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MATHEMATICAL ASPECTS IN CURRENT QUANTUM INFORMATION THEO

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Notes Taken by: Gunjan sapra.

Title:

* Free probability
* Random matrices
* positive maps

(joint work with Ion Nechita & P. Hayden, can be found on 150508042)

Q: Why Random matrices theory is a powerful tool to create positive map but not completely positive.

Notation $\Rightarrow X^{M} = X^{M*} \in M_n(\mathbb{C})$.

$\lambda_1^{(N)} \geq \lambda_2^{(N)} \dots \geq \lambda_N^{(N)} \in \mathbb{R}^N$ are eigenvalues of X^M.

We consider eigenvalue counting measure

$$\mu_{X(N)} = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i^{(N)}}$$

It can be interpreted in two ways:

(i) $\mu_{X(N)}([a,b]) = \%$ of eigenvalues in $[a,b]$.

(ii) $\int x^l \, d\mu_X^{(N)}(\lambda) = \frac{1}{N} \underbrace{\mathrm{Tr}\, X^{(N)\,l}}_{\text{Normalized Trace.}}$

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Let μ be a probability distribution on $\mathbb{R}$.
We say that

$$X^{(N)} \xrightarrow{\text{dist}} \mu \overset{\text{def}^n}{\iff} \mu_{X^{(M)}} \to \mu.$$

Normalized Trace

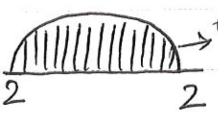
Wigner ⇒ GUE - Gaussian unitary ensembles.

<u>defn</u> ⇒ a matrix

$$X^{(N)} \overset{\mathcal{L}}{\sim} \frac{1}{2} e^{-\frac{N}{2} \operatorname{Tr} H^2} \, dH$$

dH ↓ uniform lebesgue measure on self adjoint matrices

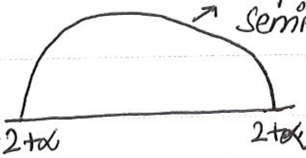
Wigner proved that (for above GUE)
as $N \to \infty$

$X^{(N)} \to$ semi-circle  → ellipse of area 1.

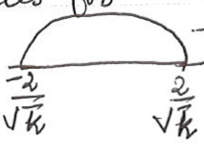
Result (1980) ⇒ With probability 1,

$$\| X^{(N)} \|_\infty \to 2$$

Remarks:

(1) Consider $X^{(N)} + \alpha \, Id_N \to$  —(1)

(2) If $N = kn$ where k is fixed.
Let $\pi^{(N)}$ be a self adjoint projection of rank n.
We will check the eigenvalues for

$$\pi^{(N)} X^{(N)} \pi^{(N)} \to \left(1 - \frac{1}{k}\right) \delta_0 + \frac{1}{k}$$  —semi-circle —(2)

Comibe (1) & (2), we get new examples of positive maps.

Idea:- View $X^{(N)} \in M_k(\mathbb{C}) \otimes M_n(\mathbb{C})$ & think of it as choi map of $\Phi^{(N)}: M_k(\mathbb{C}) \to M_n(\mathbb{C})$.

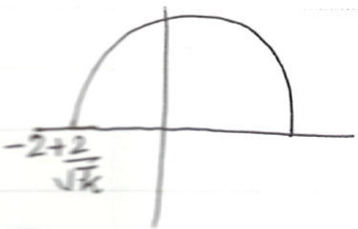
$k=3$, $N=3n$

$$X^{(N)} = \begin{bmatrix} \Phi^N(E_{11}) & \Phi^N(E_{12}) & \cdots \\ & & \\ & & \cdots \end{bmatrix}$$

Thm ⇒ let $k \geq 2$, $\epsilon > 0$ & $N = kn$ consider $X^{(N)} = GUE + \frac{2}{\sqrt{k}}(1+\epsilon) I_N$
Then, with probability 1, as $N \to \infty$, the associated

$\Phi^{(N)}: M_k(\mathbb{C}) \to M_n(\mathbb{C})$ is positive but not completely positive.

Outline of proof ⇒
Consider shifted semi-circle



⇒ $\Phi^{(N)}$ is not completely positive

$\Phi^{(N)}$ is positive, for example $\Phi^{(N)}(E_{11}) \geq 0$.
We need to use union bound arguement + Continuity

Dual version (positive maps $\Leftrightarrow$ entanglement detection)

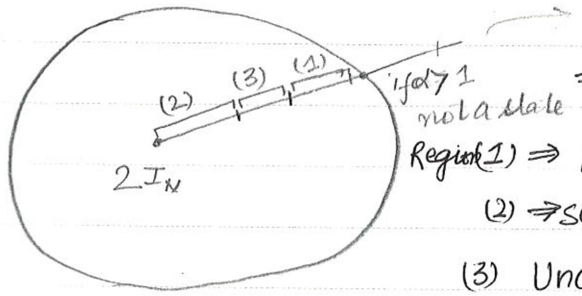
Thm $\Rightarrow$ With probability 1, as $N \to \infty$

(i) $2 I_N + \alpha\,GUE$ is separable as soon as

$$\frac{2}{\alpha} > 2 + \frac{4(k-1)}{\sqrt{k}}, \quad k \geq 2.$$

(ii) It is PPT $\Leftarrow$ $1 > \alpha > \frac{4}{\sqrt{k}}$

(iii) If $\alpha > 1$, not even a state



With probability (1), as $N \to \infty$ $\Rightarrow$ $\Phi^{(N)}$ is indecomposable.

Region (1) $\Rightarrow$ PPT

(2) $\Rightarrow$ separable

(3) Uncertain & size of the region doesn't depend on k.

(Free Probability construction)

$$(M, \tau) = \left(M_k(\mathbb{C}), \frac{1}{k} Tr\right) * (L^\infty(\mathbb{R}, \mu), E)$$

$\downarrow$ (E_{ij}) $\qquad\qquad$ $\downarrow$ With compact support.

Define $\Phi_\mu : M_k \to E_{11} M E_{11}$

$$E_{ij} \mapsto E_{i1} X_\mu E_{j1}$$

Φ_μ is Completely positive $\Leftrightarrow$ $\mathrm{supp}\,\mu \subseteq [0, \infty)$

Φ_μ is positive $\Leftrightarrow$ $\mathrm{supp}\,\mu^{\boxplus k} \subseteq [0, \infty)$