Hypercontractivity and log Sobolev inequalities for completely bounded norms

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Outline

- Introduce (classical) Markov semigroups
- Hypercontractivity and log-Sobolev inequalities
- Applications in
  - estimating mixing time
  - local state transformation
- Quantum hypercontractivity and log-Sobolev inequalities
- Completely bounded (CB) norm
  - CB-hypercontractivity and CB-log-Sobolev inequalities

Markov semigroups

- (Ω, π): finite probability space with $\pi(x) > 0$ for all $x \in \Omega$
- $L^2(\Omega, \pi)$: space of real functions on Ω

$$\langle f, g \rangle = \mathbb{E}[f\, g], \qquad \|f\|_2 = \left(\mathbb{E}[f^2]\right)^{\frac{1}{2}}$$

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- Markov semigroup: $P_t : L^2(\Omega, \pi) \to L^2(\Omega, \pi), \qquad \forall t \geq 0$
  - $P_0 = I$
  - $t \mapsto P_t$ continuous
  - $P_s P_t = P_{s+t}$
  - P_t is stochastic: $\qquad P_t 1 = 1, \qquad \& \qquad f \geq 0 \Rightarrow P_t f \geq 0$

Lindblad operator

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Reversible

We assume that $\mathcal{L}$ is self-adjoint as an operator acting on $L^2(\Omega, \pi)$.

- Reversibility $\Rightarrow$ π is an invariant measure: $\quad \pi P_t = \pi$
- Reversibility $\Rightarrow$ $\mathcal{L}$ is positive semidefinite

Dirichlet form

$$\mathcal{E}(f,g) := \langle f, \mathcal{L}g \rangle = \mathbb{E}[f\mathcal{L}g] = \mathbb{E}[g\mathcal{L}f] = -\frac{\mathrm{d}}{\mathrm{d}t}\langle f, P_t g \rangle \Big|_{t=0}$$

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- $p \to \|f\|_p$ is non-decreasing

Hypercontractivity inequalities

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- By the convexity of $x \mapsto x^q$: $\qquad \|P_t\|_{q\to q} \leq 1, \qquad \forall q \geq 1$
- Do we have $\quad \|P_t\|_{q\to p} \leq 1 \quad$ for some $p > q$?
- An inequality of the above form is called a hypercontractivity inequality

Hypercontractivity inequality $\Rightarrow$ Log-Sobolev inequality

Theorem I

$p(t)$ smooth increasing function with $p(0) = q$. Let $c = (q-1)/p'(0)$

$$\|P_t\|_{q \to p(t)} \leq 1, \quad \forall t \geq 0$$

$$\Rightarrow \mathrm{Ent}(f) \leq c\, q\hat{q}\, \mathcal{E}\left(f^{1/\hat{q}},\, f^{1/q}\right), \quad \forall f > 0$$

$$\mathrm{Ent}(f) = \mathbb{E}(f \log f) - \mathbb{E}f \log \mathbb{E}f$$

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- Best constant c in LS inequality: α_q
- $\alpha_q = \alpha_{\hat{q}}$

Log-Sobolev inequality$\Rightarrow$ Hypercontractivity inequality

Theorem II

$p(t)$ smooth increasing function with $p(0) = q$. Let $c(t) = (p(t) - 1)/p'(t)$.

$$\mathrm{Ent}(f) \leq c(t)\, p(t)\hat{p}(t)\, \mathcal{E}\Big(f^{/\hat{p}(t)},\, f^{1/(p(t))}\Big), \quad \forall f > 0, \forall t$$

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Comparing LS constants

Theorem

For $1 \leq q \leq p \leq 2$

$$q\hat{q}\,\mathcal{E}\left(f^{1/\hat{q}}, f^{1/q}\right) \geq p\hat{p}\,\mathcal{E}\left(f^{1/\hat{p}}, f^{1/p}\right)$$

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- $q \mapsto \alpha_q$ is non-decreasing on $[1, 2]$
- α_2 is the largest log-Sobolev constant

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Corollary

$$\|P_t\|_{q\to p} \leq 1, \qquad \forall p, q \text{ s.t.} \qquad \frac{p-1}{q-1} \leq e^{t/\alpha_2}.$$

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- $\widetilde{\mathcal{L}} = \hat{\mathcal{L}}_1 + \cdots + \hat{\mathcal{L}}_m$ Lindblad operator for $(\widetilde{\Omega}, \widetilde{\mathcal{L}})$
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- Operator norm is multiplicative:

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- Classical conditional entropy is a convex combination of entropies & entropy is sub-additivity

Application: bounding the mixing time

Spectral gap

$$\tau_{\mathsf{mix}} := \min\{t : \|\mu P_t - \pi\|_{\mathsf{TV}} \leq \frac{1}{e}, \forall \mu\}$$

$$\|\rho\|_{\mathsf{TV}} = \sum_x |\rho(x)|$$

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$$\begin{aligned}
\|\mu P_t - \pi\|_{\text{TV}} &\le \|\frac{\mu P_t}{\pi} - 1\|_2 \\
&= \|P_t f - \mathbb{E} f\|_2 && (f = \mu/\pi) \\
&\le e^{-\lambda t}\|f - \mathbb{E} f\|_2 \\
&\le e^{-\lambda t}\sqrt{\frac{1}{\pi_{\min}}} && (\pi_{\min} = \min_{x\in\Omega} \pi(x))
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$$\tau_{\mathsf{min}} = O(\frac{1}{\lambda}\log\frac{1}{\pi_{\mathsf{min}}})$$

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Log-Sobolev constant

$$\|\mu P_t - \pi\|_{TV} \leq 2D(\mu P_t \| \pi) \qquad \text{(Pinsker's inequality)}$$

$$= 2\mathrm{Ent}(f_t) \qquad (f_t = \frac{\mu P_t}{\pi})$$

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$$\tau_{\mathsf{min}} = O(\alpha_1 \log\log \frac{1}{\pi_{\mathsf{min}}}) = O(\alpha_2 \log\log \frac{1}{\pi_{\mathsf{min}}})$$

Application: bounding the mixing time

Random transposition

- Start with $1, 2, \ldots, n$
- In each time step choose random i, j and exchange them
- Using $\tau_{\text{min}} = O(\frac{1}{\lambda} \log \frac{1}{\pi_{\text{min}}})$

$$\tau_{mix} = O(n^2 \log n)$$

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Application: Non-interactive correlation distillation

- Given two bipartite distributions π_{AB} and μ_{CD}
- Are there n and stochastic maps $T : A^n \to C$ and $S : B^n \to D$ s.t.

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- Define $V : L^2(\mu_C) \to L^2(\mu_D)$ similarly.

Theorem [Ahlswede, Gács '76]

If there are $p > q$ such that $\|U\|_{q\to p} \leq 1$ and $\|V\|_{q\to p} > 1$, then the answer is no.

Quantum hypercontractivity & log-Sobolev inequalities

- Depolorizing channels form a quantum Markov semigroup:

$$\mathcal{L}(X) := X - \mathrm{tr}X\frac{I}{d}, \qquad e^{-t\mathcal{L}}(\rho) = e^{-t}\rho + (1 - e^{-t})\frac{I}{d}$$

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- Easier when

$$\pi = \text{maximally mixed} = \frac{I}{d}$$

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- Semigroup of maps $\Phi_t : \mathcal{M}_d \to \mathcal{M}_d$

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- completely positive
  self-adjoint (reversible)
  trace preserving & unital $\quad (\pi = I/d)$

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- Tensorization doesn't hold!
    - $\|\Phi_t\|_{q\to p}$ is not multiplicative!
    - Quantum conditional entropy is not a convex combination of entropies!

Completely bounded norm

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CB-norm is multiplicative for CP maps

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(t, q)-norm

$$\|X_{RS}\|_{(t,q)} := \sup\backslash\inf_{\sigma_R}\|(\sigma_R^{-1/2r} \otimes I_S)X_{RS}(\sigma_R^{-1/2r} \otimes I_S)\|_q$$

$$\frac{1}{r} = \frac{1}{t} - \frac{1}{q}$$

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- Choice of t is arbitrary. Usually $t = q$

CB-log-Sobolev inequality

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$$\le c q \hat{q}\, \widehat{\mathrm{tr}}\left(X_{RS}^{1/\hat{q}} (\mathcal{I}_R \otimes \mathcal{L})(X_{RS}^{1/q})\right)$$

- [SB, King '15] CB-hypercontractivity $\Leftrightarrow$ CB-log-Sobolev inequality
- $\alpha_2^{\mathrm{CB}} \ge \alpha_q^{\mathrm{CB}}, \qquad \forall q$
- Tensorization holds!
  - CB-norm is multiplicative on CP maps!
  - CB-log-Sobolev inequality is already in terms of conditional entropy!

Application: bounding the mixing time

- [Kastoryano, Temme '13]: α_1 gives a bound on the mixing time of Φ_t

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- [Kastoryano, Temme '13]: α_1 gives a bound on the mixing time of Φ_t
- α_1^{CB} gives a bound on the mixing time of $\Phi_t \otimes \cdots \otimes \Phi_t$

Application: local state transformation

- Given ρ_{AB} and σ_{CD}
- Question: Is there n and $\Phi : A^n \to C$ and $\Psi : B^n \to D$ such that

$$\Phi \otimes \Psi(\rho_{AB}^{\otimes n}) = \sigma_{CD}?$$

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- If $\exists p, q$ such that

$$\|\Gamma_{\rho_B}^{\frac{1}{p'}} \circ \Lambda_\rho \circ \Gamma_{\rho_A^*}^{\frac{1}{q}}\|_{\mathrm{CB},q\to p} \leq 1 \qquad \& \qquad \|\Gamma_{\sigma_B}^{\frac{1}{p'}} \circ \Lambda_\sigma \circ \Gamma_{\sigma_A^*}^{\frac{1}{q}}\|_{\mathrm{CB},q\to p} > 1$$

  the answer is NO!

  - $\Lambda_\rho : A \to B$ is the map whose Choi matrix is ρ_{AB}
  - $\Gamma_M(X) = M^{1/2} X M^{1/2}$
  - M^* is the entry-wise complex conjugate of M

Other appliactions

- Objectives
- Confirmed Participants
- Press Release
- Meeting Facilities
- Schedule (PDF)
- Abstracts (PDF)
- Mailing List
- Workshop Videos
- Workshop Files
- Final Report (PDF)
- Testimonials

Hypercontractivity and Log Sobolev Inequalities in Quantum Information Theory (15w5098)

Arriving in Banff, Alberta Sunday, February 22 and departing Friday February 27, 2015

Organizers

Patrick Hayden (Stanford University)

Christopher King (Northeastern University)

Ashley Montanaro (University of Bristol)

Mary Beth Ruskai (delocalized)

Conclusion

- CB version of log-Sobolev inequality characterizes CB-hypercontractivity inequalities
- Application: mixing time [Kastoryano & Temme '13]
  - CB-log-Sobolev inequalities can be used to bound mixing times of multipartite systems
- Application: non-interactive correlation simulation
  - Computing the CB-hypercontractivity ribbon [Delgosha, B. '14]
- Open problem: compute the CB-log-Sobolev constant for depolorizing channels
- Open problem: generalize to non-unital channels

For further reading

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Impossibility of Local State Transformation via Hypercontractivity
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