On quantum correlations.

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- *PLAN:*

- Classical probability; correlations, canonical form of two-points correlation function.

- Quantization - definitions: C^* (W^*) case.

- What is lost during the quantization procedure?

- Decomposition theory ($\equiv$ integral representation).

- Coefficient of quantum correlations.

- Entanglement of Formation ($\equiv$ EoF).

- Classical probability theory:

- The pair $(\Omega, \mathcal{F})$ consists of a set Ω and a σ-algebra $\mathcal{F}$.

- **Definition 1.** *A probability space is a triple $(\Omega, \mathcal{F}, p)$ where Ω is a space (sample space), $\mathcal{F}$ is a σ-algebra (a family of events), and p is a probability measure on $(\Omega, \mathcal{F})$.*

- **Definition 2.** *A* ***correlation coefficient*** *$C(X, Y)$ is defined as*

$$C(X,Y) = \frac{E(XY) - E(X)E(Y)}{(E(X^2) - E(X)^2)^{\frac{1}{2}}(E(Y^2) - E(Y)^2)^{\frac{1}{2}}}, \tag{1}$$

where $E(X) = \int X dp$, X a stochastic variable.

- Note that $C(X,Y)$ provides a nice classification. Firstly: $C(X,Y) \in [-1,1]$. Secondly, if $C(X,Y)$ is equal to 0 then stochastic variables X and Y are *uncorrelated*. Further, if $C(X,Y) \in (0,1]$, then X, Y are said to be *correlated* and finally when $C(X,Y) \in [-1,0)$, stochastic variables X and Y are said to be *anti correlated*.

- $E(X,Y)$ plays a crucial role in the definition of $C(X,Y)$.

- We will need the notion of Dirac's (point) measure δ_a, where $a \in E$. Such measures are determined by the condition:

$$\delta_a(f) = f(a). \tag{2}$$

- We say that a measure μ has a finite support if it can be written as a linear (finite) combination of δ_a's.

- The well known fact is, Chapter 3, Section 2 , Corollaire 3 in N. Bourbaki *Livre VI. Intégration*:

  Theorem 3. *Any positive finite measure μ on E is a limit point, in the vague topology, of a convex hull of positive measures having a finite support contained in the support of μ.*

- **Remark 4.** *1. This result will be not valid in the non-commutative setting. It is taken from the (classical) measure theory.*
  2. A slightly stronger formulation can be find in Meyer. Namely, every probability measure λ in $\mathfrak{M}(\Omega)$ is a weak limit of discrete (with finite support) measures belonging to the collection of probability measures in $\mathfrak{M}(\Omega)$ which have the same barycenter as λ ($\mathfrak{M}(\Omega)$ stands for the collection of Radon measures on Ω).
  3. The statement of Theorem 3 can be rephrased by saying that a classical measure has the weak-* Riemann approximation property.

- Classical composite systems

- *A composite system is characterized by the triple* $(\Gamma \equiv \Gamma_1 \times \Gamma_2,\ \mu,\ T_t)$, *where the probability measure* μ *is defined on the Cartesian product of two measurable spaces* $(\Gamma_1 \times \Gamma_2, \mathcal{F}_1 \times \mathcal{F}_2)$, *and finally,* T_t *is a global evolution defined on* Γ.

- We recall that there is the identification

$$C(\Gamma_1 \times \Gamma_2) = C(\Gamma_1) \otimes C(\Gamma_2), \tag{3}$$

where on the right hand side of (3) $\otimes$ stands for the tensor product,

- We will identify the function f_1 (defined on Γ_1) with the function $f_1 \otimes \mathbb{1}_{\Gamma_2}$ (defined on $\Gamma_1 \times \Gamma_2$); and analogously for f_2.

- We wish to study the functionals $\varphi(\cdot)$ given by

$$\varphi(f_1 \otimes f_2) = \varphi(f_1 f_2) \equiv \varphi_\mu(f_1 f_2) \equiv \int_{\Gamma_1 \times \Gamma_2} f_1(\gamma_1) f_2(\gamma_2) d\mu, \qquad (4)$$

where $f_i \in C(\Gamma_i)$, $i = 1, 2.$

- Now taking into account the weak-* Riemann approximation property, see Theorem 3, one has

$$\begin{aligned} \varphi(f_1 f_2) &= \lim_{n\to\infty} \int_{\Gamma_1 \times \Gamma_2} f_1(\gamma_1) f_2(\gamma_2) d\mu_n \\ &= \lim_{n\to\infty} \int_{\Gamma_1 \times \Gamma_2} f_1(\gamma_1) f_2(\gamma_2) (\sum_n \lambda_n d\delta^{(n)}_{(a_{1,n}, a_{2,n})}), \end{aligned} \qquad (5)$$

where $\delta_{(a,b)}$ stands for the Dirac's measure supported by (a, b), $\lambda_n \geq 0$ and $\sum_n \lambda_n = 1.$

- Note, that for a point measure, one has

$$\delta_{(a,b)} = \delta_a \times \delta_b.$$

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$$\begin{aligned}
\varphi_\mu(f_1 f_2) &= \lim_{n\to\infty} \sum_n \lambda_n \int_{\Gamma_1} f_1(\gamma_1) d\delta^{(n)}_{a_{1,n}}(\gamma_1) \int_{\Gamma_2} f_2(\gamma_2) d\delta^{(n)}_{a_{2,n}}(\gamma_2) \\
&= \lim_{n\to\infty} \sum_n \lambda_n \varphi_{\delta_{a_{1,n}}}(f_1) \varphi_{\delta_{a_{2,n}}}(f_2) \\
&= \lim_{n\to\infty} \sum_n \lambda_n (\varphi_{\delta_{a_{1,n}}} \otimes \varphi_{\delta_{a_{2,n}}})(f_1 \otimes f_2),
\end{aligned} \tag{6}$$

for any $f_i \in C(\Gamma_i)$, $i = 1, 2$.

- Consequently

$$\varphi_\mu(f_1 \otimes f_2) = \lim_{n\to\infty} \sum_n \lambda_n (\varphi_{\delta_{a_{1,n}}} \otimes \varphi_{\delta_{a_{2,n}}})(f_1 \otimes f_2) \tag{7}$$

for any $f_i \in C(\Gamma_i)$, $i = 1, 2$.

- **Corollary 5.** *For a classical case, any two point correlation function of bipartite system is given by the limit of a convex combination of product states.*

- This is taken as the basic feature of classical correlations.

- Quantization: C^*-algebra approach.

- "quantization" consists in replacing classical coordinate q_k and canonically conjugate momentum p_l by self-adjoint operators satisfying CCR relations:

$$[q_k, q_l] = 0 = [p_k, p_l], \quad [p_k, q_l] = \frac{h}{2\pi i}\delta_{kl}, \tag{8}$$

$k, l = 1, ..., n.$

- No any finite dimensional realization!

- (v.Neumann, Rellich, Stone, Weyl) under natural requirements (irreducibility, sufficient regularity) CCR relations fix the representation of operators p_k, q_l up to unitary equivalence provided that n is **finite!**

- For Statistical Mechanics, QFT - *n is infinite!*

- There are non-equivalent representations.

- typical algebras for infinite systems: factor III.

- (D. Kastler and his school) There is a C^*-algebra carrying some of the main features attached to the concept of Weyl quantization.

- C^***-algebra formalism is appearing.**

- Essential point for quantum probability:

- **Theorem 6.** *(Markov-Riesz-Kakutani) If φ is a linear, positive, continuous form on $C_{\mathfrak{K}}(\Gamma)$ (continuous functions with compact support) then there exist a unique positive Borel measure μ on E such that*

$$\varphi(f) = \int_E f d\mu \qquad f \in C_{\mathfrak{K}}(\Gamma). \tag{9}$$

- Let $\mathfrak{A}$ be an abelian C*-algebra with unit $\mathbb{1}$. Then, Gelfand-Neimark theorem says that $\mathfrak{A}$ can be identified with the (abelian) C^*-algebra of all complex valued continuous functions on Γ, where Γ is a compact Hausdorff space.

- Quantization of probability calculus: drop abelian!

- Noncommutative case

- C^*-algebra case: $(\mathfrak{A}_1 \otimes \mathfrak{A}_2, \mathfrak{S})$, where $\mathfrak{S}$ stands for the set of states.

- W^*-algebra case: to take in account normal states one has (Sakai; Effros, Ruan)

Theorem 7. *Let $\mathfrak{M} \subseteq B(\mathcal{H})$ and $\mathfrak{N} \subseteq B(\mathcal{K})$ be two von Neumann algebras. Denote by $\mathfrak{M}_*$ the predual of $\mathfrak{M}$, i.e. such Banach space that $(\mathfrak{M}_*)^*$ is isomorphic to $\mathfrak{M}$, i.e. $(\mathfrak{M}_*)^* \cong \mathfrak{M}$. There is an isometry*

$$(\mathfrak{M} \otimes \mathfrak{N})_* = \mathfrak{M}_* \otimes_\pi \mathfrak{N}_*, \tag{10}$$

where the von Neumann algebra $\mathfrak{M} \otimes \mathfrak{N}$ is the weak closure of the set $\{A \otimes B; A \in \mathfrak{M}, B \in \mathfrak{N}\}$. In particular,

$$B(\mathcal{H} \otimes \mathcal{K})_* = B(\mathcal{H})_* \otimes_\pi B(\mathcal{K})_*. \tag{11}$$

- $\otimes_\pi$ stands for the operator space projective tensor product:

- (1) a matrix norm $\|\cdot\|$ on a linear space V is an assignment of a norm $\|\cdot\|$ on the matrices $M_n(V)$ for $\forall_{n\in\mathbb{N}}$.

- (2) an operator space is a linear space V together with a matrix norm $\|\cdot\|$ for which: $\|v \oplus w\|_{n+m} = max\{\|v\|_m, \|w\|_n\}$ and $\|\alpha v \beta\|_n \leq \|\alpha\|\|v\|_m\|\beta\|$ where $v \in M_n(V)$, $w \in M_n(V)$, $\alpha \in M_{n,m}$, $\beta \in M_{n,m}$.

- (3) given an element $u \in M_n(V \otimes W)$ define

$$\|u\|_\pi = \inf\{\|\alpha\|\|v\|\|w\|\|\beta\|; u = \alpha(v \otimes w)\beta\}$$

where $v \in M_p(V)$, $w \in M_q(W)$, $\alpha \in M_{n,p\times q}$, and $\beta \in M_{p\times q,n}$.

- Lack of the weak* Riemann approximation property (for products!).

- Namely, one has (see Exercise 11.5.11 in Kadison, Ringrose)

  Example 8. *Let* $\mathfrak{A}_1 = B(\mathcal{H})$ *and* $\mathfrak{A}_2 = B(\mathcal{K})$ *where* $\mathcal{H}$ *and* $\mathcal{K}$ *are 2-dimensional Hilbert spaces. Consider the vector state* $\omega_x(\cdot) = (x, \cdot\, x)$ *with* $x = \frac{1}{\sqrt{2}}(e_1 \otimes f_1 + e_2 \otimes f_2)$ *where* $\{e_1, e_2\}$ *and* $\{f_1, f_2\}$ *are orthonormal bases in* $\mathcal{H}$ *and* $\mathcal{K}$ *respectively. Let* ρ *be any state in the norm closure of the convex hull of product states, i.e.* $\rho \in \overline{conv}(\mathfrak{S}_1 \otimes \mathfrak{S}_2)$. *Then, one can show that*

  $$\|\omega_x - \rho\| \geq \frac{1}{4}. \tag{12}$$

- **Remark 9.** *One should note that* ω_x *can always be approximated by a finite linear combination of simple tensors. However, here we wish to approximate* ω_x *by a convex combination of positive (normalized) functionals (cf Theorem 3) and this makes the difference.*

- Consequently, contrary to the classical case (see Corollary 5) even in the simplest non-commutative case, the space of all states of $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ is not norm closure of $conv(\mathfrak{S}_1 \otimes \mathfrak{S}_2)$.

- *It means, in mathematical terms, that for non-commutative case,* ***for product structures****, the weak* Riemann approximation property of a (classical) measure does not hold.*

- Thus, we are in position to give the following definitions:

Definition 10. – *C^*-algebra case.*

Let $\mathfrak{A}_i$, $i = 1, 2$ be a C^*-algebra, $\mathfrak{S}$ the set of all states on $\mathfrak{A} \equiv \mathfrak{A}_1 \otimes \mathfrak{A}_1$, i.e. the set of all normalized positive forms on $\mathfrak{A}$. The subset $\overline{conv}(\mathfrak{S}_1 \otimes \mathfrak{S}_2)$ in $\mathfrak{S}$ will be called the set of separable states and will be denoted by $\mathfrak{S}_{sep}$. The closure is taken with respect to the norm of $\mathfrak{A}^*$. The subset $\mathfrak{S} \setminus \mathfrak{S}_{sep} \subset \mathfrak{S}$ is called the subset of entangled states.

– *W^*-algebra case, (cf. Theorem 7.)*

Let $\mathfrak{M}_i$, $i = 1, 2$ be a W^*-algebra, $\mathfrak{M} = \mathfrak{M}_1 \otimes \mathfrak{M}_2$ be the spacial tensor product of $\mathfrak{M}_1$ and $\mathfrak{M}_2$, $\mathfrak{S}$ the set of all states on $\mathfrak{M}$, and $\mathfrak{S}^n$ the set of all normal states on $\mathfrak{M}$, i.e. the set of all normalized, weakly*-continuous positive forms on $\mathfrak{M}$ (equivalently, the set of all density matrices). The subset $\overline{conv}^{\pi}(\mathfrak{S}_1^n \otimes \mathfrak{S}_2^n)$ in $\mathfrak{S}^n$ will be called the set of separable states and will be denoted by $\mathfrak{S}_{sep}^n$. The closure is taken with respect to the projective operator space norm on $\mathfrak{M}_{1,*} \odot \mathfrak{M}_{2,*}$. The subset $\mathfrak{S}^{\mathfrak{n}} \setminus \mathfrak{S}_{sep}^n \subset \mathfrak{S}^{\mathfrak{n}}$ is called the subset of normal entangled states.

- Further differences between commutative and noncommutative cases.

- **Fact 11.** *1. classical case.*
  Let δ_a be a Dirac's measure on a product measure space, i.e. δ_a is given on $\Gamma_1 \times \Gamma_2$. Note that the marginal of the point measure δ_a gives another point measure, i.e. $\delta_a|_{\Gamma_1} = \delta_{a_1}$. Here we put $a \in \Gamma_1 \times \Gamma_2$, $a = (a_1, a_2)$. The same in "physical terms" reads: a reduction of a pure state is again a pure state.
  2. *non-commutative case.*
  Let $\mathcal{H}$ and $\mathcal{K}$ are finite dimensional Hilbert spaces. Without loss of generality we can assume that dim$\mathcal{H}$=dim$\mathcal{K} = n$. Let $\omega_x(\cdot) = (x, \cdot\, x)$ be a state on $B(\mathcal{H}) \otimes B(\mathcal{K})$ where x is assumed to be of the form

$$x = \frac{1}{\sqrt{n}} \left(\sum_i e_i \otimes f_i \right). \tag{13}$$

Here $\{e_i\}$ *and* $\{f_i\}$ *are basis in* $\mathcal{H}$ *and* $\mathcal{K}$ *respectively. Then, we have*

$$\begin{aligned}\omega_x(A\otimes\mathbb{1}) &= \frac{1}{n}\left(\sum_i e_i\otimes f_i, A\otimes\mathbb{1}\sum_j e_j\otimes f_j\right) \\ &= \frac{1}{n}\sum_{i,j}(e_i, Ae_j)(f_i, f_j) = \mathbf{Tr}_{\mathcal{H}}\frac{1}{n}\mathbb{1}A \equiv \mathbf{Tr}_{\mathcal{H}}\varrho_0 A,\end{aligned} \tag{14}$$

where $\varrho_0 = \frac{1}{n}\mathbb{1}$ *is "very non pure" state. In other words, the non-commutative counterpart of the marginal of a point measure (pure state) does not need to be again a point measure (pure state). Consequently, the second crucial ingredient of the discussion leading to Corollary 5 is not valid in non-commutative case.*

- The next difficulty follows from the geometrical characterization of the set of states. Namely,

- **Proposition 12.** *Let $\mathfrak{A}$ be a C*-algebra. Then the following conditions are equivalent*

  1. *The state space $\mathfrak{S}_{\mathfrak{A}}$ is a simplex.*
  2. *$\mathfrak{A}$ is abelian algebra.*
  3. *Positive elements $\mathfrak{A}^+$ of $\mathfrak{A}$ form a lattice.*

- Therefore in quantum case the set of states is not a simplex (contrary to the classical case). Consequently, in quantum case, all possible decompositions of a given state should be taken into account. In general, there are many such decompositions.

- G. Choquet:

- Let K be a base of a convex cone C with apex at the origin. The cone C gives rise to the order $\leq$ ($a \leq b$ if and only if $b - a \in C$). K is said to be a simplex if C equipped with the order $\leq$ is a lattice (Lattice is a partially ordered set in which every two elements have a supremum and an infimum).

- To sum up: as the family of states in quantum mechanics does not form a simplex, a state can be decomposed in many ways.

- Intuition: $2D$ ball (non-simplex) versus a triangle (simplex).

- A decomposition of a state can be realized using measure-theoretical approach (decomposition theory $\equiv$ integral representations).

- It should be noted that extreme points of some subsets of states can exhibit "bad" measure-theoretic properties. To avoid such cases, an auxiliary condition, Ruelle's separability condition SC, should be imposed (this point is essential for EoF).

- Decompositions supported by extreme points are essential for EoF; not for coefficient of quantum correlations!

- Fortunately, all essential physical models, satisfy SC condition. Consequently, the program of decomposing of states can be carried out.

- Quantum correlations

- One can perform the quantization of the coefficient of correlations:

Definition 13.

$$C_q(A,A') = \frac{\langle (A - \langle A \rangle)\,(A' - \langle A' \rangle) \rangle}{\left\langle (A - \langle A \rangle)^2 \right\rangle^{\frac{1}{2}} \left\langle (A' - \langle A' \rangle)^2 \right\rangle^{\frac{1}{2}}} \tag{15}$$

where $< A >= \phi(A)$*;* $A \in \mathfrak{A}$*,* $\phi \in \mathfrak{S}$*.*

- BUT, the coefficient C_q is not able to distinguish correlations of quantum nature from that of classical nature.

- Thus, a new measure of quantum correlation should be introduced.

- To this end, we will look for the best approximation of a given state ω by separable states, like in Kadison-Ringrose example.

- However, a given (non pure) state ω, in general, can possess various decompositions. Thus, we should use the decomposition theory.

- To proceed with the study of coefficient of (quantum) correlations for a quantum composite system specified by $(\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2, \mathfrak{S}_{\mathfrak{A}})$, where $\mathfrak{A}_i$ are C^*-algebras, we will consider restriction maps

$$(r_1\omega)(A) = \omega(A \otimes \mathbb{1}) \tag{16}$$
$$(r_2\omega)(B) = \omega(\mathbb{1} \otimes B), \tag{17}$$

where $\omega \in \mathfrak{S}_{\mathfrak{A}}$, $A \in \mathfrak{A}_1$, and $B \in \mathfrak{A}_2$.

- $r_i : \mathfrak{S}_{\mathfrak{A}} \to \mathfrak{S}_{\mathfrak{A}_i}$ and the restriction map r_i is continuous (in weak-$*$ topology), $i = 1, 2$.

- To proceed with the decomposition procedure we start with a measure on the state space $\mathfrak{S}$ (from $M_\omega(\mathfrak{S}) \equiv \{\mu : \omega = \int_{\mathfrak{S}} \nu d\mu(\nu)\}$).

- Define
$$\mu_i(F_i) = \mu(r_i^{-1}(F_i)) \tag{18}$$
for $i = 1, 2$, where F_i is a Borel subset in $\mathfrak{S}_{\mathfrak{A}_i}$.

- The formula (18) provides the well defined measures μ_i on $\mathfrak{S}_{\mathfrak{A}_i}$, $i = 1, 2$.

- Having two measures μ_1, μ_2 on $\mathfrak{S}_1$, and $\mathfrak{S}_2$ respectively, we want to "produce" a new measure $\boxtimes\mu$ on $\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}$. To this end, firstly, let us consider the case of finitely supported probability measure μ:
$$\mu = \sum_{i=1}^{N} \lambda_i \delta_{\rho_i}, \tag{19}$$

where $\lambda_i \geq 0$, $\sum_{i=1}^{N} \lambda_i = 1$, and δ_{ρ_i} denotes the Dirac's measure.

- Define

$$\mu_1 = \sum_{i=1}^{N} \lambda_i \delta_{r_1 \rho_i} \tag{20}$$

$$\mu_2 = \sum_{i=1}^{N} \lambda_i \delta_{r_2 \rho_i}. \tag{21}$$

Then

$$\boxtimes \mu = \sum_{i=1}^{N} \lambda_i \delta_{r_1 \rho_i} \times \delta_{r_2 \rho_i} \tag{22}$$

gives a well defined measure on $\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}$. Here $\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}$ is understood as a measure space obtained as a product of two measure spaces $\mathfrak{S}_{\mathfrak{A}_1}$ and $\mathfrak{S}_{\mathfrak{A}_2}$. A measure structure on $\mathfrak{S}_{\mathfrak{A}_i}$ is defined as the Borel structure determined by the corresponding weak-$*$ topology on $\mathfrak{S}_{\mathfrak{A}_i}$, $i = 1, 2$.

- Take an arbitrary measure μ from M_ω. By Theorem 3 there exists a net of discrete measures (having a finite support) μ_k such that $\mu_k \to \mu$, and the convergence is understood in the weak-* topology on $\mathfrak{S}_{\mathfrak{A}}$.

- Defining μ_1^k (μ_2^k) analogously as μ_1 (μ_2 respectively) one has $\mu_1^k \to \mu_1$ and $\mu_2^k \to \mu_2$, where again the convergence is taken in the weak-* topology on $\mathfrak{S}_{\mathfrak{A}_1}$ ($\mathfrak{S}_{\mathfrak{A}_2}$ respectively).

- Then define, for each k, $\boxtimes\mu^k$ as it was done in (22). $\left\{\boxtimes\mu^k\right\}$ is convergent (in weak $*$-topology) to a measure on $\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}$.

- Consequently, taking the weak-* limit we arrive at the measure $\boxtimes\mu$ on $\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}$. It follows that $\boxtimes\mu$ does not depend on the chosen approximation procedure.

- Now, we are in position to give the definition of the coefficient of quantum correlations, $d(\omega, A_1, A_2) \equiv d(\omega, A)$, where $A_i \in \mathfrak{A}_i$.

- **Definition 14.** *Let a quantum composite system* $(\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2, \mathfrak{S}_{\mathfrak{A}})$ *be given. Take a* $\omega \in \mathfrak{S}_{\mathfrak{A}}$*. We define the coefficient of quantum correlations as*

$$d(\omega, A) = \inf_{\mu \in M_\omega(\mathfrak{S}_{\mathfrak{A}})} \left| \int_{\mathfrak{S}_{\mathfrak{A}}} \xi(A) d\mu(\xi) - \int_{\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}} \xi(A)(d \boxtimes \mu)(\xi) \right|. \quad (23)$$

- Following the strategy of Kadison-Ringrose example, an evaluation of a distance between the given state ω and the set of approximative separable states is done.

- It is a simple matter to see that $d(\omega, A)$ is equal to 0 if the state ω is a separable one. The converse statement is much less obvious. However, we are able to prove it. Namely:

- **Theorem 15.** *Let $\mathfrak{A}$ be the tensor product of two C*-algebras $\mathfrak{A}_1$, $\mathfrak{A}_2$. Then state $\omega \in \mathfrak{S}_{\mathfrak{A}}$ is separable if and only if $d(\omega, A) = 0$ for all $A \in \mathfrak{A}_1 \otimes \mathfrak{A}_2$*

- The basic idea of the proof of the statement that $d(\omega, A) = 0$ implies separability of ω relies on the study of continuity properties of the function

$$M_\omega(\mathfrak{S}_{\mathfrak{A}}) \ni \mu \mapsto \int_{\mathfrak{S}_{\mathfrak{A}}} \xi(A) d\mu(\xi) - \int_{\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}} \xi(A)(d \boxtimes \mu)(\xi) \tag{24}$$

and the proof falls naturally into few steps.

- $M_\omega(\mathfrak{S}_{\mathfrak{A}})$ is a compact set.

- The mapping $M_\omega(\mathfrak{S}_{\mathfrak{A}}) \ni \mu \mapsto \boxtimes\mu \in M^+(\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2})$ is weakly continuous.

- The continuity proved in the second step implies that the function (24) is a real valued, continuous function defined on a compact space. Hence, by Weierstrass theorem, infimum is attainable. Therefore, the condition $d(\omega, A) = 0$ means that

$$\omega(A) = \int_{\mathfrak{S}_{\mathfrak{A}}} \xi(A) d\mu_0(\xi) = \int_{\mathfrak{S}_{\mathfrak{A}_1} \times \mathfrak{S}_{\mathfrak{A}_2}} \xi(A) d \boxtimes \mu_0(\xi), \tag{25}$$

  for all $A = A_1 \otimes A_2$. But, this means the separability of ω.

- Theorem 15 may be summarized by saying that any separable state contains "classical" correlations only. Therefore, **an entangled state contains "non-classical" (or pure quantum) correlations**.

- To comment the question of separability of normal states we have two remarks:

Remark 16. *1. (indirect way)*

As we have considered C^*-algebra case, taking a normal state $\varphi \in \mathfrak{S}^n_{\mathfrak{M}} \equiv \mathfrak{S}_{\mathfrak{M}} \cap \mathfrak{M}_* \subset \mathfrak{S}_{\mathfrak{M}}$, we can apply Theorem 15 for its analysis. If $d(\varphi, A) = 0$ we are getting a "separable" decomposition of φ. However, still one must check whether components of the decomposition are normal or not.

2. (a possibility for a direct way)

One can try to modify the results obtained for C^*-algebra case to that which are relevant for W^*-algebra case. However, there are two essential differences. The first is given by Definition 10 – the closure of convex hull should be carried out with respect to the projective operator space norm topology.

The second difference leads to a great problem. Namely, the set $\mathfrak{S}^n_{\mathfrak{M}}$ (normal states) is compact, in general, with respect to another topology than that which gives compactness of $\mathfrak{S}_{\mathfrak{M}}$.

- Entanglement of Formation (Benett, DiVicenzo, Smolin, Wootters; WAM)

- **Definition 17.** *Let ω be a state, $\omega \in F \subset \mathfrak{S}_{\mathfrak{A}_1 \otimes \mathfrak{A}_2}$ and F satisfy separability condition SC. The entanglement of formation EoF is defined as*
$$E_{\mathbb{F}}(\omega) = \inf_{\mu \in M_\omega(\mathfrak{S}_{\mathfrak{A}_1 \otimes \mathfrak{A}_2})} \int \mathbb{F}(r\varphi) d\mu(\varphi) \qquad (26)$$
where $\mathbb{F}$ is a concave non-negative continuous function which vanishes on pure states and only on pure states

- **Theorem 18.** *Let SC hold. $E(\omega) = 0$ if and only if $\omega \in F$ is separable.*

- It is worth pointing out that Entanglement of Formation, EoF, is not only a nice indicator of separability. It possesses also many useful properties like convexity, semi-continuity and others.

- Existence of $\mathbb{F}$: finite dimensional case - one can take as $\mathbb{F}$ the von Neumann entropy $S(\varrho)$.

- However, for a general case (infinite dimensional) $S(\varrho)$ is only semicontinuous and $\{\varrho : S(\varrho) < \infty\}$ is merely a meager set (set of first category). General case - Orlicz spaces!

- **Definition 19.** *Ruelle's SC condition*
  Let $\mathfrak{A}$ be a C-algebra with unit, and $\mathfrak{F}$ a subset of the state space $\mathfrak{S}_{\mathfrak{A}}$. $\mathfrak{F}$ is said to satisfy separability condition (SC) if there exists a sequence of sub-C*-algebras $\{\mathfrak{A}_n\}$ such that $\bigcup_{n=1}^{\infty} \mathfrak{A}_n$ is dense in $\mathfrak{A}$ and each $\mathfrak{A}_n$ contains a two-sided, closed, separable ideal $\mathcal{I}_n$ such that*

$$\mathfrak{F} = \{\omega,\ \omega \in \mathfrak{S}_{\mathfrak{A}},\ \|\omega|_{\mathcal{I}_n}\| = 1,\ n \geq 1\}.$$

- Final remarks:

- The presented "tools": coefficient of quantum correlations and EoF, in a sense, are complementary each other.

- ω extreme, then μ unique, then μ is either of the form $\boxtimes\mu$ or not; for EoF extremality of ω leads to a great simplifications - $\inf$ can be dropped.

- EoF gives a possibility to speak about "witness of entanglement", i.e. there are observables which determine the value of EoF.

- Details are in: W. A. Majewski, *Quantum correlations; quantum probability approach*, arXiv:1407.4754v4[quant-ph]