A mathematical toolbox for resource theories

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What are resources?

Resource objects are things which can be converted into each other via processes, such as:

$$\text{timber} + \text{nails} \longrightarrow \text{table}$$

The details vary with the context:

- Industrial chemistry: $N_2 + 3\,H_2 \xrightarrow{\text{Haber process}} 2\,NH_3$
- Thermodynamics: $\text{hot gas} + \text{cold gas} \xrightarrow{\text{Carnot process}} \text{gas} + \text{work}$
- Communication: $\text{noisy channel} \xrightarrow{\text{Channel coding}} \text{perfect channel}$
- Energy economics: $\text{nuclear fuel} + \text{reactor} \xrightarrow{\text{fission}} \text{electricity} + \text{nuclear waste} + \text{reactor}$

Persistent pattern: **convertibility** and **combinability** of resource objects. One investigates problems about catalysis, rates, ...

A mathematical theory of resources

The **convertibility** and **combinability** is formalized by a mathematical structure:

Definition

An ordered commutative monoid $(A, +, 0, \geq)$ consists of a set A with the structure of

- a commutative monoid $(A, +, 0)$,
- a partial order $(A, \geq)$,
- such that addition is monotone,

$$x \geq y \quad \Rightarrow \quad x + z \geq y + z.$$

A mathematical theory of resources

Intuition:

- "$+$" describes how resource objects combine, with trivial resource object 0.
- "$a \geq b$" means that: there is a process which turns a into b.
- If $a \geq b$ and $b \geq a$, then $a = b$. Mutually interconvertible resource objects are considered equal.

Example: the resource theory of chemistry

Let **Chem** be the ordered commutative monoid in which the resource objects $a, b, \ldots \in$ **Chem** are collections of molecules like

$$2\,\mathrm{H_2O_2}, \qquad 2\,\mathrm{H_2O} + \mathrm{O_2}, \qquad \ldots,$$

with addition given by union of collections.

Declare $a \geq b$ to hold if your laboratory can perform a chemical reaction of type $a \to b$.

Example: the resource theory of chemistry

We have

$$2\,\mathrm{H_2O_2} \not\geq 2\,\mathrm{H_2O} + \mathrm{O_2},$$

but

$$2\,\mathrm{H_2O_2} + \mathrm{MnO_2} \geq 2\,\mathrm{H_2O} + \mathrm{O_2} + \mathrm{MnO_2}.$$

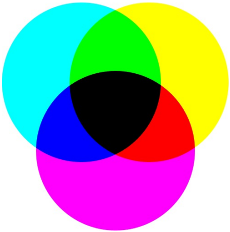
This is an example of **catalysis**, a phenomenon which may occur in any resource theory.

Example: the resource theory of paint

Let **Paint** be the ordered commutative monoid in which a resource object $a \in$ **Paint** is a collection of buckets, where each bucket in a contains paint of a certain colour.

The binary operation $+$ joins collections of buckets. The empty collection of buckets is 0.

The paints in different buckets can be mixed:



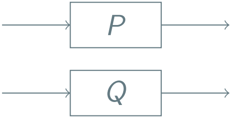
Declare $a \geq b$ to hold if b can be obtained from a by mixing paints and/or discarding buckets.

Example: the resource theory of communication

The resource objects of `CommCh` are channels (stochastic matrices) $P, Q, \ldots$ with arbitrary finite input and output alphabet.

Take $P \geq Q$ if there exist encoding and decoding channels enc and dec such

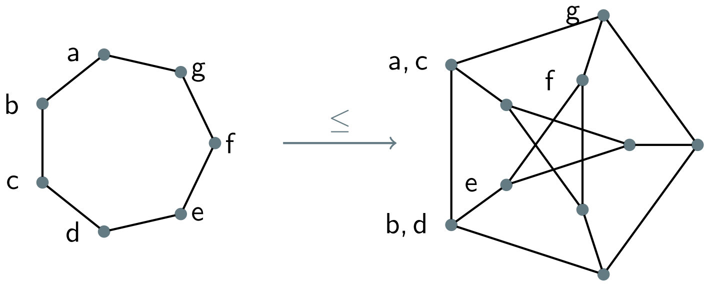
Channels combine by using them in parallel,



The binary operation "+" often has a multiplicative character!

Example: the ordered commutative monoid of graphs

The elements of `Grph` are finite graphs, ordered via the existence of a homomorphism:



They combine via disjunctive product, $V(G+H) := V(G) \times V(H)$,

$$(v, w) \sim (v', w') \quad :\Longleftrightarrow \quad v \sim v' \;\vee\; w \sim w'.$$

Example: the ordered commutative monoid of graphs

Theorem

Catalysis exists in $\mathtt{Grph}$. For example,

$$3C_5 \not\geq K_{11}, \qquad 3C_5 + (3C_5 \vee K_{11}) \geq K_{11} + (3C_5 \vee K_{11}).$$

Taking the distinguishability graph of a channel is a homomorphism

$$\mathtt{CommCh} \to \mathtt{Grph}.$$

This is closely related to zero-error communication.

Example: representation theory

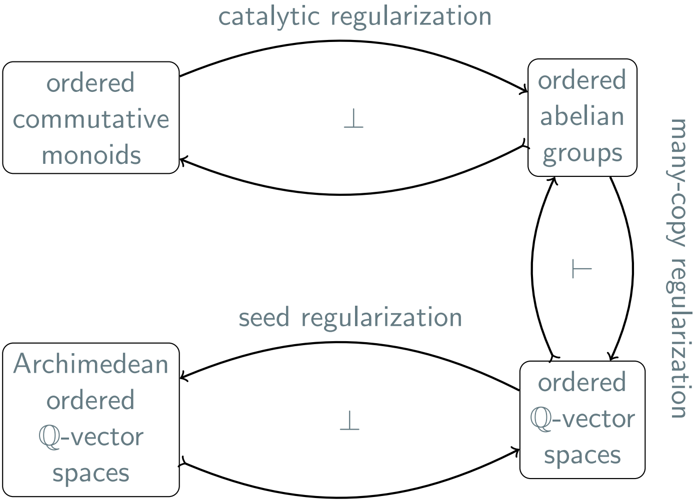
Let G be a locally compact group. The elements of $\mathrm{Rep}(G)$ are the unitary representations $\pi : G \to \mathcal{U}(H)$ for any $\mathcal{H}$.

Declare $\pi \geq \rho$ to hold whenever ρ is weakly contained in π. This turns $\mathrm{Rep}(G)$ into a semilattice with the direct sum as join.

Two representations combine via the tensor product, $\pi + \rho := \pi \otimes \rho$.

Regularizations

Ordered commutative monoids are very nasty beasts. So let's turn them into better-behaved gadgets!



The catalytic regularization

On an ordered commutative monoid A, introduce the **catalytic ordering**,

$$x \geq_{\mathrm{cat}} y \qquad :\Longleftrightarrow \qquad \exists z,\ x + z \geq y + z.$$

In this new ordering, A is guaranteed to be cancellative,

$$x + z \geq_{\mathrm{cat}} y + z \qquad \Longrightarrow \qquad x \geq_{\mathrm{cat}} y.$$

Now define an ordered abelian group $\mathtt{OAG}(A)$ as consisting of the formal differences $x - x'$, ordered as

$$x - x' \geq y - y' \qquad :\Longleftrightarrow \qquad x + y' \geq_{\mathrm{cat}} y + x'.$$

$\mathtt{OAG}(A)$ is an ordered generalization of the **Grothendieck group** of a commutative monoid.

The many-copy regularization

Let G be an ordered abelian group G. Then

$$x \geq y \qquad \Longleftrightarrow \qquad x - y \geq 0.$$

Hence it is sufficient to consider the **positive cone** G_+.

Then introduce the **many-copies ordering**,

$$x \geq_\infty 0 \qquad :\Longleftrightarrow \qquad \exists n \in \mathbb{N},\ nx \geq 0.$$

In this new ordering, G is guaranteed to be torsion-free,

$$nx \geq_\infty 0 \qquad \Longrightarrow \qquad x \geq_\infty 0.$$

Now define an ordered $\mathbb{Q}$-vector space $\mathtt{OVS}_\mathbb{Q}(G)$ as consisting of the formal fractions $\frac{1}{n}x$, ordered as

$$\tfrac{1}{n}x \geq 0 \qquad :\Longleftrightarrow \qquad x \geq_\infty 0.$$

The seed regularization

Suppose that V is an ordered $\mathbb{Q}$-vector space that has an **order unit** $u \in V_+$, i.e. for all $x \in V$ there is $\lambda \in \mathbb{Q}$ such that

$$x + \lambda u \geq 0.$$

In this case, the seed regularization has a simple description: introduce a new ordering on V by putting

$$x \geq_{\text{seed}} 0 \qquad :\Longleftrightarrow \qquad \exists \varepsilon > 0,\ x + \varepsilon u \geq 0.$$

This results in the Archimedean ordered $\mathbb{Q}$-vector space $\mathtt{AOVS}_{\mathbb{Q}}(V)$.

Now we are in the realm of functional analysis!

Rates and the rate theorem

Frequently, we would like to produce many copies of a resource object y from many copies of a resource object x. Mass production increases efficiency!

In the asymptotic limit, how many copies of y can be produced per copy of x? This is a question about **rates**:

$$R_{\max}(x \to y) = \sup \left\{ \frac{m}{n} \;\middle|\; nx \geq my \right\}$$

$$R_{\min}(x \to y) = \inf \left\{ \frac{m}{n} \;\middle|\; nx \geq my \right\}$$

Rates and the rate formula

There is a better-behaved notion of **regularized rate**: r is a regularized rate if for every $\varepsilon > 0$ there are $k, m, n \in \mathbb{N}$ with $|r - \frac{m}{n}| < \varepsilon$ and $k \leq n\varepsilon$ such that

$$nx + ku \geq my.$$

Applying the Hahn-Banach theorem to $\mathtt{AOVS}_{\mathbb{Q}}(A)$ yields:

Theorem (Rate formula)

If $x, y \geq 0$, then

$$R^{\mathrm{reg}}_{\min}(x \to y) = 0, \qquad R^{\mathrm{reg}}_{\max}(x \to y) = \inf_f \frac{f(x)}{f(y)}.$$

where $f : A \to \mathbb{R}$ ranges over all (extremal) homomorphisms.

Application to entanglement theory

Nielsen's Theorem: pure bipartite states under LOCC are described by `Major`, the ordered commutative monoid of finite probability spaces ordered by majorization.

Conjecture

The extremal functionals $\texttt{Major} \to \mathbb{R}$ are precisely the Rényi entropies,

$$H_\alpha(p) = \frac{1}{1-\alpha} \log \left(\sum_i p_i^\alpha \right)$$

If this is true, then the rate formula yields

$$R_{\max}^{\mathrm{reg}}(p \to q) = \inf_\alpha \frac{H_\alpha(p)}{H_\alpha(q)}.$$

The trouble with epsilonification

In most resource theories that come up in information theory, one does not require to outcome of a process to coincide *exactly* with a desired resource b. It is enough if arbitrarily good approximations to b can be produced.

Goal: incorporate this into the formalism and redevelop classical Shannon theory.

Problem: so far, all approaches to **epsilonification** have failed.

The trouble with epsilonification

Possible solution: maybe the standard paradigms used in information theory are not quite "right".

Conventionally, $x \geq_\varepsilon y$ means:

$$\exists y' \approx_\varepsilon y, \quad x \geq y'.$$

But maybe it should mean:

$$\forall x' \approx_\varepsilon x \quad \exists y' \approx_\varepsilon y, \quad x' \geq y'.$$

The trouble with epsilonification

Advantages of this alternative definition:

- Nicely symmetric.
- Guarantees composability on the nose,

$$x \geq_\varepsilon y, \qquad y \geq_\varepsilon z \qquad \Longrightarrow \qquad x \geq_\varepsilon z.$$

- More realistic.

This results in a family of ordered commutative monoids A_ε indexed by $\varepsilon > 0$. Regularize to $\mathtt{AOVS}_\mathbb{Q}(A_\varepsilon)$ and take the limit

$$\lim_{\varepsilon \to 0} \mathtt{AOVS}_\mathbb{Q}(A_\varepsilon).$$

Details to be worked out. Opinions?