A class of gapped Hamiltonians on quantum spin chains and its classification

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Hamiltonian

Quantum spin chain $\mathcal{A}_{\mathbb{Z}} := \bigotimes_{\mathbb{Z}} \mathrm{Mat}_n(\mathbb{C})$

α_x, $x \in \mathbb{Z}$: space translation

Subsystems : for $\Lambda \subset \mathbb{Z}$, $\mathcal{A}_{\Lambda} := \bigotimes_{\Lambda} \mathrm{Mat}_n(\mathbb{C})$

$$\mathcal{A}_{\mathrm{loc}} := \cup_{\Lambda, |\Lambda| < \infty} \mathcal{A}_{\Lambda}$$

Hamiltonian

Construct a sequence of matrices:

Fix some $m \in \mathbb{N}$ and a self-adjoint element $h \in \mathcal{A}_{[0,m-1]}$.

Local Hamiltonian on an interval $[1, N]$:

$$H_{[1,N]}(h) = \sum_{x:[x,x+m-1]\subset[1,N]} \alpha_x(h),$$

Obtain a sequence of matrices

$$H(h) := \left(H_{[1,N]}(h)\right)_N \qquad \text{Hamiltonian}$$

Gapped Hamiltonian

Definition

A Hamiltonian $H(h) := \left(H_{[1,N]}(h)\right)_{N\in\mathbb{N}}$ *associated with* h *is gapped if there exists a* $\gamma > 0$ *such that*

$$\sigma\left(H_{[1,N]}(h)\right) \cap \left[\inf\left(\sigma\left(H_{[1,N]}(h)\right)\right), \inf\left(\sigma\left(H_{[1,N]}(h)\right)\right) + \gamma\right] = \left\{\inf\left(\sigma\left(H_{[1,N]}(h)\right)\right)\right\}$$

for all $N \in \mathbb{N}$.

C^1-classification

Definition

Let $H(h_0), H(h_1)$ be gapped Hamiltonians. We say that $H(h_0), H(h_1)$ are C^1-equivalent if the following conditions are satisfied.

1. *There exists a continuous and piecewise C^1-path of $h : [0,1] \to \mathcal{A}_{\mathrm{loc},sa}$ such that $h(0) = h_0$, $h(1) = h_1$.*
2. *There are $\gamma > 0$ and finite intervals $I(t) = [a(t), b(t)]$, whose endpoints $a(t), b(t)$ smoothly depending on $t \in [0,1]$, such that for any $N \in \mathbb{N}$ and $t \in [0,1]$*

$$\sigma\left(H_{[1,N]}(h(t))\right) \cap I(t) = \left\{\inf \sigma\left(H_{[1,N]}(h(t))\right)\right\},$$

$$\sigma\left(H_{[1,N]}(h(t))\right) \cap I(t)^c \subset [b(t)+\gamma, \infty).$$

Remark

Ground state structure is an invariant of C^1-classification (Bachmann, Michalakis, Nachtergaele, Sims (2011))

C^1-classification'

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1. *There exists a continuous and piecewise C^1-path of $h : [0,1] \to \mathcal{A}_{\mathrm{loc},sa}$ such that $h(0) = h_0$, $h(1) = h_1$.*
2. *There are $\gamma > 0$ and finite intervals $I(t) = [a(t), b(t)]$, $t \in [0,1]$, satisfying the followings:*

   (i) the endpoints $a(t), b(t)$ smoothly depends on $t \in [0,1]$,

   (ii) there exists a sequence $\{\varepsilon_N\}_{N\in\mathbb{N}}$ of positive numbers with $\varepsilon_N \to 0$, for $N \to \infty$, such that

$$\sigma\left(H_{[1,N]}(h(t))\right) \cap I(t) \subset \inf \sigma\left(H_{[1,N]}(h(t))\right) + [0, \varepsilon_N],$$

$$\sigma\left(H_{[1,N]}(h(t))\right) \cap I(t)^c \subset [b(t) + \gamma, \infty).$$

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However, it is hard to show the existence of the gap in general.

If the space dimension is more than one, even the examples of gapped Hamiltonians are quite limited.

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Recipe of MPS Hamiltonians

Local Hamiltonian on an interval $[1, N]$:

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We would like to decide this h, from some given set of matrices $(B_1, B_2, \ldots, B_n)$.

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2. Define a subspace $\mathcal{G}_{m,\mathbb{B}}$ of $\bigotimes_{i=0}^{m-1} \mathbb{C}^n$ by the range of the following map $\Gamma_{m,\mathbb{B}} : \mathrm{Mat}_k(\mathbb{C}) \to \bigotimes_{i=0}^{m-1} \mathbb{C}^n$,

$$\Gamma_{m,\mathbb{B}}(X) = \sum_{\mu_0,\ldots,\mu_{m-1} \in \{1,\cdots,n\}} \left(\mathrm{Tr}\, X \left(B_{\mu_0} B_{\mu_1} \cdots B_{\mu_{m-1}} \right)^* \right) \bigotimes_{i=0}^{m-1} \psi_{\mu_i}.$$

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This $h_{m,\mathbb{B}}$ is the interaction given by this recipe.

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This is sufficient to guarantee the gap.

However, there is a gapped Hamiltonian which does not belong to the same equivalence class as the ones of MPS-Hamiltonians with injectivity.
(Bachmann-Nachtergaele '12)

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Main Statement of this talk:
A Hamiltonian whose ground state structure satisfies "five qualitative conditions" is equivalent to a MPS-Hamiltonian.

Ground state structure

For a finite interval I

$\mathcal{S}_I(h)$: the set of all states on $\mathcal{A}_I$ with support under the spectral projection of $H_I(h)$ onto the lowest eigenvalue.

For $\Gamma = (-\infty, -1],\ [0, \infty),\ \mathbb{Z}$,

$\mathcal{S}_\Gamma(h)$: the set of all wk^*-accumulation points of elements in $\mathcal{S}_I(h)$, $I \subset \Gamma$ as $I \uparrow \Gamma$

Assumption

We may assume $h \geq 0$.

- A1 There exists $N_1, d_0 \in \mathbb{N}$ such that $1 \leq \dim \ker H_{[0,N-1]}(h) \leq d_0$ for all $N_1 \leq N \in \mathbb{N}$.
- A2 $H(h)$ is gapped.
- A3 $\mathcal{S}_{\mathbb{Z}}(H(h))$ consists of a unique state ω_∞ on $\mathcal{A}_{\mathbb{Z}}$,

Assumption

A4 Let G_N be the spectral projection of $H_{[1,N]}((h))$ onto the lowest eigenvalue. There exist $0 < C_1$, $0 < s_1 < 1$, $N_2 \in \mathbb{N}$ and factor states $\omega_R \in \mathcal{S}_{[0,\infty)}(H(h))$,

$$\left| \frac{\mathrm{Tr}_{[1,N]}\left(G_N A\right)}{\mathrm{Tr}_{[1,N]}\left(G_N\right)} - \omega_R(A) \right| \leq C_1 s_1^{N-l} \left\| A \right\|$$

for all $l \in \mathbb{N}$, $A \in \mathcal{A}_{[0,l-1]}$, and $N \geq \max\{l, N_2\}$, and

$$\inf \left\{ \sigma\left(\omega_R|_{\mathcal{A}_{[0,l-1]}} \right) \setminus \{0\} \mid l \in \mathbb{N} \right\} > 0,$$

(Similar property holds for the left infinite chain.)

Assumption

A5 For any $\psi \in \mathcal{S}_{[0,\infty)}(H)$ there exists an $l_\psi \in \mathbb{N}$ such that $\left\|\psi - \psi \circ \alpha_{l_\psi}\right\| < 2$.

(Similar property holds for the left infinite chain.)

Classification

Theorem (O)

Suppose that the properties [A1]-[A5] holds for h. Then there exists an n-tuple of matrices $\mathbb{B} \in \mathrm{Class(A)}$ *and* $m \in \mathbb{N}$ *such that*

1. $H_{[1,N]}(h_{m,\mathbb{B}})$ *is gapped.*
2. $H_{[1,N]}(h_{m,\mathbb{B}})$ *and* $H_{[1,N]}(h)$ *are in the same equivalence class with respect to the* C^1*-classification'.*

A Hamiltonian whose ground state structure satisfies [A1]-[A5] is equivalent to a MPS-Hamiltonian with respect to the C^1-classification'.

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Class A?

Define for each $l \in \mathbb{N}$,

$$\mathcal{K}_l(\mathbb{B}) := \mathrm{span}\left\{ B_{\mu_1} B_{\mu_2} \ldots B_{\mu_l} \mid \mu_1, \ldots, \mu_l \in \{1, \ldots n\} \right\}.$$

The injectivity condition is equivalent to

$$\mathcal{K}_l(\mathbb{B}) = \mathrm{Mat}_k(\mathbb{C}), \quad \text{for } l \text{ large enough.}$$

Class A is a kind of extension of this.

Class(A)

Class(A) is a set of n-tuples of matrices $\mathbb{B}$ which satisfies

$$\mathcal{K}_l(\mathbb{B}) = \mathrm{Mat}_{n_{\mathbb{B}}}(\mathbb{C}) \otimes \mathcal{D}_{\mathbb{B}} \Lambda_{\mathbb{B}}^l, \quad \text{for } l \text{ large enough},$$

where

- $n_{\mathbb{B}} \in \mathbb{N}$ and $k_{R,\mathbb{B}}, k_{L,\mathbb{B}} \in \mathbb{N} \cup \{0\}$,
- $\mathbb{B}$ is an element of $\mathrm{Mat}_{n_{\mathbb{B}}}(\mathbb{C}) \otimes \mathrm{Mat}_{k_{L,\mathbb{B}}+k_{R,\mathbb{B}}+1}(\mathbb{C})$,
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- $\mathcal{D}_{\mathbb{B}}$ is a subalgebra of upper triangular matrices satisfying some additional conditions.

GSS of $\mathrm{Class}(\mathrm{A})$

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2. The edge states has the following structures:

$$S_{[0,\infty)}(h_{m,\mathbb{B}}) \simeq \text{state space over } \mathrm{Mat}_{n_0(k_{R,\mathbb{B}}+1)}(\mathbb{C}),$$

$$S_{(-\infty,-1]}(h_{m,\mathbb{B}}) \simeq \text{state space over } \mathrm{Mat}_{n_0(k_{L,\mathbb{B}}+1)}(\mathbb{C}).$$

If $k_{L,\mathbb{B}} \neq k_{R,\mathbb{B}}$, then the ground state structure is asymmetric. (Injective case : $k_{L,\mathbb{B}} = k_{R,\mathbb{B}} = 0$ symmetric)

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3. If we consider only the subclass of $\mathrm{Class}\, A$ with non-degenerate $\Lambda_{\mathbb{B}}$, we can carry out the C^1-classification of this subclass. The complete invariant is the pair of the numbers $n_0(k_{R,\mathbb{B}} + 1), n_0(k_{L,\mathbb{B}} + 1)$.

Thank you!

Invariant of C^1-classification

Theorem (Bachmann, Michalakis, Nachtergaele, Sims (2011))

Suppose that two gapped Hamiltonians $H(h_0), H(h_1)$ *are* C^1*-equivalent. Then, for* $\Gamma = (-\infty, -1]$, $\Gamma = [0, \infty)$ *and* $\Gamma = \mathbb{Z}$, *there exists a quasi-local automorphism* α_Γ *of* $\mathcal{A}_\Gamma$ *such that*

$$\mathcal{S}_\Gamma(h_0) = \mathcal{S}_\Gamma(h_1) \circ \alpha_\Gamma.$$