X-parts of multi-qubit states and witnesses

Kyung Hoon Han
(joint work with Seung-Hyeok Kye)

University of Suwon

2016. 2.18

0. Plan

$$\begin{array}{ccccc}
\text{fully separable} & \Longrightarrow & \text{fully bi-separable} & \Longrightarrow & \text{bi-separable} \\
 & & \Downarrow & & \Downarrow \\
 & & \text{PPT} & \Longrightarrow & \text{PPT mixture}
\end{array}$$

The vertical arrows in the above diagram become equivalences for X-shaped multi-qubit states.

Necessary conditions will be described by X-parts

We have the following dual diagram:

$$\begin{array}{ccccc} \text{fully separable}^d & \Longleftarrow & \text{fully bi-separable}^d & \Longleftarrow & \text{bi-separable}^d \\ & & \Uparrow & & \Uparrow \\ & & \text{PPT}^d & \Longleftarrow & \text{PPT mixture}^d \end{array}$$

The vertical arrows are again equivalences for X-shaped multi-qubit witnesses

Necessary conditions will be described by X-parts

1. Full Separability

In this talk, every state is assumed to be unnormalized.

A multi-partite state $\varrho \in \bigotimes_{i=1}^{n} M_{d_i}$ is said to be *fully separable* if it is the sum of

$$A_1 \otimes A_2 \otimes \cdots \otimes A_n, \qquad A_i \in M_{d_i}^{+}.$$

For a given subset $S \subset \{1, 2, \cdots, n\}$, the partial transpose $T(S)$ on $\bigotimes_{i=1}^{n} M_{d_i}$ is the linear map satisfying

$$(a_1 \otimes a_2 \otimes \cdots \otimes a_n)^{T(S)} := b_1 \otimes b_2 \otimes \cdots \otimes b_n, \quad \text{with } b_i = \begin{cases} a_i^{\mathrm{t}}, & i \in S, \\ a_i, & i \notin S. \end{cases}$$

1. For a given bi-partition $S \sqcup T$ of the set $\{1, 2, \ldots, n\}$, a multi-partite state $\varrho \in \bigotimes_{i=1}^{n} M_{d_i}$ may be considered as in the tensor product $(\bigotimes_{i \in S} M_{d_i}) \otimes (\bigotimes_{i \in T} M_{d_i})$ of two matrix algebras, and is said to be *S-T PPT* if it is PPT;
2. A multi-partite state $\varrho \in \bigotimes_{i=1}^{n} M_{d_i}$ is called *PPT* if it is S-T PPT for all bi-partitions $S \sqcup T = \{1, 2, \cdots, n\}$.

Equivalently, ϱ is *S-T PPT* if $\varrho^{T(S)}$ is positive, and PPT if $\varrho^{T(S)}$ is positive for every $S \subset \{1, 2, \cdots, n\}$.
It is obvious that every fully separable state is PPT.

A multi-qubit state $\varrho \in \bigotimes_{i=1}^{n} M_{d_i}$ $(d_i = 2)$ is said to be X-shaped if it has of the form

$$\varrho = \begin{pmatrix} a_1 & & & & & & & z_1 \\ & a_2 & & & & & z_2 & \\ & & \ddots & & & \iddots & & \\ & & & a_{2^{n-1}} & z_{2^{n-1}} & & & \\ & & & \bar{z}_{2^{n-1}} & b_{2^{n-1}} & & & \\ & & \iddots & & & \ddots & & \\ & \bar{z}_2 & & & & & b_2 & \\ \bar{z}_1 & & & & & & & b_1 \end{pmatrix}.$$

We denote the above by $\varrho = X(a, b, z)$ briefly and the argument of z_i by θ_i.
It is easy to check that $\sqrt{a_i b_i} \geqslant |z_i|$ and each partial transpose fixes diagonal entries and permutes anti-diagonal entries.

Theorem

Let ϱ be a three qubit PPT state whose X*-part is given by* $X(a, b, z)$. *Suppose that* $X(a, b, z)$ *is not diagonal and its rank is less than or equal to six. We have the following:*

(i) *there exist* $r > 0$ *and distinct* $i_1, i_2, i_3, i_4 \in \{1, 2, 3, 4\}$ *such that*

$$\sqrt{a_i b_i} = r = |z_i| \quad \text{and} \quad \sqrt{a_j b_j} \geqslant r \geqslant |z_j|, \qquad i = i_1, i_2,\ j = i_3, i_4;$$

(ii) *if ϱ is fully separable, then we have*

$$a_1 b_2 b_3 a_4 \geqslant r^4, \quad b_1 a_2 a_3 b_4 \geqslant r^4 \quad \text{and} \quad |z_{i_3}| = |z_{i_4}|, \quad \theta_1 + \theta_4 = \theta_2 + \theta_3;$$

(iii) *the converse of* (ii) *holds when ϱ is* X*-shaped.*

Example

$$\varrho = \begin{pmatrix} 1 & * & * & * & * & * & * & 1 \\ * & 1 & * & * & * & * & 1 & * \\ * & * & 1 & * & * & -1 & * & * \\ * & * & * & 1 & 1 & * & * & * \\ * & * & * & 1 & 1 & * & * & * \\ * & * & -1 & * & * & 1 & * & * \\ * & 1 & * & * & * & * & 1 & * \\ 1 & * & * & * & * & * & * & 1 \end{pmatrix}$$

is not fully separable because it violates angle condition

$$(\theta_1 + \theta_4 = 0 + 0 \neq 0 + \pi = \theta_2 + \theta_3).$$

Example

$$\varrho = \begin{pmatrix} 1 & * & * & * & * & * & * & 1 \\ * & 1 & * & * & * & * & 1 & * \\ * & * & 1 & * & * & 1/2 & * & * \\ * & * & * & 1 & 1/3 & * & * & * \\ * & * & * & 1/3 & 1 & * & * & * \\ * & * & 1/2 & * & * & 1 & * & * \\ * & 1 & * & * & * & * & 1 & * \\ 1 & * & * & * & * & * & * & 1 \end{pmatrix}$$

is not fully separable because it violates modulus condition

$$(|z_3| = 1/2 \neq 1/3 = |z_4|).$$

Example

$$\varrho = \begin{pmatrix} 1 & * & * & * & * & * & * & 1 \\ * & 1 & * & * & * & * & 1 & * \\ * & * & 2\lambda & * & * & 1 & * & * \\ * & * & * & 2 & 1 & * & * & * \\ * & * & * & 1 & 1 & * & * & * \\ * & * & 1 & * & * & 1/\lambda & * & * \\ * & 1 & * & * & * & * & 1 & * \\ 1 & * & * & * & * & * & * & 1 \end{pmatrix}$$

If $\lambda > 2$, then ϱ is not fully separable because it violates diagonal condition

$$(a_1 b_2 b_3 a_4 = 2/\lambda \ngeq 1^4).$$

If $\lambda < 1/2$, then ϱ is not fully separable because it violates diagonal condition

$$(b_1 a_2 a_3 b_4 = 2\lambda \ngeq 1^4).$$

Example

If $1/2 \leqslant \lambda \leqslant 2$, then ϱ satisfies all diagonal, modulus, angle conditions, hence undecidable. If ϱ is X-shaped, that is,

$$\varrho = \begin{pmatrix} 1 &&&&&&& 1 \\ & 1 &&&&& 1 & \\ && 2\lambda &&& 1 && \\ &&& 2 & 1 &&& \\ &&& 1 & 1 &&& \\ && 1 &&& 1/\lambda && \\ & 1 &&&&& 1 & \\ 1 &&&&&&& 1 \end{pmatrix}$$

then, it is fully separable.

The diagonal, modulus, angle conditions are necessary when the rank of X-part $\leqslant 6$. We give the necessary condition for general rank.

Theorem

Let ϱ be a three qubit state whose X-part is given by $X(a, b, z)$. If ϱ is fully separable, then the inequality

$$4\sqrt{a_i b_i} \geqslant \sqrt{2}|z_1 e^{\mathrm{i}\theta} + z_2| + \sqrt{2}|z_3 e^{\mathrm{i}\theta} - z_4|$$

holds for each $i = 1, 2, 3, 4$ and $\theta \in \mathbb{R}$.

Corollary

Let ϱ be a three qubit state whose X-part is given by $X(a, b, z)$ with $\min \sqrt{a_i b_i} = (1 + \varepsilon)r$, $0 \leqslant \varepsilon < (-1 + \sqrt{2})/2$ and $z_i = re^{\mathrm{i}\theta_i}$. If ϱ is fully separable, then we have

$$|(\theta_1 + \theta_4) - (\theta_2 + \theta_3)| \leqslant \arcsin(4\varepsilon(1 + \varepsilon)).$$

When $\varepsilon = 0$, the above reduces to the angle condition $\theta_1 + \theta_4 = \theta_2 + \theta_3$.

2. Dual of Full Separability

Theorem

Suppose that the X-part of a three qubit Hermitian matrix W is $X(s,t,u)$ and $\langle \varrho, W\rangle \geqslant 0$ for all three qubit fully separable states ϱ. Then, the inequality

$$\sqrt{(s_1+t_4|\alpha|^2)(s_4+t_1|\alpha|^2)}+\sqrt{(s_2+t_3|\alpha|^2)(s_3+t_2|\alpha|^2)} \geqslant |u_1\bar{\alpha}+\bar{u}_4\alpha| + |u_2\bar{\alpha}+\bar{u}_3\alpha|$$

holds for each $\alpha \in \mathbb{C}$. The converse holds when W is X-shaped.

Example

Let W be an X-shaped Hermitian matrix $X(s,t,u)$ with $u_i = \sqrt{2}e^{\mathrm{i}\theta_i}$ and $\theta_1 + \theta_4 = \theta_2 + \theta_3 + \pi$. Then,

(i) if $s_{i_0} t_{i_0} \geqslant 16$ for some i_0, then $\langle \varrho, W \rangle \geqslant 0$ for all fully separable ϱ;

(ii) if $\sqrt[4]{s_1 s_4 t_1 t_4} + \sqrt[4]{s_2 s_3 t_2 t_3} \geqslant 2$, then $\langle \varrho, W \rangle \geqslant 0$ for all fully separable ϱ.

$$
\begin{array}{ccccc}
\text{fully separable} & \Longrightarrow & \text{fully bi-separable} & \Longrightarrow & \text{bi-separable} \\
 & & \Downarrow & & \Downarrow \\
 & & \text{PPT} & \Longrightarrow & \text{PPT mixture}
\end{array}
$$

$$
\begin{array}{ccccc}
\text{fully separable}^{d} & \Longleftarrow & \text{fully bi-separable}^{d} & \Longleftarrow & \text{bi-separable}^{d} \\
 & & \Uparrow & & \Uparrow \\
 & & \text{PPT}^{d} & \Longleftarrow & \text{PPT mixture}^{d}
\end{array}
$$

3. Full Bi-Separability and PPT

1. For a given bi-partition $S \sqcup T$ of the set $\{1, 2, \ldots, n\}$, a multi-partite state $\varrho \in \bigotimes_{i=1}^{n} M_{d_i}$ may be considered as in the tensor product $(\bigotimes_{i\in S} M_{d_i}) \otimes (\bigotimes_{i\in T} M_{d_i})$ of two matrix algebras, and is said to be *S-T bi-separable* if it is separable.
2. A multi-partite state $\varrho \in \bigotimes_{i=1}^{n} M_{d_i}$ is called *fully bi-separable* if it is in the intersection of S-T bi-separable matrices through all bi-partitions $S \sqcup T = \{1, 2, \cdots, n\}$.

Obviously,

$$\text{fully separable} \Rightarrow \text{fully bi-separable} \Rightarrow \text{PPT}$$

Theorem

Suppose that an X-shaped n-qubit state $\varrho = X(a, b, z)$ and a bi-partition $\{1, \cdots, n\} = S \sqcup T$ are given. If $1 \notin S$ then the following are equivalent:

(i) ϱ *is* S-T *bi-separable;*

(ii) ϱ *is* S-T *PPT.*

Since we may interchange the roles of S and T in the bi-partition $\{1, \cdots, n\} = S \sqcup T$, the assumption $1 \notin S$ is actually superfluous.

Theorem

Suppose that the X-part of a multi-qubit state ϱ is given by $X(a, b, z)$. If ϱ is PPT, then the inequality

$$\sqrt{a_i b_i} \geqslant |z_j|$$

holds for every choice of $i, j = 1, 2, \ldots, 2^{n-1}$.

Theorem

Let $\varrho = X(a, b, z)$ be an X-shaped n-qubit state. Then the following are equivalent:

(i) ϱ *is fully bi-separable;*

(ii) ϱ *is PPT;*

(iii) *the inequality*

$$\sqrt{a_i b_i} \geqslant |z_j|$$

holds for every choice of $i, j = 1, 2, \ldots, 2^{n-1}$.

Examples of fully bi-separable states which is not fully separable were first discovered by T, Vértesi and N. Brunner

(*Quantum Nonlocality Does Not Imply Entanglement Distillability*, Phys. Rev. Lett. **108**, (2012)).

Combining the previous results, we get a simple and large family of fully bi-separable states which is not fully separable.

Theorem

Let ϱ be a three qubit X-shaped state $X(a, b, z)$ with rank $\leqslant 6$. The following are equivalent:

(i) *ϱ is PPTES;*

(ii) *ϱ is fully bi-separable but not fully separable;*

(iii) *there exist $r > 0$ and distinct $i_1, i_2, i_3, i_4 \in \{1, 2, 3, 4\}$ such that*

$$\sqrt{a_i b_i} = r = |z_i|, \quad \text{and} \quad \sqrt{a_j b_j} \geqslant r \geqslant |z_j|, \qquad i = i_1, i_2, \; j = i_3, i_4$$

and ϱ violates one of conditions

$$a_1 b_2 b_3 a_4 \geqslant r^4, \quad b_1 a_2 a_3 b_4 \geqslant r^4 \quad \text{and} \quad |z_{i_3}| = |z_{i_4}|, \quad \theta_1 + \theta_4 = \theta_2 + \theta_3.$$

4. Dual of PPT

Definition

A Hermitian matrix D is said to be *decomposable* if it is in the convex hull of the convex cones

$$\mathbb{T}^S := \{A \in \bigotimes_{i=1}^n M_{d_i} : A^{T(S)} \text{ is positive}\}$$

through subsets S of $\{1, \cdots, n\}$.

The decomposability is dual to PPT.

Theorem

Suppose that the X-part *of W is given by $X(s,t,u)$. If W is decomposable, then the inequality*

$$\sum_{i=1}^{2^{n-1}} \sqrt{s_i t_i} \geqslant \sum_{i=1}^{2^{n-1}} |u_i|$$

holds.

Theorem

For an X-shaped n-qubit Hermitian $W = X(s, t, u)$ *with nonnegative diagonals, the following are equivalent:*

(i) W *is decomposable;*

(ii) $\langle \varrho, W \rangle \geqslant 0$ *for every PPT state* ϱ*;*

(iii) $\langle \varrho, W \rangle \geqslant 0$ *for every fully bi-separable state* ϱ*;*

(iv) *the inequality*

$$\sum_{i=1}^{2^{n-1}} \sqrt{s_i t_i} \geqslant \sum_{i=1}^{2^{n-1}} |u_i|$$

holds.

Example

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 3 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

is decomposable where $\cdot$ denotes zero.

$$
\begin{array}{ccccc}
\text{fully separable} & \Longrightarrow & \text{fully bi-separable} & \Longrightarrow & \text{bi-separable} \\
 & & \Downarrow & & \Downarrow \\
 & & \text{PPT} & \Longrightarrow & \text{PPT mixture}
\end{array}
$$

$$
\begin{array}{ccccc}
\text{fully separable}^{d} & \Longleftarrow & \text{fully bi-separable}^{d} & \Longleftarrow & \text{bi-separable}^{d} \\
 & & \Uparrow & & \Uparrow \\
 & & \text{PPT}^{d} & \Longleftarrow & \text{PPT mixture}^{d}
\end{array}
$$

5. Bi-Separability and PPT Mixture

1. A multi-partite state ϱ is called *bi-separable* if it is in the convex hull of S-T bi-separable states through all bi-partitions $S \sqcup T = \{1, \cdots, n\}$.
2. A multi-partite state ϱ is called *a PPT mixture* if it is in the convex hull of S-T PPT states through all bi-partitions $S \sqcup T = \{1, \cdots, n\}$.
3. A state is said to be *genuinely entangled* if it is not bi-separable.

It is obvious that every bi-separable state is a PPT mixture.

By Gao and Hong (*Separability criteria for several classes of n-partite quantum states*, Eur. Phys. J. D **61** (2011)), and Gühne and Seevinck (*Separability criteria for genuine multiparticle entanglement*, New J. Phys. **12** (2010)), it was shown that if an arbitrary multi-qubit state ϱ whose diagonal and anti-diagonal parts are given by $X(a, b, z)$ is bi-separable then the inequality

$$\sum_{j \neq i} \sqrt{a_j b_j} \geqslant |z_i|$$

holds for each $1 \leqslant i \leqslant 2^{n-1}$.

By Rafsanjani, Huber, Broadbent and J. H. Eberly (*Genuinely multipartite concurrence of N-qubit X matrices*, Phys. Rev. A **86** (2012)), it was shown that this inequality is equivalent to bi-separability for X-shaped states.

They can be improved.

Theorem

Let ϱ be a multi-qubit state whose diagonal and anti-diagonal parts are given by $X(a, b, z)$. If ϱ is a PPT mixture, then the inequality

$$\sum_{j \neq i} \sqrt{a_j b_j} \geqslant |z_i|$$

holds for each $1 \leqslant i \leqslant 2^{n-1}$.

Theorem

For an X-shaped multi-qubit state $\varrho = X(a, b, z)$, the following are equivalent:

(i) *ϱ is bi-separable;*

(ii) *ϱ is a PPT mixture;*

(iii) *the inequality*

$$\sum_{j\neq i} \sqrt{a_j b_j} \geqslant |z_i|$$

holds for each $1 \leqslant i \leqslant 2^{n-1}$.

6. Genuine Entanglement Witnesses

We call a non-positive (non positive semi-definite) Hermitian matrix W in $\bigotimes_{i=1}^n M_{d_i}$ *genuine entanglement witness* if

$$\langle \varrho, W \rangle := \mathrm{Tr}\,(\varrho W^{\mathrm{t}}) \geqslant 0$$

for every bi-separable state ϱ.
Non-positivity condition of W guarantees the existence of a state ϱ with $\langle \varrho, W \rangle < 0$, and the above condition tells us that this ϱ must be genuinely entangled. By duality, any genuine entanglement is detected by a genuine entanglement witness.

Theorem

Suppose that the X*-part of W is given by $X(s,t,u)$. If $\langle \varrho, W\rangle \geqslant 0$ for every n qubit bi-separable state ϱ, then the inequality*

$$\sqrt{s_i t_i} + \sqrt{s_j t_j} \geqslant |u_i| + |u_j|$$

holds for every choice of $i, j = 1, 2, \ldots, 2^{n-1}$ with $i \neq j$.

Theorem

Suppose that $W = X(s, t, u)$ is an X-shaped multi-qubit Hermitian matrix with nonnegative diagonals. Then the following are equivalent:

(i) $\langle \varrho, W \rangle \geqslant 0$ *for every n qubit bi-separable state ϱ;*

(ii) $\langle \varrho, W \rangle \geqslant 0$ *for every n qubit PPT mixture ϱ;*

(iii) *the inequality*

$$\sqrt{s_i t_i} + \sqrt{s_j t_j} \geqslant |u_i| + |u_j|$$

holds for every choice of $i, j = 1, 2, \ldots, 2^{n-1}$ with $i \neq j$.

Corollary

Suppose that W is an X-shaped matrix. Then, W is a genuine entanglement witness if and only if the inequality

$$\sqrt{s_i t_i} + \sqrt{s_j t_j} \geqslant |u_i| + |u_j|$$

holds for every choice of $i, j = 1, 2, \ldots, 2^{n-1}$ with $i \neq j$ and the inequality

$$\sqrt{s_{i_0} t_{i_0}} < |u_{i_0}|$$

holds for some i_0

For a given genuine entanglement witness W, we consider the set G_W of all genuine entanglement ϱ which are detected by W in the sense of $\langle \varrho, W \rangle < 0$. We say that W is *optimal* if the set G_W is maximal.

We say that a vector $\xi \in \bigotimes_{1\leqslant i\leqslant n}\mathbb{C}^{d_i}$ is a *bi-product vector* if there is a partition $S \sqcup T = [\{1, \cdots, n\}$ such that ξ is a product vector as an element of $\left(\bigotimes_{i\in S}\mathbb{C}^{d_i}\right) \otimes \left(\bigotimes_{i\in T}\mathbb{C}^{d_i}\right)$. For a given genuine entanglement witness W, we denote by P_W the set of all bi-product vectors ξ such that

$$\langle\bar{\xi}|W|\bar{\xi}\rangle = \langle|\xi\rangle\langle\xi|, W\rangle = 0.$$

We say that W has the *spanning property* if the set P_W spans the whole space $\bigotimes_{i=1}^{n}\mathbb{C}^{d_i}$.
By Lewenstein, Kraus, Cirac and Horodecki (*Optimization of entanglement witness*, Phys. Rev. A **62** (2000)), the spanning property implies the optimality.

Theorem

Suppose that $W = X(s,t,u)$ is an X-shaped n-qubit genuine entanglement witness. Then the following are equivalent:

(i) *W is an optimal genuine entanglement witness;*

(ii) *W is a genuine entanglement witness with the spanning property;*

(iii) *there exists $i_0 \in \{1, \cdots, 2^{n-1}\}$ and positive number $r > 0$ with the properties:*

- $s_{i_0} = t_{i_0} = 0$ *and* $|u_{i_0}| = r$,
- $\sqrt{s_i t_i} = r$ *and* $u_i = 0$ *for* $i \neq i_0$.

Example

A typical example of three qubit optimal genuine entanglement witness is given by

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & e^{i\theta} \\ \cdot & s_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & s_3 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & s_4 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1/s_4 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1/s_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1/s_2 & \cdot \\ e^{-i\theta} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

where $\cdot$ denotes zero.

7. Summary

$$\begin{array}{ccccc} \text{fully separable} & \Longrightarrow & \text{fully bi-separable} & \Longrightarrow & \text{bi-separable} \\ & & \Downarrow & & \Downarrow \\ & & \text{PPT} & \Longrightarrow & \text{PPT mixture} \end{array}$$

The vertical arrows in the above diagram become equivalences for X-shaped multi-qubit states.

Necessary conditions by X-parts

1. fully separable
   - $a_1 b_2 b_3 a_4 \geqslant r^4, \quad b_1 a_2 a_3 b_4 \geqslant r^4, |z_{i_3}| = |z_{i_4}|, \theta_1 + \theta_4 = \theta_2 + \theta_3$ for three qubit states with rank $\leqslant 6$. Multi-index is used to generalize it to multi-qubit states with rank $\leqslant 2^n - 2$.
   - $4\sqrt{a_i b_i} \geqslant \sqrt{2}|z_1 e^{i\theta} + z_2| + \sqrt{2}|z_3 e^{i\theta} - z_4|$ for three qubit states. Multi-index is used to generalize it to multi-qubit states.
2. PPT : $\sqrt{s_i t_i} \geqslant |u_j|$;
3. PPT mixture : $\sum_{j \neq i} \sqrt{a_j b_j} \geqslant |z_i|$.

We have the following dual diagram:

$$\begin{array}{ccccc} \text{fully separable}^d & \Longleftarrow & \text{fully bi-separable}^d & \Longleftarrow & \text{bi-separable}^d \\ & & \Uparrow & & \Uparrow \\ & & \text{PPT}^d & \Longleftarrow & \text{PPT mixture}^d \end{array}$$

The vertical arrows are again equivalences for X-shaped multi-qubit witnesses

Necessary conditions by X-parts

1. PPT mixture^d : $\sqrt{s_i t_i} + \sqrt{s_j t_j} \geqslant |u_i| + |u_j|$;
2. PPT^d : $\sum_{i=1}^{2^{n-1}} \sqrt{s_i t_i} \geqslant \sum_{i=1}^{2^{n-1}} |u_i|$;
3. fully separable^d :

$$\sqrt{(a_1 + b_4|\alpha|^2)(a_4 + b_1|\alpha|^2)} + \sqrt{(a_2 + b_3|\alpha|^2)(a_3 + b_2|\alpha|^2)} \geqslant |z_1\bar{\alpha} + \bar{z}_4\alpha| + |z_2\bar{\alpha} + \bar{z}_3\alpha|.$$

This talk is based on

- K. H. Han and S.-H, Kye, *Various notions of positivity for bi-linear maps and applications to tri-partite entanglement*, J. Math. Phys. **57** (2016), 015205.
- K. H. Han and S.-H, Kye, *Construction of multi-qubit optimal genuine entanglement witnesses*, arXiv:1510.03620.
- K. H. Han and S.-H, Kye, *in preparation* (the section on the full separability)