Freeness and the Transpose

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based on joint work with M. Popa, E. Redelmeier, and R. Speicher

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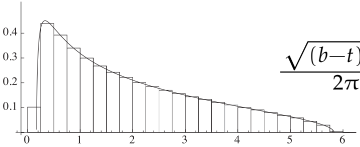
random matrices and entangled states

- $G = (g_{ij})_{ij}$, g_{ij} is a complex Gaussian random variable with $\mathrm{E}(g_{ij}) = 0$ and $\mathrm{E}(|g_{ij}|^2) = 1$, all entries independent
- G a $N \times M$ rectangular complex Gaussian matrix
- **complex** Wishart matrix $W_n = \frac{1}{N} G G^* \geqslant 0$
- *assume* $M/N \to c$, $0 < c < \infty$ as $M, N \to \infty$
- $\mu_N(\omega) = N^{-1}(\lambda_1(\omega) + \cdots + \lambda_N(\omega))$ where $\lambda_1(\omega) \leqslant \cdots \leqslant \lambda_N(\omega)$ are the eigenvalues of $W(\omega)$

Marchenko-Pastur law

- μ_N is a random measure on $\mathbb{R}$ which converges as $M, N \to \infty$ to a deterministic measure ν_c



$$\frac{\sqrt{(b-t)(t-a)}}{2\pi t}\, dt$$ (plus possibly an atom at 0)

Block Version

for $1 \leqslant i \leqslant d_1$, G_i is a $d_2 \times p$ Gaussian random matrix as above

$$W = \frac{1}{d_1 d_2} \begin{pmatrix} G_1 \\ \hline \vdots \\ \hline G_{d_1} \end{pmatrix} \left(\begin{array}{c|c|c} G_1^* & \cdots & G_{d_1}^* \end{array} \right) = \frac{1}{d_1 d_2} (G_i G_j^*)_{ij}$$

$$W \in M_{d_1}(\mathbf{C}) \otimes M_{d_2}(\mathbf{C})$$

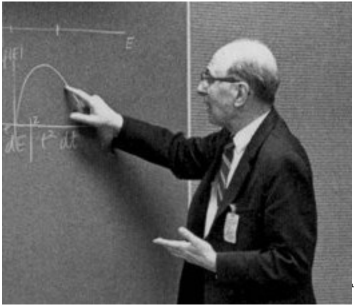
$W(i,j) = \frac{1}{d_1 d_2} G_i G_j^* = (i,j)$ block of W

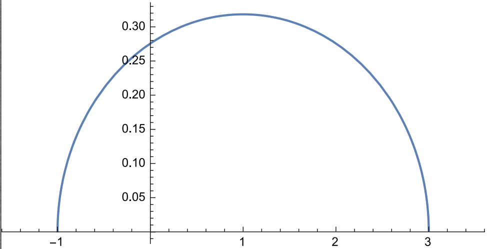
$W^{\urcorner} =$ transpose blocks but leave blocks intact

$W^{\Gamma} =$ leave blocks in place but apply transpose inside block

Aubrun's example of entanglement (2012)

- showed that $W^{\urcorner}$ and W^{Γ} have limiting eigenvalue distributions (which are non-random) as $p/(d_1 d_2) \to c$





$$\frac{\sqrt{4-\left(\frac{x-c}{\sqrt{c}}\right)^2}}{2\pi}\, d\left(\frac{x-c}{\sqrt{c}}\right) \text{ on } [c-2\sqrt{c}, c+2\sqrt{c}]$$

So W does not have positive partial transpose for $0 < c < 4$

Voiculescu's freeness (1983)

- $\mathbb{F}_2 = \langle u, v \rangle$ = free group on generators u and v
- $\mathcal{A} = \mathbf{C}[\mathbb{F}_2] = \{\sum_{g \in \mathbb{F}_2} \alpha_g g \mid \text{sum is finite}\}$, $\varphi(\sum_{g \in \mathbb{F}_2} \alpha_g g) = \alpha_e$
- $x_1 = u + u^{-1}$, $\mathcal{A}_1 = \text{alg}(1, x_1)$ $x_2 = v + v^{-1}$, $\mathcal{A}_2 = \text{alg}(1, x_2)$

if $a_1, \ldots a_n \in \mathcal{A}$,

- without constant term (i.e. $\varphi(a_i) = 0$), and
- $a_i \in \mathcal{A}_{j_i}$ and $j_1 \neq j_2 \neq \cdots \neq j_n$,

then $a_1 \cdots a_n$ has no constant term.

and in general

- $(\mathcal{A}, \varphi)$ is a unital algebra and a linear functional $\varphi : \mathcal{A} \to \mathbf{C}$ with $\varphi(1) = 1$
- $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}$ are *free* with respect to φ if whenever $a_1, \ldots, a_n \in \mathcal{A}$
  - (*i*) $\varphi(a_i) = 0$ for $1 \leqslant i \leqslant n$,
  - (*ii*) $a_i \in \mathcal{A}_{j_i}$ and $j_1 \neq j_2 \neq \cdots \neq j_n$,

  then $\varphi(a_1 \cdots a_n) = 0$

Voiculescu's free world

- Voiculescu produced a parallel universe by finding a free analogue for a long list of objects in classical probability;
- there is an important wormhole that connects the two universes;

[Voi91] independent matrices are asymptotically free provided that one ensemble is unitarily invariant

- in higher order freeness the transpose plays an important role

the transpose of the GUE (joint with M. Popa)

- $X = \frac{1}{\sqrt{N}}(x_{ij})$ where $\mathrm{E}(x_{ij}) = 0$, $\mathrm{E}(|x_{ij}|^2) = 1$ and $\{x_{ii}\}_i \cup \{\mathrm{Re}(x_{ij})\}_{i<j} \cup \{\mathrm{Im}(x_{ij})\}_{i<j}$ are independent real Gaussian random variables
- $Y_1 = \frac{\sqrt{2}}{\sqrt{N}}\big(\mathrm{Re}(x_{ij})\big)_{ij}$ $Y_2 = \frac{\sqrt{2}}{\sqrt{N}}\big(i\,\mathrm{Im}(x_{ij})\big)_{ij}$
- Y_1 and Y_2 are real/imaginary and self-adjoint, independent and asymptotically semi-circular
- X and X^{T} are not independent, but
- $X = \frac{1}{\sqrt{2}}(Y_1 + Y_2)$, $X^{\mathrm{T}} = \frac{1}{\sqrt{2}}(Y_1 - Y_2)$ are asymptotically free

THM: unitarily invariant ensembles are asymptotically free from their transpose

back to the Wishart case

(joint work with M. Popa (San Antonio, TX)

$$W = \frac{1}{d_1 d_2} \begin{pmatrix} G_1 \\ \hline \vdots \\ \hline G_{d_1} \end{pmatrix} \left(\begin{array}{c|c|c} G_1^* & \cdots & G_{d_1}^* \end{array} \right) = \frac{1}{d_1 d_2} (G_i G_j^*)_{ij}$$

THM: the matrices $\{W, W^{\urcorner}, W^{\Gamma}, W^{T}\}$ form an asymptotically free family

THM: we don't need to assume that the entries are Gaussian

Partial Transposes of Unitary Matrices

- let U be a $d_1 d_2 \times d_1 d_2$ Haar distributed random unitary matrix. We wish to investigate the joint $*$-distribution of the matrices $\{U, U^{\urcorner}, U^{\Gamma}, U^{\mathrm{T}}\}$. This amounts to finding the joint distribution of the following eight operators

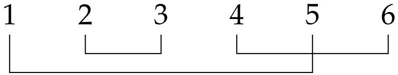
$$U, U^*, U^{\urcorner}, (U^{\urcorner})^*, U^{\Gamma}, (U^{\Gamma})^*, U^{\mathrm{T}}, (U^{\mathrm{T}})^*.$$

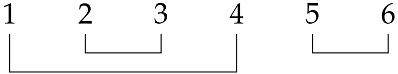
- ϵ is left partial transpose 'bit', η is the right partial transpose 'bit', θ is the adjoint 'bit'

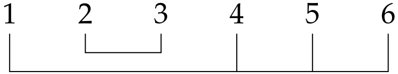
$$U^{(\epsilon,\eta,\theta)} = \begin{cases} U & \epsilon = 1, \eta = 1, \theta = 1 \\ U^* & \epsilon = 1, \eta = 1, \theta = -1 \\ U^{\urcorner} & \epsilon = -1, \eta = 1, \theta = 1 \\ (U^{\urcorner})^* & \epsilon = -1, \eta = 1, \theta = -1 \\ U^{\Gamma} & \epsilon = 1, \eta = -1, \theta = 1 \\ (U^{\Gamma})^* & \epsilon = 1, \eta = -1, \theta = -1 \\ U^{\mathrm{T}} & \epsilon = -1, \eta = -1, \theta = 1 \\ (U^{\mathrm{T}})^* & \epsilon = -1, \eta = -1, \theta = -1 \end{cases}$$

$$\mathrm{E}(\mathrm{Tr}(U^{(\epsilon_1,\eta_1,\theta_1)}\dots U^{(\epsilon_n,\eta_n,\theta_n)}))$$
$$= \sum_{p,q\in\mathcal{P}_2^\theta(n)} \mathrm{Wg}(p,q) d_1^{\#(\theta\epsilon\gamma\delta\gamma^{-1}\epsilon\theta\vee\delta q\delta p)} d_2^{\#(\theta\eta\gamma\delta\gamma^{-1}\eta\theta\vee\delta q\delta p)}$$

- $\mathcal{P}_2^\theta(n)$ is the set of pairings of $\{1,\dots,n\}$ that connect a $\theta = 1$ to a $\theta = -1$
- Wg is a matrix indexed by pairs of pairings (= inverse of $N^{\#(p\vee q)}$), see the work of Benoît Collins)

$p =$


$q =$


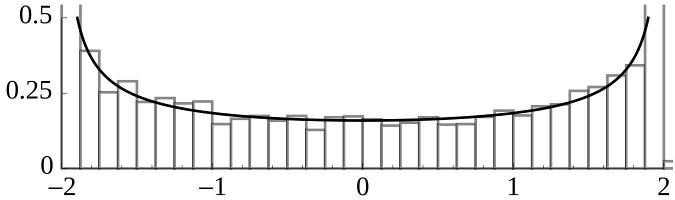
$p \vee q =$


conclusion

(joint with E. Redelmeier)

THM: the families $\{U, U^*\}, \{U^{\urcorner}, (U^{\urcorner})^*\}, \{U^{\Gamma}, (U^{\Gamma})^*\}$, and $\{U^{\mathrm{T}}, (U^{\mathrm{T}})^*\}$ are asymptotically free

THM: $U + U^*$ is asymptotically arcsine but $(U + U^*)^{\urcorner}$ is asymptotically semi-circular



methodology—classical cumulants

- μ is probability measure on $\mathbb{R}$ with Fourier transform $\hat{\mu}$: $\varphi(z) = \int e^{itz}\,dt$, we can expand the logarithm as

$$\log(\varphi(it)) = \sum_{n\geqslant 1} k_n \frac{t^n}{n!}$$

  (*provided the measure has moments of all orders*).
- the numbers $\{k_n\}_n$ are the cumulants of μ
- if $d\mu(t) = \frac{e^{-t^2/2}}{\sqrt{2\pi}}\,dt$ is the normal distribution then $\log(\varphi(it)) = t^2/2$ so $k_2 = 1$ and all other k's are 0.
- if $X : \Omega \to \mathbb{R}$ is a random variable and $\mu_X(E) = \mathrm{P}(X^{-1}(E))$ is the distribution of X, μ_X is a probability measure on $\mathbb{R}$ with cumulants $\{k_n^{(X)}\}_n$, we call these the cumulants of X
- X_1 and X_2 are independent $\Rightarrow k_n^{(X_1+X_2)} = k_n^{(X_1)} + k_n^{X_2)}$ for all n

the R-transform & free cumulants (*Voiculescu, 1982*)

- $(\mathcal{A}, \varphi)$ unital $*$-algebra with state, $a = a^* \in \mathcal{A}$;
- $G_a(z) = \int (z-t)^{-1}\, d\mu_a(t) = \varphi((z-a)^{-1})$, $z \in \mathbf{C}^+$.
- $R_a(z) = G_a^{\langle -1 \rangle}(z) - \dfrac{1}{z} = \sum_{n \geqslant 0} \kappa_{n+1} z^n$
  is the R-transform of a; and $\{\kappa_n\}_n$ are the *free* cumulants of a.
- a_1 and a_2 freely independent $\Rightarrow \kappa_n^{(a_1+a_2)} = \kappa_n^{(a_1)} + \kappa_n^{(a_2)}$ for all n
- a_1 and a_2 are free $\Leftrightarrow \forall n, \forall i_1, \ldots, i_n$, $\kappa_n(a_{i_1}, \ldots, a_{i_n}) = 0$ unless $i_1 = \cdots = i_n$ (*vanishing of mixed cumulants—Nica-Speicher*)

cumulants for the partial transpose of a Wishart matrix

- $R(z) = c + cz$, so $\kappa_1 = \kappa_2 = c$
- $G^{\langle -1 \rangle}(z) = \frac{1}{z} + c + cz$
- $G(z) = \sqrt{c}\frac{\left(\frac{z-c}{\sqrt{c}}\right) - \sqrt{\left(\frac{z-c}{\sqrt{c}}\right)^2 - 4}}{2}$
- in the limit, the spectral measure of W^Γ is a shifted semi-circle

why do we get semi-circular operators?

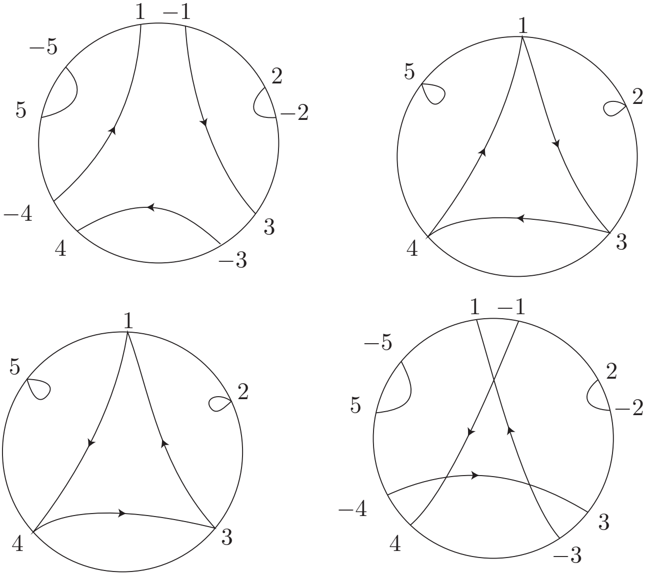
- in both the Wishart and Haar unitary case we reduce the calculation of either

$$E(\mathrm{Tr}(W^{(\epsilon_1,\eta_1)} \cdots W^{(\epsilon_n,\eta_n)}))$$

  or

$$E(\mathrm{Tr}(U^{(\epsilon_1,\eta_1,\theta_1)} \cdots U^{(\epsilon_n,\eta_n,\theta_n)}))$$

  to a sum over permutations (or graphs in the non-Gaussian case)
- the only terms which survive in the limit are the ones which admit planar diagrams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