT state $\rho = \frac{1}{1-\epsilon}(\sigma - \epsilon|f, g\rangle\langle f, g|)$ is n -undistillable.

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Proof.

For any pure state $|\psi\rangle$ of Schmidt rank two, we have

$$\begin{aligned} (1 - \epsilon)^n \langle \psi | (\rho^\Gamma)^{\otimes n} | \psi \rangle &:= \langle \psi | (\sigma^\Gamma)^{\otimes n} | \psi \rangle + \sum_{k=1}^n c_k \epsilon^k \\ &\geq p^n \langle \psi | (I_9 - |\Psi\rangle\langle\Psi|)^{\otimes n} | \psi \rangle + \sum_{k=1}^n c_k \epsilon^k \end{aligned}$$

where c_k are complex numbers and p is the minimum positive eigenvalue of σ^Γ . Since the first summand is positive and has nothing to do with ϵ , the assertion holds. \square

Distilling two-qutrit NPT states of rank four

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Distilling two-qutrit NPT states of rank four

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Lemma

$$\min_{\psi \in \text{sr}_2} \langle \psi | (I_9 - |\Psi\rangle\langle\Psi|)^{\otimes n} | \psi \rangle \geq \frac{1}{3^n},$$

where sr_2 is the set of bipartite pure states of Schmidt rank two, and $|\Psi\rangle$ is the two-qutrit maximally entangled state.

Comparing with Werner states

- The comparison between our suspicious two-qutrit NPT states ρ of rank five and the “critical” NPT Werner states

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rank of partial transpose	9	9
parameters	b, θ, ϵ, n	n
construction	edge PPT states	$U \otimes U$ -invariant

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- The condition of rank nine prevents the further investigation in both cases.

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- Is there an undistillable suspicious two-qutrit NPT state $\rho = \frac{1}{1-\epsilon}(\sigma - \epsilon|f, g\rangle\langle f, g|)$ by a constant $\epsilon > 0$?

End

- Thanks for your attention!