X-parts of multi-qubit states and witnesses

Kyung Hoon Han (joint work with Seung-Hyeok Kye)

University of Suwon

2016. 2.18

Kyung Hoon Han (joint work with Seung-H X-parts of multi-qubit states and witnesses

$$\begin{array}{ccc} \mbox{fully separable} & \Longrightarrow & \mbox{fully bi-separable} & \Longrightarrow & \mbox{bi-separable} \\ & & \Downarrow & & \\ & & & \downarrow \\ & & & PPT & \implies & PPT \mbox{ mixture} \end{array}$$

The vertical arrows in the above diagram become equivalences for X-shaped multi-qubit states.

Necessary conditions will be described by X-parts

We have the following dual diagram:

fully separable^d
$$\leftarrow$$
 fully bi-separable^d \leftarrow bi-separable^d
 \uparrow \uparrow
PPT^d \leftarrow PPT mixture^d

The vertical arrows are again equivalences for X-shaped multi-qubit witnesses

Necessary conditions will be described by X-parts

1. Full Separability

In this talk, every state is assumed to be unnormalized.

A multi-partite state $\varrho \in \bigotimes_{i=1}^{n} M_{d_i}$ is said to be *fully separable* if it is the sum of

$$A_1 \otimes A_2 \otimes \cdots \otimes A_n, \qquad A_i \in M_{d_i}^+.$$

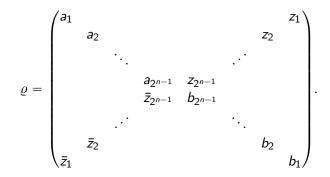
For a given subset $S \subset \{1, 2, \dots, n\}$, the partial transpose T(S) on $\bigotimes_{i=1}^{n} M_{d_i}$ is the linear map satisfying

$$(a_1 \otimes a_2 \otimes \cdots \otimes a_n)^{T(S)} := b_1 \otimes b_2 \otimes \cdots \otimes b_n, \text{ with } b_i = \begin{cases} a_i^{\mathrm{t}}, & i \in S, \\ a_i, & i \notin S. \end{cases}$$

1

- For a given bi-partition S □ T of the set {1,2,...,n}, a multi-partite state *ρ* ∈ ⊗ⁿ_{i=1} M_{di} may be considered as in the tensor product (⊗_{i∈S} M_{di}) ⊗ (⊗_{i∈T} M_{di}) of two matrix algebras, and is said to be S-T PPT if it is PPT;
- ② A multi-partite state $\rho \in \bigotimes_{i=1}^{n} M_{d_i}$ is called *PPT* if it is *S*-*T* PPT for all bi-partitions *S* ⊔ *T* = {1, 2, · · · , n}.

Equivalently, ϱ is *S*-*T PPT* if $\varrho^{T(S)}$ is positive, and PPT if $\varrho^{T(S)}$ is positive for every $S \subset \{1, 2, \dots, n\}$. It is obvious that every fully separable state is PPT. A multi-qubit state $\rho \in \bigotimes_{i=1}^{n} M_{d_i}$ $(d_i = 2)$ is said to be X-shaped if it has of the form



We denote the above by $\rho = X(a, b, z)$ briefly and the argument of z_i by θ_i .

It is easy to check that $\sqrt{a_i b_i} \ge |z_i|$ and each partial transpose fixes diagonal entries and permutes anti-diagonal entries.

Let ϱ be a three qubit PPT state whose X-part is given by X(a, b, z). Suppose that X(a, b, z) is not diagonal and its rank is less than or equal to six. We have the following:

(i) there exist r > 0 and distinct $i_1, i_2, i_3, i_4 \in \{1, 2, 3, 4\}$ such that

$$\sqrt{a_i b_i} = r = |z_i|$$
 and $\sqrt{a_j b_j} \ge r \ge |z_j|$, $i = i_1, i_2, j = i_3, i_4$;

(ii) if ϱ is fully separable, then we have

 $a_1b_2b_3a_4 \ge r^4$, $b_1a_2a_3b_4 \ge r^4$ and $|z_{i_3}| = |z_{i_4}|$, $\theta_1 + \theta_4 = \theta_2 + \theta_3$;

(iii) the converse of (ii) holds when ϱ is X-shaped.

$$\varrho = \begin{pmatrix} 1 & * & * & * & * & * & * & 1 \\ * & 1 & * & * & * & * & 1 & * \\ * & * & 1 & * & * & -1 & * & * \\ * & * & * & 1 & 1 & * & * & * \\ * & * & * & 1 & 1 & * & * & * \\ * & * & * & 1 & 1 & * & * & * \\ * & * & * & * & 1 & 1 & * & * \\ * & 1 & * & * & * & * & * & 1 & * \\ 1 & * & * & * & * & * & * & * & 1 \end{pmatrix}$$

is not fully separable because it violates angle condition

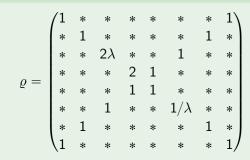
$$(\theta_1 + \theta_4 = 0 + 0 \neq 0 + \pi = \theta_2 + \theta_3).$$

$$\varrho = \begin{pmatrix} 1 & * & * & * & * & * & * & 1 \\ * & 1 & * & * & * & * & 1 & * \\ * & * & 1 & * & * & 1/2 & * & * \\ * & * & 1 & 1/3 & * & * & * \\ * & * & 1/3 & 1 & * & * & * \\ * & * & 1/2 & * & 1 & * & * \\ * & 1 & * & * & * & 1 & * \\ 1 & * & * & * & * & * & 1 \end{pmatrix}$$

is not fully separable because it violates modulus condition

$$(|z_3| = 1/2 \neq 1/3 = |z_4|).$$

Kyung Hoon Han (joint work with Seung-H X-parts of multi-qubit states and witnesses



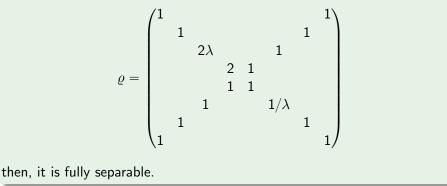
If $\lambda > 2$, then ϱ is not fully separable because it violates diagonal condition

$$(a_1b_2b_3a_4=2/\lambda \geqslant 1^4).$$

If $\lambda < 1/2,$ then ϱ is not fully separable because it violates diagonal condition

$$(b_1a_2a_3b_4=2\lambda \geqslant 1^4).$$

If $1/2 \le \lambda \le 2$, then ρ satisfies all diagonal, modulus, angle conditions, hence undecidable. If ρ is X-shaped, that is,



The diagonal, modulus, angle conditions are necessary when the rank of X-part ≤ 6 . We give the necessary condition for general rank.

Theorem

Let ρ be a three qubit state whose X-part is given by X(a, b, z). If ρ is fully separable, then the inequality

$$4\sqrt{a_ib_i} \ge \sqrt{2}|z_1e^{\mathrm{i}\theta} + z_2| + \sqrt{2}|z_3e^{\mathrm{i}\theta} - z_4|$$

holds for each i = 1, 2, 3, 4 and $\theta \in \mathbb{R}$.

Corollary

Let ϱ be a three qubit state whose X-part is given by X(a, b, z) with $\min \sqrt{a_i b_i} = (1 + \varepsilon)r$, $0 \le \varepsilon < (-1 + \sqrt{2})/2$ and $z_i = re^{i\theta_i}$. If ϱ is fully separable, then we have

$$|(\theta_1 + \theta_4) - (\theta_2 + \theta_3)| \leq \arcsin(4\varepsilon(1 + \varepsilon)).$$

When $\varepsilon = 0$, the above reduces to the angle condition $\theta_1 + \theta_4 = \theta_2 + \theta_3$.

2. Dual of Full Separability

Theorem

Suppose that the X-part of a three qubit Hermitian matrix W is X(s, t, u)and $\langle \varrho, W \rangle \ge 0$ for all three qubit fully separable states ϱ . Then, the inequality

$$\sqrt{(s_1 + t_4 |\alpha|^2)(s_4 + t_1 |\alpha|^2)} + \sqrt{(s_2 + t_3 |\alpha|^2)(s_3 + t_2 |\alpha|^2)} \ge |u_1 \bar{\alpha} + \bar{u}_4 \alpha| + |u_2 \bar{\alpha} + \bar{u}_3 \alpha|$$

holds for each $\alpha \in \mathbb{C}$. The converse holds when W is X-shaped.

Let W be an X-shaped Hermitian matrix X(s, t, u) with $u_i = \sqrt{2}e^{i\theta_i}$ and $\theta_1 + \theta_4 = \theta_2 + \theta_3 + \pi$. Then,

(i) if $s_{i_0}t_{i_0} \ge 16$ for some i_0 , then $\langle \varrho, W \rangle \ge 0$ for all fully separable ϱ ;

(ii) if $\sqrt[4]{s_1s_4t_1t_4} + \sqrt[4]{s_2s_3t_2t_3} \ge 2$, then $\langle \varrho, W \rangle \ge 0$ for all fully separable ϱ .

$$\begin{array}{cccc} \mbox{fully separable} & \Longrightarrow & \mbox{fully bi-separable} & \Longrightarrow & \mbox{bi-separable} & & \Downarrow \\ & & & & & \Downarrow \\ & & & PPT & \Longrightarrow & PPT & \mbox{mixture} \\ \end{array}$$

$$\begin{array}{cccc} \mbox{fully separable}^d & \longleftarrow & \mbox{fully bi-separable}^d & \longleftarrow & \mbox{bi-separable}^d \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ \end{array}$$

3. Full Bi-Separability and PPT

- For a given bi-partition S □ T of the set {1,2,...,n}, a multi-partite state *ρ* ∈ ⊗ⁿ_{i=1} M_{di} may be considered as in the tensor product (⊗_{i∈S} M_{di}) ⊗ (⊗_{i∈T} M_{di}) of two matrix algebras, and is said to be S-T bi-separable if it is separable.
- A multi-partite state *ρ* ∈ ⊗ⁿ_{i=1} *M*_{di} is called *fully bi-separable* if it is in the intersection of *S*-*T* bi-separable matrices through all bi-partitions *S* ⊔ *T* = {1, 2, · · · , *n*}.

Obviously,

fully separable \Rightarrow fully bi-separable \Rightarrow PPT

Suppose that an X-shaped n-qubit state $\varrho = X(a, b, z)$ and a bi-partition $\{1, \dots, n\} = S \sqcup T$ are given. If $1 \notin S$ then the following are equivalent: (i) ϱ is S-T bi-separable; (ii) ϱ is S-T PPT.

Since we may interchange the roles of S and T in the bi-partition $\{1, \dots, n\} = S \sqcup T$, the assumption $1 \notin S$ is actually superfluous.

Suppose that the X-part of a multi-qubit state ρ is given by X(a, b, z). If ρ is PPT, then the inequality

$$\sqrt{a_i b_i} \geqslant |z_j|$$

holds for every choice of $i, j = 1, 2, \ldots, 2^{n-1}$.

Let $\rho = X(a, b, z)$ be an X-shaped n-qubit state. Then the following are equivalent:

- (i) ϱ is fully bi-separable;
- (ii) *ρ* is PPT;
- (iii) the inequality

 $\sqrt{a_i b_i} \geqslant |z_j|$

holds for every choice of $i, j = 1, 2, ..., 2^{n-1}$.

Examples of fully bi-separable states which is not fully separable were first discovered by T, Vértesi and N. Brunner

(Quantum Nonlocality Does Not Imply Entanglement Distillability, Phys. Rev. Lett. 108, (2012)).

Combining the previous results, we get a simple and large family of fully bi-separable states which is not fully separable.

Let ρ be a three qubit X-shaped state X(a, b, z) with rank ≤ 6 . The following are equivalent:

- (i) *ρ* is PPTES;
- (ii) ρ is fully bi-separable but not fully separable;
- (iii) there exist r>0 and distinct $i_1,i_2,i_3,i_4\in\{1,2,3,4\}$ such that

$$\sqrt{a_i b_i} = r = |z_i|, \text{ and } \sqrt{a_j b_j} \ge r \ge |z_j|, \quad i = i_1, i_2, \ j = i_3, i_4$$

and ϱ violates one of conditions

$$a_1b_2b_3a_4 \geqslant r^4, \quad b_1a_2a_3b_4 \geqslant r^4 \quad \text{and} \quad |z_{i_3}| = |z_{i_4}|, \quad \theta_1 + \theta_4 = \theta_2 + \theta_3.$$

4. Dual of PPT

Definition

A Hermitian matrix D is said to be *decomposable* if it is in the convex hull of the convex cones

$$\mathbb{T}^{S} := \{A \in \bigotimes_{i=1}^{n} M_{d_{i}} : A^{T(S)} \text{ is positive}\}\$$

through subsets S of $\{1, \cdots, n\}$.

The decomposability is dual to PPT.

Suppose that the X-part of W is given by X(s, t, u). If W is decomposable, then the inequality

$$\sum_{i=1}^{2^{n-1}} \sqrt{s_i t_i} \ge \sum_{i=1}^{2^{n-1}} |u_i|$$

holds.

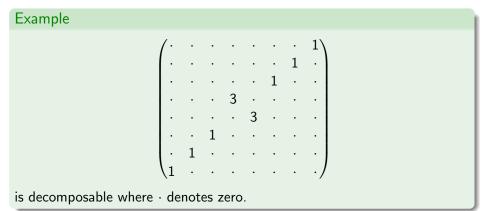
For an X-shaped n-qubit Hermitian W = X(s, t, u) with nonnegative diagonals, the following are equivalent:

- (i) W is decomposable;
- (ii) $\langle \varrho, W \rangle \ge 0$ for every PPT state ϱ ;
- (iii) $\langle \varrho, W \rangle \ge 0$ for every fully bi-separable state ϱ ;

(iv) the inequality

$$\sum_{i=1}^{2^{n-1}} \sqrt{s_i t_i} \ge \sum_{i=1}^{2^{n-1}} |u_i|$$

holds.



$$\begin{array}{rccc} \text{fully separable} & \Longrightarrow & \text{fully bi-separable} & \Longrightarrow & \text{bi-separable} \\ & & & \downarrow \\ & & & PPT & \implies & PPT & \text{mixture} \end{array}$$
$$\begin{array}{rccc} \text{fully separable}^d & \longleftarrow & \text{bi-separable}^d \\ & & & \uparrow \\ & & & PPT^d & \longleftarrow & PPT & \text{mixture}^d \end{array}$$

Kyung Hoon Han (joint work with Seung-H X-parts of multi-qubit states and witnesses

5. Bi-Separability and PPT Mixture

- A multi-partite state *ρ* is called *bi-separable* if it is in the convex hull of S-T bi-separable states through all bi-partitions S ⊔ T = {1, · · · , n}.
- A multi-partite state *ρ* is called a PPT mixture if it is in the convex hull of S-T PPT states through all bi-partitions S □ T = {1, · · · , n}.
- A state is said to be *genuinely entangled* if it is not bi-separable.
- It is obvious that every bi-separable state is a PPT mixture.

By Gao and Hong (Separability criteria for several classes of n-partite quantum states, Eur. Phys. J. D 61 (2011)), and Gühne and Seevinck (Separability criteria for genuine multiparticle entanglement, New J. Phys. 12 (2010)), it was shown that if an arbitrary multi-qubit state ϱ whose diagonal and anti-diagonal parts are given by X(a, b, z) is bi-separable then the inequality

$$\sum_{j\neq i} \sqrt{\mathsf{a}_j \mathsf{b}_j} \geqslant |\mathsf{z}_i|$$

holds for each $1 \leq i \leq 2^{n-1}$.

By Rafsanjani, Huber, Broadbent and J. H. Eberly (*Genuinely multipartite concurrence of* N-qubit X matrices, Phys. Rev. A **86** (2012)), it was shown that this inequality is equivalent to bi-separability for X-shaped states.

They can be improved.

Let ρ be a multi-qubit state whose diagonal and anti-diagonal parts are given by X(a, b, z). If ρ is a PPT mixture, then the inequality

$$\sum_{j\neq i} \sqrt{\mathsf{a}_j \mathsf{b}_j} \geqslant |\mathsf{z}_i|$$

holds for each $1 \leq i \leq 2^{n-1}$.

For an X-shaped multi-qubit state $\rho = X(a, b, z)$, the following are equivalent:

- (i) ϱ is bi-separable;
- (ii) *ρ* is a PPT mixture;

(iii) the inequality

$$\sum_{j\neq i}\sqrt{a_jb_j} \geqslant |z_i|$$

holds for each $1 \leq i \leq 2^{n-1}$.

6. Genuine Entanglement Witnesses

We call a non-positive (non positive semi-definite) Hermitian matrix W in $\bigotimes_{i=1}^{n} M_{d_i}$ genuine entanglement witness if

 $\langle \varrho, W \rangle := \operatorname{Tr}(\varrho W^{\mathrm{t}}) \ge 0$

for every bi-separable state ϱ .

Non-positivity condition of W guarantees the existence of a state ϱ with $\langle \varrho, W \rangle < 0$, and the above condition tells us that this ϱ must be genuinely entangled. By duality, any genuine entanglement is detected by a genuine entanglement witness.

Suppose that the X-part of W is given by X(s, t, u). If $\langle \varrho, W \rangle \ge 0$ for every n qubit bi-separable state ϱ , then the inequality

$$\sqrt{s_i t_i} + \sqrt{s_j t_j} \ge |u_i| + |u_j|$$

holds for every choice of $i, j = 1, 2, ..., 2^{n-1}$ with $i \neq j$.

Suppose that W = X(s, t, u) is an X-shaped multi-qubit Hermitian matrix with nonnegative diagonals. Then the following are equivalent: (i) $\langle \varrho, W \rangle \ge 0$ for every n qubit bi-separable state ϱ ; (ii) $\langle \varrho, W \rangle \ge 0$ for every n qubit PPT mixture ϱ ;

(iii) the inequality

$$\sqrt{s_i t_i} + \sqrt{s_j t_j} \ge |u_i| + |u_j|$$

holds for every choice of $i, j = 1, 2, ..., 2^{n-1}$ with $i \neq j$.

Corollary

Suppose that W is an X-shaped matrix. Then, W is a genuine entanglement witness if and only if the inequality

 $\sqrt{s_i t_i} + \sqrt{s_j t_j} \ge |u_i| + |u_j|$

holds for every choice of $i, j = 1, 2, ..., 2^{n-1}$ with $i \neq j$ and the inequality

$$\sqrt{s_{i_0}t_{i_0}} < |u_{i_0}|$$

holds for some i₀

For a given genuine entanglement witness W, we consider the set G_W of all genuine entanglement ρ which are detected by W in the sense of $\langle \rho, W \rangle < 0$. We say that W is *optimal* if the set G_W is maximal.

We say that a vector $\xi \in \bigotimes_{1 \leq i \leq n} \mathbb{C}^{d_i}$ is a *bi-product vector* if there is a partition $S \sqcup T = [\{1, \dots, n\}$ such that ξ is a product vector as an element of $(\bigotimes_{i \in S} \mathbb{C}^{d_i}) \otimes (\bigotimes_{i \in T} \mathbb{C}^{d_i})$. For a given genuine entanglement witness W, we denote by P_W the set of all bi-product vectors ξ such that

$$\langle \bar{\xi} | W | \bar{\xi} \rangle = \langle | \xi \rangle \langle \xi |, W \rangle = 0.$$

We say that W has the *spanning property* if the set P_W spans the whole space $\bigotimes_{i=1}^{n} \mathbb{C}^{d_i}$.

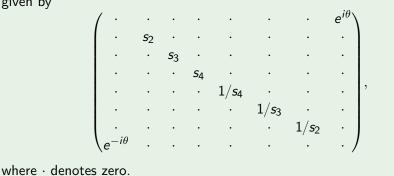
By Lewenstein, Kraus, Cirac and Horodecki (*Optimization of entanglement witness*, Phys. Rev. A 62 (2000)), the spanning property implies the optimality.

Suppose that W = X(s, t, u) is an X-shaped n-qubit genuine entanglement witness. Then the following are equivalent:

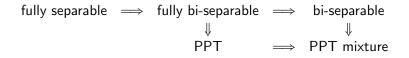
- (i) W is an optimal genuine entanglement witness;
- (ii) W is a genuine entanglement witness with the spanning property;
- (iii) there exists $i_0 \in \{1, \dots, 2^{n-1}\}$ and positive number r > 0 with the properties:

$$s_{i_0} = t_{i_0} = 0$$
 and $|u_{i_0}| = r$,
 $\sqrt{s_i t_i} = r$ and $u_i = 0$ for $i \neq i_0$

A typical example of three qubit optimal genuine entanglement witness is given by



7. Summary



The vertical arrows in the above diagram become equivalences for X-shaped multi-qubit states.

Necessary conditions by X-parts

fully separable

- ▶ $a_1b_2b_3a_4 \ge r^4$, $b_1a_2a_3b_4 \ge r^4$, $|z_{i_3}| = |z_{i_4}|$, $\theta_1 + \theta_4 = \theta_2 + \theta_3$ for three qubit states with rank ≤ 6 . Multi-index is used to generalize it to multi-qubit states with rank $\le 2^n 2$.
- ► $4\sqrt{a_ib_i} \ge \sqrt{2}|z_1e^{i\theta} + z_2| + \sqrt{2}|z_3e^{i\theta} z_4|$ for three qubit states. Multi-index is used to generalize it to multi-qubit states.

2 PPT :
$$\sqrt{s_i t_i} \ge |u_j|;$$

3 PPT mixture :
$$\sum_{j \neq i} \sqrt{a_j b_j} \ge |z_i|$$
.

We have the following dual diagram:

fully separable^d
$$\leftarrow$$
 fully bi-separable^d \leftarrow bi-separable^d
 \uparrow \uparrow
PPT^d \leftarrow PPT mixture^d

The vertical arrows are again equivalences for X-shaped multi-qubit witnesses

Necessary conditions by X-parts

- PPT mixture^d : $\sqrt{s_i t_i} + \sqrt{s_j t_j} \ge |u_i| + |u_j|;$ • PPT^d : $\sum_{i=1}^{2^{n-1}} \sqrt{s_i t_i} \ge \sum_{i=1}^{2^{n-1}} |u_i|;$
- fully separable^d :

$$\sqrt{(a_1 + b_4 |\alpha|^2)(a_4 + b_1 |\alpha|^2)} + \sqrt{(a_2 + b_3 |\alpha|^2)(a_3 + b_2 |\alpha|^2)} \ge |z_1 \bar{\alpha} + \bar{z}_4 \alpha| + |z_2 \bar{\alpha} + \bar{z}_3 \alpha|.$$

This talk is based on

- K. H. Han and S.-H, Kye, Various notions of positivity for bi-linear maps and applications to tri-partite entanglement, J. Math. Phys. 57 (2016), 015205.
- K. H. Han and S.-H, Kye, *Construction of multi-qubit optimal genuine entanglement witnesses*, arXiv:1510.03620.
- K. H. Han and S.-H, Kye, *in preparation* (the section on the full separability)