Conditions for Separability of Matrices in $\mathbb{M}_2 \otimes \mathbb{M}_n$

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(2) § 1. Introduction.

The tensor product $\mathbb{M}_m \otimes \mathbb{M}_n$ is identified with the space $\mathbb{M}_m(\mathbb{M}_n)$ of $m \times m$ matrices with entries in $\mathbb{M}_n$.

Here for $X = [\xi_{jk}]_{j,k} \in \mathbb{M}_m, \ Y \in \mathbb{M}_n$

$$X \otimes Y \sim [\xi_{jk} Y]_{j,k} \in \mathbb{M}_m(\mathbb{M}_n).$$

The important fact is that

$$0 \le X \in \mathbb{M}_m, \ 0 \le Y \in \mathbb{M}_n \implies X \otimes Y \in \mathbb{M}_m(\mathbb{M}_n)^+.$$

(3) The cone

$$\mathfrak{P}_+ := \{\sum X_k \otimes Y_k;\ 0 \leq X_k \in \mathbb{M}_m,\ 0 \leq Y_k \in \mathbb{M}_n\},$$

however, does not cover the cone $\mathfrak{P}_0 := \mathbb{M}_m(\mathbb{M}_n)^+$.

$0 \leq \mathbf{S} \in \mathbb{M}_m(\mathbb{M}_n)$ is said to be separable if it belongs to the cone $\mathfrak{P}_+$.

The standard cone $\mathfrak{P}_0$ is selfdual with respect to the duality induced by the inner product:

$$\langle X|Y\rangle := \mathrm{Tr}(X^* Y) \qquad \forall\ X, Y \in \mathbb{M}_n.$$

The dual cone of $\mathfrak{P}_+$ will be denoted by $\mathfrak{P}_-$.

$$\mathfrak{P}_+ \subset \mathfrak{P}_0 \subset \mathfrak{P}_-.$$

(4) Given selfadjoint $\mathbf{S} = [S_{jk}]_{j,k} \in \mathbb{M}_m(\mathbb{M}_n)$, our problem is to enumerate reasonable conditions on S_{jk} $(j, k = 1, \ldots, m)$ which guarantee separability of $\mathbf{S}$.

In this talk we restrict ourselves to the simplest case

$$m = 2, \quad \text{that is,} \quad \mathbf{S} = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \quad A, B, C \in \mathbb{M}_n.$$

(5) **Notations.** $x \in \mathbb{C}^n$ is a column n-vector, so that x^* is row n-vector.

(Matrix multiplication)

$$xy^* \in \mathbb{M}_n \quad \text{and} \quad y^*x = \langle y|x\rangle \text{ (inner product)}$$

○ For $x_j, y_j \in \mathbb{C}^n$ $(j = 1, 2, \ldots, m)$

$$[x_1, \ldots, x_m] \cdot [y_1, \ldots, y_m]^* = \sum_{j=1}^{m} x_j y_j^* \in \mathbb{M}_n,$$

$$[y_1, \ldots, y_m]^* \cdot [x_1, \ldots, x_m] = [\langle y_j|x_k\rangle]_{j,k} \in \mathbb{M}_m.$$

(6) A vector in $\mathbb{C}^2(\mathbb{C}^n) = \mathbb{C}^2 \otimes \mathbb{C}^n$ is called a product vector if it is of the form $\boldsymbol{\xi} \otimes x$ with $\boldsymbol{\xi} \in \mathbb{C}^2$, $x \in \mathbb{C}^n$, that is

$$\boldsymbol{\xi} \otimes x = \begin{bmatrix} \xi_1 x \\ \xi_2 x \end{bmatrix} \text{ with } \boldsymbol{\xi} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}.$$

◯ Extreme rays of the convex cone $\mathfrak{P}_+$ consist of

$$t(\boldsymbol{\xi} \otimes x)(\boldsymbol{\xi} \otimes x)^* \quad (t \geq 0), \quad \exists\, \boldsymbol{\xi} \in \mathbb{C}^2, x \in \mathbb{C}^2.$$

◯ Extreme rays of the cone $\mathfrak{P}_0$ consist of

$$t \cdot \begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}^* (t \geq 0), \quad \exists\, x, y \in \mathbb{C}^n.$$

◯ Extreme rays of the cone $\mathfrak{P}_-$? ? ?

(7) Recall the Choi corredpondence between a linear map $\Phi : \mathbb{M}_2 \longmapsto \mathbb{M}_n$ and its Choi matrix $\mathbf{C}_\Phi \in \mathbb{M}_2(\mathbb{M}_n)$

$$\mathbf{C}_\Phi := \begin{bmatrix} \Phi(E_{1,1}) & \Phi(E_{1,2}) \\ \Phi(E_{2,1}) & \Phi(E_{2,2}) \end{bmatrix}$$

where $E_{j,k}$ $(j, k = 1, 2)$ are matrix units of $\mathbb{M}_2$.

Theorem of Choi:

$$\Phi \quad \textbf{completely positive} \quad \Longleftrightarrow \quad \mathbf{C}_\Phi \in \mathfrak{P}_0.$$

The following is also immediate:

○ Φ **positive** $\Longleftrightarrow$ $\mathbf{C}_\Phi \in \mathfrak{P}_-$.

(8) ○ $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathfrak{P}_0 \quad \Longleftrightarrow$

$$\left\langle \begin{bmatrix} x \\ y \end{bmatrix} \middle| \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle \geq 0 \quad \forall\, x, y \in \mathbb{C}^n$$

$$\Longleftrightarrow \quad A, C \geq 0, \quad \langle x|Ax\rangle \cdot \langle y|Cy\rangle \geq |\langle x|By\rangle|^2 \ \forall\, x, y$$

○ $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathfrak{P}_- \quad \Longleftrightarrow$

$$\left\langle \begin{bmatrix} \xi x \\ \eta x \end{bmatrix} \middle| \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \begin{bmatrix} \xi x \\ \eta x \end{bmatrix} \right\rangle \geq 0 \quad \forall\, \xi, \eta \in \mathbb{C}, \ x$$

$$\Longleftrightarrow \quad A, C \geq 0, \quad \langle x|Ax\rangle \cdot \langle x|Cx\rangle \geq |\langle x|Bx\rangle|^2 \ \forall\, x.$$

○ No corresponding formula for $\mathfrak{P}_+$!!

(9) $0 \le \mathbf{S} \in \mathbb{M}_2(\mathbb{M}_n)$ is said to be totally separable if $\exists$ $\boldsymbol{\xi}_j \in \mathbb{C}^2,\ x_j \in \mathbb{C}^n,$ such that

$$\langle \boldsymbol{\xi}_j \otimes x_j | \boldsymbol{\xi}_k \otimes x_k \rangle = 0 \quad \forall\, j \neq k$$

and

$$\mathbf{S} = \sum_{j=1}^{N} (\boldsymbol{\xi}_j \otimes x_j)(\boldsymbol{\xi}_j \otimes x_j)^* = \sum_{j=1}^{N} (\boldsymbol{\xi}_j \boldsymbol{\xi}_j^*) \otimes (x_j x_j^*).$$

◯ **S** is totally separable if and only if the eigenspace corresponding to each positive eigenvalue admits a **CONS** consisting of product vectors.

◯ Totally separable $\Longrightarrow$ separable.

(10) ○ $\mathfrak{P}_0$ is stable under the functional calculus :

$$f(\mathfrak{P}_0) \subset \mathfrak{P}_0 \quad \forall\, f(t) \geq 0 \text{ on } [0, \infty).$$

○ Only functions $f(t) \geq 0$ on $[0, \infty)$ for which

$$f(\mathfrak{P}_+) \subset \mathfrak{P}_+$$

are of the form $f(t) = \alpha t$ with $\alpha \geq 0$.

○ **S** totally separable $\Longrightarrow$

$f(\mathbf{S})$ toally separable $\forall\, f(t) \geq 0$ (with $f(0) = 0$.)

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(11) § 2. Separability

A necessary condition for $0 \leq \mathbf{S} = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ to be separable is (PPT) (positive partial transpose)

$$\mathbf{S}^\tau := \begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geq 0 \quad (\text{because} \quad (X \otimes Y)^\tau = X^T \otimes Y.)$$

○ (Woronowicz) When $n \leq 3$,

$$(\text{PPT}) \quad \Longrightarrow \quad \text{separable}$$

(12) ◯ $0 \leq X \in \mathbb{M}_n \quad \Longrightarrow$

$$\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & X \end{bmatrix} \quad \text{separable.}$$

◯ $X \in \mathbb{M}_2, \quad \mathbf{Y} \in \mathbb{M}_{n,m},$

$\mathbf{T} \in \mathbb{M}_2 \otimes \mathbb{M}_m$ separable

$$\Longrightarrow$$

$$\mathbf{S} := (X \otimes \mathbf{Y}) \cdot \mathbf{T} \cdot (X \otimes \mathbf{Y})^* \quad \text{separable.}$$

(13) **Proposition 1.** Let $\mathbf{S} = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathfrak{P}_-$.

(i) A, B, B^*, C commuting (normal !)

$\Longrightarrow$ $\mathbf{S}$ totally separable.

(ii) $\mathbf{S}$ separable $\Longrightarrow$

$\exists\ \tilde{A} \in \mathbb{M}_{n,m}$, normal $N \in \mathbb{M}_m$, $0 \le \tilde{C} \in \mathbb{M}_n$ such that

$$\mathbf{S} = (I_2 \otimes \tilde{A}) \begin{bmatrix} I & N^* \\ N & NN^* \end{bmatrix} (I_2 \otimes \tilde{A})^* + \begin{bmatrix} 0 & 0 \\ 0 & \tilde{C} \end{bmatrix}.$$

(14) **(Proof.)** (i) Use simultaneous diagonalization of commuting normal matrices, A, B, B^* and C !

$$A = \sum_{j=1}^{n} \lambda_j (x_j x_j^*), \; B = \sum_{j=1}^{n} \xi_j (x_j x_j^*), \; C = \sum_{j=1}^{n} \mu_j (x_j x_j^*)$$

$$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} = \sum_{j=1}^{n} \begin{bmatrix} \lambda_j & \xi_j \\ \overline{\xi_j} & \mu_j \end{bmatrix} \otimes (x_j x_j^*).$$

$\mathbf{S} \in \mathfrak{P}_-$ implies $\begin{bmatrix} \lambda_j & \xi_j \\ \overline{\xi_j} & \mu_j \end{bmatrix} \geq 0$. Use representation

$$\begin{bmatrix} \lambda_j & \xi_j \\ \overline{\xi_j} & \mu_j \end{bmatrix} = \boldsymbol{\xi}_{\mathbf{j}} \boldsymbol{\xi}_j^* + \boldsymbol{\eta}_j \boldsymbol{\eta}_j^* \quad \exists \; \boldsymbol{\xi}_j, \; \boldsymbol{\eta}_j \in \mathbb{C}^2.$$

(15) **Proof** of (ii).

Use representation for separable $\mathbf{S}$ =

$$\sum_{j=1}^{m_1}(\begin{bmatrix}1\\ \xi_j\end{bmatrix}\otimes x_j))(\begin{bmatrix}1\\ \xi_j\end{bmatrix}\otimes x_j)^*+\sum_{k=1}^{m_2}(\begin{bmatrix}0\\ \eta_k\end{bmatrix}\otimes y_k)(\begin{bmatrix}0\\ \eta_k\end{bmatrix}\otimes y_k)^*.$$

Let $\tilde{A} = [x_1, \dots, x_{m_1}] \in \mathbb{M}_{n,m_1}$, $\tilde{C} = \sum_{k=1}^{m_2} |\eta_k|^2 y_k y_k^* \in \mathbb{M}_n$, $N = \mathrm{diag}(\xi_1, \dots, \xi_{m_1})$.

Then

$$\mathbf{S} = \begin{bmatrix}\tilde{A}\\ \tilde{A}N\end{bmatrix} \cdot \begin{bmatrix}\tilde{A}\\ \tilde{A}N\end{bmatrix}^* + \begin{bmatrix}0 & 0\\ 0 & \tilde{C}\end{bmatrix}.$$

(16) **(Positiivity)**

$$\bigcirc \quad \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0 \iff$$

$$A, C \geq 0, \; B = A^{1/2} W C^{1/2} \quad \exists \, \|W\| \leq 1.$$

$$\iff \quad \mathrm{ran}(B) \subset \mathrm{ran}(A),$$

$$\ker(B) \supset \ker(C) \quad \text{and} \quad B^* A^{-1} B \leq C.$$

Here A^{-1} is the generalized inverse of $A \geq 0$.

$$\ker(A^{-1}) = \ker(A) \quad \text{and} \quad \mathrm{ran}(A^{-1}) = \mathrm{ran}(A)$$

$$A^{-1} \cdot A = \text{projection to } \mathrm{ran}(A).$$

(17) (Positivity) + (PPT)

○ $$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0 \quad \text{and} \quad \begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geq 0 \quad \iff$$

$B = A^{1/2} N A^{1/2} \quad \exists^1 \ N$ such that

$$\mathrm{ran}(N) \subset \mathrm{ran}(A) \qquad \text{and} \qquad \ker(N) \geq \ker(A)$$

$$A^{1/2} N^* N A^{1/2} \leq C \qquad \text{and} \qquad A^{1/2} N N^* A^{1/2} \leq C.$$

(18) **Theorem 1. (Rank condition 1.)**

Let $\mathbf{S} = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$ and $\mathbf{S}^\tau = \begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geq 0.$

$\mathrm{rank}(\mathbf{S}) = \mathrm{rank}(A) \implies \mathbf{S}$ separable.

(19) **(Proof.)** Use the identity

$$\mathbf{S} = \begin{bmatrix} I & 0 \\ B^*A^{-1} & I \end{bmatrix} \cdot \begin{bmatrix} A & 0 \\ 0 & C - B^*A^{-1}B \end{bmatrix} \cdot \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}$$

to conclude

$$B^*A^{-1}B = A^{1/2}N^*NA^{1/2} = C \geq A^{1/2}NN^*A^{1/2}$$

hence $N^*N \geq NN^*$ and N is normal.

Then $\mathbf{S} =$

$$(I_2 \otimes A^{1/2}) \otimes \begin{bmatrix} I & N \\ N^* & N^*N \end{bmatrix} (I_2 \otimes A^{1/2}) + \begin{bmatrix} 0 & 0 \\ 0 & C - B^*A^{-1}B \end{bmatrix}.$$

Theorem 2. (Rank condition 2).

Let $\mathbf{S} = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$ and $\mathbf{S}^\tau = \begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geq 0$.

$$\mathrm{rank}(\mathbf{S}) \leq 4 \quad \textit{or} \quad \mathrm{rank}(A) \leq 3$$

$$\Longrightarrow \quad \mathbf{S} \quad \text{separable.}$$

(21) **(Proof)** Use Theorem 1 if

$$\mathrm{rank}(A) = 4 = \mathrm{rank}(\mathbf{S}).$$

If $\mathrm{rank}(A) \le 3$,
$\mathrm{ran}(A^{1/2}NA^{1/2}), \mathrm{ran}(A^{1/2}N^*A^{1/2}) \subset \mathrm{ran}(A)$

$$\begin{bmatrix} A & A^{1/2}NA^{1/2} \\ A^{1/2}N^*A^{1/2} & A^{1/2}N^*NA^{1/2} \end{bmatrix} \ge 0 \qquad C \ge A^{1/2}N^*NA^{1/2},$$

$$\begin{bmatrix} A & A^{1/2}N^*A^{1/2} \\ A^{1/2}NA^{1/2} & A^{1/2}NN^*A^{1/2} \end{bmatrix} \ge 0 \qquad C \ge A^{1/2}NN^*A^{1/2}.$$

Use the following lemma to apply Woronowicz.

(22) **Lemma.**
(Schur complement theorem)

For any subspace $\mathcal{M} \subset \mathbb{C}^n$ and $0 \leq C \in \mathbb{M}_n$

$$\max\{X;\ C \geq X \geq 0,\ \mathrm{ran}(X) \subset \mathcal{M}\}$$

$$= \begin{bmatrix} C_{11} - C_{12}C_{22}^{-1}C_{21} & 0 \\ 0 & 0 \end{bmatrix}$$

where $\qquad C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$

according to the decomposition $\quad \mathbb{C}^n = \mathcal{M} \oplus \mathcal{M}^{\perp}$.

(23) **(Proof continued:)** In fact, let

$$C_0 \ := \ \max\{X; C \geq X \geq 0, \ \mathrm{ran}(X) \subset \mathrm{ran}(A)\}.$$

Then $\mathbf{S}_0 := \begin{bmatrix} A & A^{1/2}NA^{1/2} \\ A^{1/2}N^*A^{1/2} & C_0 \end{bmatrix} \geq 0$
the ranges of all entries are contained in $\mathrm{ran}(A)$ of dimension 3 and $\mathbf{S}_0$ has (PPT) .

By Woronowicz, $\mathbf{S}_0$ is separable and so is

$$\mathbf{S} = \mathbf{S}_0 + \begin{bmatrix} 0 & 0 \\ 0 & C - C_0 \end{bmatrix}. \qquad \square$$

(24) **Theorem 3. (Three commuting entries)**

Let $\mathbf{S} = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$ and $\mathbf{S}^\tau = \begin{bmatrix} A & B^* \\ B & C \end{bmatrix} \geq 0.$

Some three among A, B, B^* and C are mutually commuting

$$\Longrightarrow \mathbf{S} \quad \text{separable.}$$

(25) **(Proof.)** (i) Let A, B, C be commuting.
It suffices to prove the fact :

$$C \geq B^*B, \quad C \geq BB^* \quad \text{and} \quad BC = CB$$
$$\Longrightarrow \quad \begin{bmatrix} I & B \\ B^* & C \end{bmatrix} \quad \text{separable.}$$

Consider the dilation (Halmos' method)

$$\mathbf{N} := \begin{bmatrix} B & (C - BB^*)^{1/2} \\ (C - B^*B)^{1/2} & -B^* \end{bmatrix} \quad \text{on } \mathbb{C}^n \oplus \mathbb{C}^n.$$

Because of $BC = CB$, $\mathbf{N}$ is normal, and with the canonical imbedding V,

$$\begin{bmatrix} I & B \\ B^* & C \end{bmatrix} = (I_2 \otimes V)^* \cdot \begin{bmatrix} \mathbf{I} & \mathbf{N} \\ \mathbf{N}^* & \mathbf{N}^*\mathbf{N} \end{bmatrix} \cdot (I_2 \otimes V).$$

(26) (ii) If A, B, B^* are commuting, consider

$$\mathbf{S} = \begin{bmatrix} A & B \\ B^* & B^*A^{-1}B \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & C - B^*A^{-1}B \end{bmatrix}.$$

$A,\ B,\ B^*,\ B^*A^{-1}B$ are commuting normal and

$C - B^*A^{-1}B \geq 0.$ $\square$

Corollary. Let $\mathbf{S} = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0.$

$A = C$ (Toeplitz) or $B = B^*$ (Hankel)

$\Longrightarrow$ $\mathbf{S}$ separable.

(Proof.)

(Toeplitz) $\Longrightarrow$ we may assume $A = C = I$.

(Hankel) $\Longrightarrow$ we may assume $A = I,\ B = B^*$.

(28) ◯ (Gurwicz and Barnum)

With respect to the Hilbert-Schmidt norm $\|\cdot\|_2$

$$\mathbf{S} = \mathbf{S}^*, \quad \|\mathbf{I} - \mathbf{S}\|_2 \leq 1$$

$$\Longrightarrow \quad \mathbf{S} \geq 0 \text{ and separable.}$$

Theorem 4.

Every orthoprojection of co-rank $= 1$ is separable.

($\because$ $\|\mathbf{P}\|_2 = 1$ for every projection of rank 1.)

§ 3. Total Separablility

Recall the definition.

$0 \leq \mathbf{S} \in \mathbb{M}_2(\mathbb{M}_n)$ is said to be totally separable if

$$\exists \quad \boldsymbol{\xi}_j \in \mathbb{C}^2,\ x_j \in \mathbb{C}^n \quad \text{such that}$$

$$\langle \boldsymbol{\xi}_j \otimes x_j | \boldsymbol{\xi}_k \otimes x_k \rangle = 0 \quad \forall\ j \neq k.$$

and

$$\mathbf{S} \ = \ \sum_{j=1}^{N} (\boldsymbol{\xi}_j \otimes x_j)(\boldsymbol{\xi}_j \otimes x_j)^* = \sum_{j=1}^{N} (\boldsymbol{\xi}_j \boldsymbol{\xi}_j^*) \otimes (x_j x_j^*).$$

(30) ◯ $\mathbf{S} \geq 0$ is totally separable $\iff$
the eigenspace of each non-zero eigenvalue admits a **CONS** of product eigenvectors.

◯ $\mathbf{S} \geq 0$ is totally separable $\Longrightarrow$
$\mathrm{ran}(\mathbf{S})$ admits a **CONS** of product vectors.

◯ For $\mathbf{S} \geq 0$ with $\mathrm{rank}(\mathbf{S}) = 1$
$\mathbf{S}$ separable $\iff$ totally separable.

◯ For an orthoprojection $\mathbf{P}$
$\mathbf{P}$ separable $\not\Longrightarrow$ $\mathbf{P}^{\perp} := \mathbf{I} - \mathbf{P}$ separable
(by Theorem 3.)

(31) **(Notations)**

$\xi \in \mathbb{C}^2$ and $e^{i\theta}\xi$ are identified. $\quad \xi \equiv e^{i\theta}\xi.$

○ $\forall\, \xi \quad \exists^1 \;\; \xi^\perp$

such that $\xi\| = \|\xi^\perp\|, \; \langle \xi | \xi^\perp \rangle = 0.$

In fact, $\quad$ for $\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \quad \xi^\perp = \begin{bmatrix} \overline{\xi_2} \\ -\overline{\xi_1} \end{bmatrix}$

○ Since $\langle \xi \otimes x | \zeta \otimes y \rangle = \langle \xi | \zeta \rangle \cdot \langle x | y \rangle,$

$$\xi \otimes x \;\perp\; \zeta \otimes y \quad \Longleftrightarrow \quad \zeta \equiv \xi^\perp \quad \text{or} \quad x \perp y.$$

(32) **Theorem 4.**

Any **ONS** of product vectors in $\mathbb{C}^2 \otimes \mathbb{C}^n$ can be extended to a **CONS** of product vectors.

(33) Proof of Theorem 4 is based on the following two Lemmas.

Lemma 1. Let $P_j,\ Q_j\ (j = 1, 2, \ldots, m)$ be orthoprojections in $\mathbb{M}_n$ such that

$$P_j P_k = Q_j Q_k = P_j Q_k = 0 \quad \forall\, j \neq k.$$

$\sum_{j=1}^{m} P_j \neq I$ or $\sum_{j=1}^{m} Q_j \neq I$

$\Longrightarrow \quad \exists\, k, \quad \exists\, \|x\| = 1 \quad$ such that

$P_j x = 0 \quad \forall\ j \neq k \quad \text{and} \quad Q_j x = 0 \quad \forall\, j,$

or

$Q_j x = 0 \quad \forall\ j \neq k \quad \text{and} \quad P_j x = 0 \quad \forall\, j.$

Lemma 2.

$$\left\{\zeta_j \otimes w_j \ ; \ j = 1, 2, \ldots, N\right\} \quad (\text{with } N < 2n)$$

is an **ONS** of product vectors in $\mathbb{C}^2 \otimes \mathbb{C}^n$,

$$\Longrightarrow$$

$\exists$ unit vectors $\boldsymbol{\eta} \in \mathbb{C}^2, \ x \in \mathbb{C}^n$ such that

$$\langle \zeta_j \otimes w_j | \boldsymbol{\eta} \otimes x \rangle \ = \ 0 \quad \forall\, j = 1, 2, \ldots, N.$$

(35) **(Proof)** Rewrite ζ_j $(j = 1, 2, \ldots, N)$ as

$$\overbrace{\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_1}^{p_1}, \quad \overbrace{\boldsymbol{\xi}_1^{\perp}, \ldots, \boldsymbol{\xi}_1^{\perp}}^{q_1}$$

$$\cdots\cdots$$

$$\cdots\cdots$$

$$\overbrace{\boldsymbol{\xi}_m, \ldots, \boldsymbol{\xi}_m}^{p_m}, \quad \overbrace{\boldsymbol{\xi}_m^{\perp}, \ldots, \boldsymbol{\xi}_m^{\perp}}^{q_m}$$

where

$$\boldsymbol{\xi}_j \not\equiv \boldsymbol{\xi}_k \text{ and } \boldsymbol{\xi}_j \not\equiv \boldsymbol{\xi}_k^{\perp} \quad (j \neq k)$$

(and some q_j may be 0.)

(36)

Correspondingly the original **ONS** becomes

$$\overbrace{\boldsymbol{\xi}_1 \otimes x_{1,1}, \ldots, \boldsymbol{\xi}_1 \otimes x_{1,p_1}}^{p_1}, \quad \overbrace{\boldsymbol{\xi}_1^{\perp} \otimes y_{1,1}, \ldots, \boldsymbol{\xi}_1^{\perp} \otimes y_{1,q_1}}^{q_1},$$

$$\cdots\cdots$$

$$\cdots\cdots$$

$$\overbrace{\boldsymbol{\xi}_m \otimes x_{m,1}, \ldots, \boldsymbol{\xi}_m \otimes x_{m,p_m}}^{p_m}, \quad \overbrace{\boldsymbol{\xi}_m^{\perp} \otimes y_{m,1}, \ldots, \boldsymbol{\xi}_m^{\perp} \otimes y_{m,q_m}}^{q_m}.$$

(37) Here by definition

$$\langle \xi_j | \xi_k \rangle \neq 0, \quad \langle \xi_j^\perp | \xi_k^\perp \rangle \neq 0, \langle \xi_j | \xi_k^\perp \rangle \neq 0 \quad \forall j \neq k$$

so that

$$\langle x_{j.s} | x_{k,t} \rangle = \delta_{jk}\delta_{st}, \quad \langle y_{j.s} | y_{k,t} \rangle = \delta_{jk}\delta_{st} \quad \forall\ j, k; s, t$$

and $\qquad \langle x_{j,s} | y_{k,t} \rangle = 0 \quad \forall\ j \neq k, \forall\ s, t.$

Let $\quad P_j := \sum_{s=1}^{p_1} x_{j,s} x_{j,s}^*$ and $\quad Q_j := \sum_{j=1}^{q_j} y_{j,s} y_{j,s}^*.$
Then

$$P_j P_k = Q_j Q_k = P_j Q_k = 0 \quad j \neq k,$$

and
$I \neq \sum_{j=1}^{m} P_j$ or $I \neq \sum_{j=1}^{m} Q_j$ because $N < 2n$.

(38) By Lemma 1 $\exists\ k$, and unit vector $x \in \mathbb{C}^n$ such that

$$P_j x = 0 \quad \forall\ j \neq k \quad \text{and} \quad Q_j x = 0 \quad \forall\ j \quad \text{(say)}.$$

Let $\boldsymbol{\eta} := \boldsymbol{\xi}_k^\perp$. Then

$$\langle \boldsymbol{\xi}_j^\perp \otimes y_{j,t} | \boldsymbol{\eta} \otimes x \rangle = \langle \boldsymbol{\xi}_j^\perp | \boldsymbol{\eta} \rangle \langle y_{j,t} | x \rangle = 0 \quad \forall\ j, \forall\ t.$$

$$\langle \boldsymbol{\xi}_j \otimes x_{j,t} | \boldsymbol{\eta} \otimes x \rangle = \langle \boldsymbol{\xi}_j | \boldsymbol{\eta} \rangle \langle x_{j,t} | x \rangle = 0 \quad \forall j \neq k, \ \forall\ t.$$

$$\langle \boldsymbol{\xi}_k \otimes x_{k,t} | \boldsymbol{\eta} \otimes x \rangle = \langle \boldsymbol{\xi}_k | \boldsymbol{\xi}_k^\perp \rangle \cdot \langle x_{k,t} | x \rangle = 0 \quad \forall\ t.$$

Therefore $\langle \zeta_j \otimes w_j | \boldsymbol{\eta} \otimes x \rangle = 0 \quad \forall\ j.$ □

Apply Lemma 2 step by step to get Theorem 4.

(39) **Corollary 1.** $\mathbf{S} \geq 0$ totally separable

$\Longrightarrow$

$\mathrm{ran}(\mathbf{S})$, $\ker(\mathbf{S})$ admits **CONS** of product vectors.

Corollary 2. $\mathbf{I} \geq \mathbf{S} \geq 0$ totally separable
$f(t) \geq 0$ on $[0.1]$ $\Longrightarrow$
$f(\mathbf{S})$ totally separable.

In particular, $\mathbf{I} - \mathbf{S}$ is totally separable.

(40) **Theorem 5.**

$$\mathbf{S} = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \quad \text{totally separable}$$

$$\Longrightarrow \quad B \quad \text{normal},$$

but A, B, C are not necessarily commuting.

(41) **(Proof)** When

$$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} = \sum_{k=1}^{N} (\boldsymbol{\xi}_k \boldsymbol{\xi}_k^*) \otimes (x_k x_k^*) \text{ with } \boldsymbol{\xi}_j = \begin{bmatrix} \xi_{j,1} \\ \xi_{j,2} \end{bmatrix}$$

where $\forall\, j \neq k$

$$\langle \boldsymbol{\xi}_j | \boldsymbol{\xi}_k \rangle \cdot \langle x_j | x_k \rangle = (\overline{\xi_{j,1}} \xi_{k,1} + \overline{\xi_{j,2}} \xi_{k,2}) \langle x_j | x_k \rangle = 0$$

that is,
$\overline{\xi_{j,1}} \xi_{k,1} + \overline{\xi_{j,2}} \xi_{k,2} = 0$ or $\langle x_j | x_k \rangle = 0 \; \forall\, j \neq k$.

The commutativity of B and B^* follows from this orthogonality condition. □

§ 4. Back to the dual cone $\mathfrak{P}_-$

Recall that for the cone $\mathfrak{P}_-$, dual to $\mathfrak{P}_+$

$$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathfrak{P}_- \iff$$

$$\left\langle \begin{bmatrix} \xi x \\ \eta x \end{bmatrix} \middle| \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \begin{bmatrix} \xi x \\ \eta x \end{bmatrix} \right\rangle \geq 0 \ \forall\, \xi, \eta \in \mathbb{C},\ x \in \mathbb{C}^n$$

$$\iff$$

$$A, C \geq 0,\ \langle x|Ax\rangle \cdot \langle x|Cx\rangle \geq |\langle x|Bx\rangle|^2 \quad \forall\, x \in \mathbb{C}^n.$$

(43) Let $\mathbf{S} = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathfrak{P}_-$.

○ A, B, B^*, C commuting $\implies$ $\mathbf{S}$ separable.

○ $A = C$ (Toeplitz) $\not\Longrightarrow$ $\mathbf{S} \in \mathfrak{P}_0$.

When $A = C = I$,

$$\mathbf{S} = \begin{bmatrix} I & B \\ B^* & I \end{bmatrix} \in \mathfrak{P}_- \iff$$

$$\max\{|\langle x|Bx\rangle|;\ \|x\| = 1\} \leq 1,$$

B : numerical contraction.

(44) Let $\mathbf{S} = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathfrak{P}_-$.

○ $B = B^*$ (Hankel) $\not\Longrightarrow$ $\mathbf{S} \in \mathfrak{P}_0$.

When $A = I$ & $B = B^*$

$$\begin{bmatrix} I & B \\ B & C \end{bmatrix} \in \mathfrak{P}_- \iff C = X^*X,\ B = \mathrm{Re}(X) \ \exists\, X.$$

○ $A = C$ & $B = B^*$ (Toepliz & Hankel)

$$\Longrightarrow \quad A \geq B \geq -A \quad \Longrightarrow \mathbf{S} \in \mathfrak{P}_0,$$

hence separable.

(45) **Theorem 6.** Let $\mathbf{S} = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in \mathfrak{P}_-$.

(i) $\mathbf{S}$ is Toeplitz , that is, $A = C$

$\Longleftrightarrow$

$$B = 2(A - D)^{1/2} W D^{1/2} \quad \exists\ A \geq D \geq 0,\ \|W\| \leq 1.$$

(ii) $\mathbf{S}$ is Hankel , that is, $B = B^*$

$\Longleftrightarrow$

$$B = \tfrac{1}{2}\{A^{1/2} W C^{1/2} + C^{1/2} W^* A^{1/2}\} \quad \exists\ \|W\| \leq 1.$$

(45) **(Proof.)** (i) When $A = C$, $\mathbf{S} \in \mathfrak{P}_-$ $\Longleftrightarrow$

$$F(e^{it}) := 2A + e^{it}B + e^{-it}B^* \geq 0 \quad \forall\ t \in \mathbb{R}$$

$\Longleftrightarrow$ (Fejer-Riesz Theorem) $\exists\ A_1,\ B_1$

$$F(e^{it}) := (A_1 + e^{it}B_1)^*(A_1 + e^{it}B_1) \quad \forall\ t \in \mathbb{R}$$

$\Longleftrightarrow$ $2A = A_1^*A_1 + B_1^*B_1$ and $B = A_1^*B_1$.

(ii) When $B = B^*$, $\mathbf{S} \in \mathfrak{P}_-$ $\Longleftrightarrow$

$$F(t) := A + 2tB + t^2C \geq 0 \quad \forall\ t \in \mathbb{R}$$

$\Longleftrightarrow$ (Fejer-Riesz Theorem) $\exists\ A_1,\ C_1$

$$F(t) = (A_1 + tC_1)^*(A_1 + tC_1) \quad \forall\ t \in \mathbb{R}$$

$\Longleftrightarrow$ $A = A_1^*A_1$, $C = C_1^*C_1$, $2B = A_1^*C_1 + C_1^*A_1$.

Thank you for your attention !