The Principle of Locality Made Simpler but Harder

by Man-Duen Choi

choi@math.toronto.edu

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NIMS National Institute
For Mathematical Sciences | South Korea

The Principle of Locality

(1) Made Vaque but Easy

-- as a non-mathematical talk

(2) Made Simple but Hard

-- as a mathematical talk

(3) Made Simpler but Harder

-- for math specialists

- (According to *Wiki Encyclopaedia*)

In PHYSICS, the **Principle of Locality** states that an object is influenced directly only by its immediate surroundings.

- This could be translated to a very simple mathematical statement of NO wisdom at all.

- With extravagant assumptions on the obvious truth, or fascinating explanation of the ultimate nonsense, the Principle may become

a big LAW/ THEORY/ THEOREM or

an incredible PARADOX to shake your body/heart.

Outline

- Non-mathematical motivations

  The adventure of Alice and Bob in human/quantum wonderland, with highlights on locality effects.

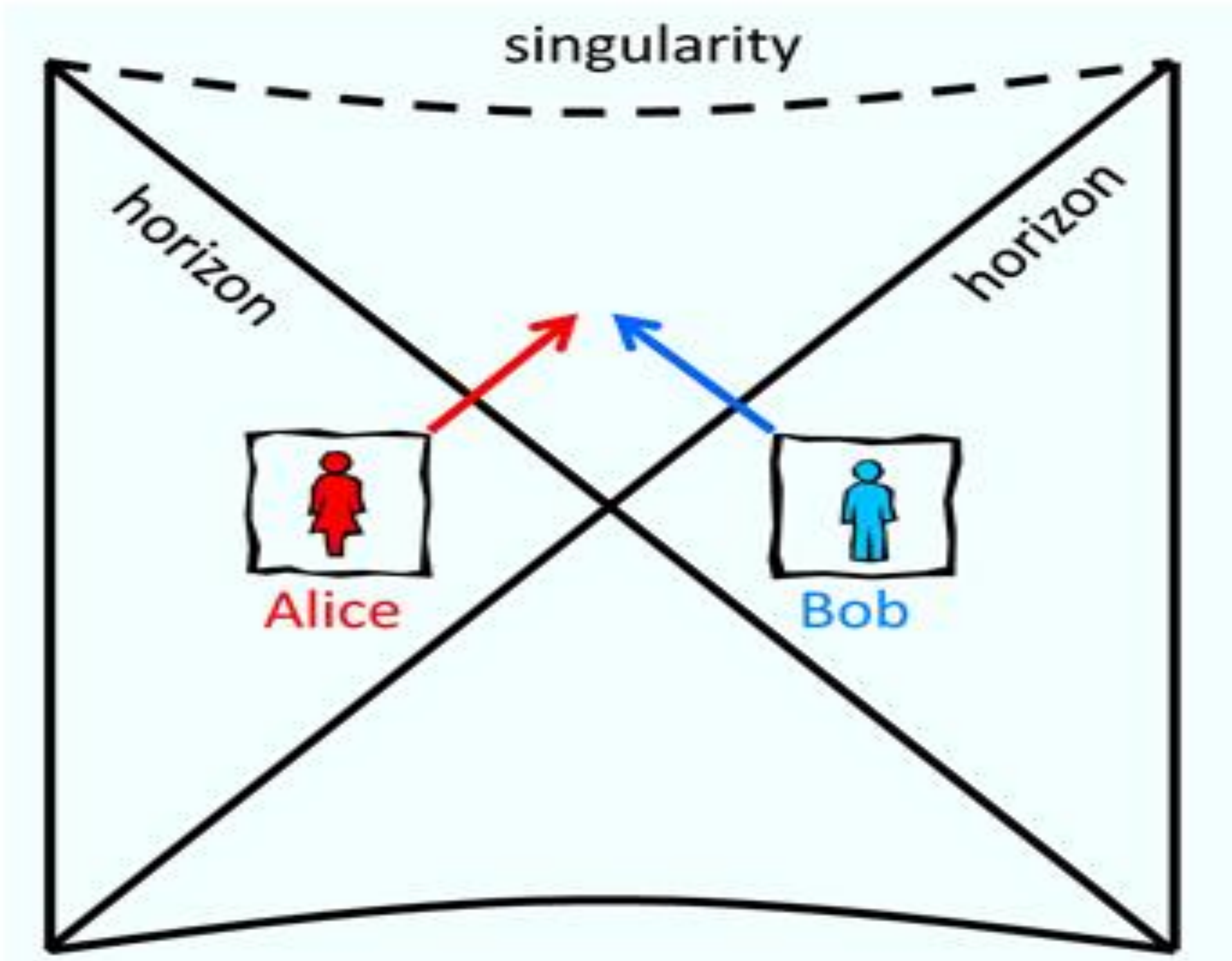
- Setup of direct sums $\oplus$
- Setup of tensor products $\otimes$
- Summary

Adventure of Alice and Bob

- Alice is a Canadian environmentalist, Bob is a System engineer of USA.

- They see everything from different perspectives. Hence, combining their local observations, they are able to provide a global view, thanks to the Principle of Locality.

Adventure of Alice and Bob



Example: North America = Canada **U** USA

- Let T be Trading (or Transportation, or Teleportation, etc) in North America.
- Alice is a Canadian, and she reads T as a single operator A in Canada, while Bob reads T as a single operator B in USA.

Together, they regard T as A $\oplus$ B.

- What is wrong? They ignore all inter-national effects, (issues of free-trade, tax-free ...)
- What is right? They exert themselves to get the best possible global view of T.

(Thanks to the Principle of Locality, their pride and prejudice could become sense and sensibility for all situations.)

Environment versus System

- Another Setup:

Whole Quantum world = Environment x System.

- Each phenomenon T, as an operator on the whole world, looks very complicated.

- Alice sees T as a single operator A on the environment space , while Bob sees T as a single operator B on the system space.
- Together (combining two different perspectives), they see T as simple as $A \otimes B$ (thanks again to the Principle of Locality).

- Similar considerations of Stress vs Strain; Motions vs Spinning; …

More Jargons *(continued)*

- Einstein was skeptical of anything sort of the spooky action at a distance. The 1935 EPR (Einstein-Podolsky-Rosen) Paradox is a disproof of the Principle of Locality.

- The Bell Theorem/Bell's Inequality (1964) shows by a quantitative measure that there exists a phenomenon T not of the form $A \otimes B$.

- Main Challenge: Any logical mathematical justification/rigor to make sense for the Principle of Locality?

Math Settings of Hilbert Spaces

Two different notions of locality, by means of

$\oplus$ and $\otimes$

- $L^2(X \,\mathbf{U}\, Y) = L^2(X) \oplus L^2(Y)$, $L^2(X\mathbf{x}Y) = L^2(X) \otimes L^2(Y)$.

❖ Often concerned about finite-dimensional Hilbert spaces as $\mathbf{C}^n$ for different positive integer n.

Thus $\mathbf{C}^n \oplus \mathbf{C}^k = \mathbf{C}^{n+k}$, $\mathbf{C}^n \otimes \mathbf{C}^k = \mathbf{C}^{nk}$.

➢ All Hilbert spaces (and all finite –dimensional C*-algebras) form a semi-ring with $\oplus$ and $\otimes$

Basic notion of **Locality**, in terms of **Direct Sums**

- $H = H_1 \oplus H_2$
  - T $\in$ *B(H)* can be written in terms of A $\in$ $B(H_1)$, B $\in$ $B(H_2)$, C $\in$ $B(H_2, H_1)$, $D \in B(H_1, H_2)$ as

$$T = \begin{bmatrix} A & C \\ D & B \end{bmatrix}$$

➢ **Main Concern:** In what ways, can whole T be influenced directly only by the locality (A,B)?

---In spite of the presence of C and D.

TRACE to be most useful for Principle of Locality

- **Def**: The *trace* τ of $A \in M_n$ is

  $\tau(A)$ = the sum of all diagonal entries of A.

- τ is the most natural linear functional on M_n satisfying $\tau(AB) = \tau(BA)$

- **Basic Fact:** $\tau(A)$ = sum of eigenvalues of A

- **Key Observation:** Given $T = \begin{bmatrix} A & C \\ D & B \end{bmatrix}$

  Then $\tau(T) = \tau(A) + \tau(B)$.

  Thus, the sum of eigenvalues of T is the sum of eigenvalues of A and eigenvalues of B.

- This is the SIMPLEST result showing the Principle of Locality is valid (by Mathematics).

THE OPERATOR NORM is always useful for locality

- **Basic Fact:** $||T|| \geq \max\{ ||A||, ||B||\}$

This provides a quick quantitative statement showing how the GLOBAL T is so different from its LOCALITY.

- **Concern**: Any better theorem to show how $||T||$ is influenced directly only by $\{ ||A||, ||B||\}$?

Obviously, we need extra assumption so that T is well behaved/disciplined/mannered.

- **Sample Theorem**: If T is positive semidefinite, then

$||T|| \leq ||A|| + ||B||$.

Many Important Mathematical results along these lines

- **Main Problem**: Let $T = \begin{bmatrix} A & C \\ D & B \end{bmatrix}$.

If T is well behaved (such as T is unitary/ positive semi-definite/ a projection /normal …), how should (A, B, C, D) be related?

- **Paraphrasing:**

Can A determine all possible (B,C,D)?
Can (A, B) determine (C, D)?
How much can (A, B) influence T?

- There are so many old and new problems/results concerning the Principle of Locality in Mathematics

Generalizations in the setting of direct sums

Such as re-definiton of

North America = Canada **U** USA **U** Mexico

- Then, in the setting of $H = H_1 \oplus H_2 \oplus H_3$

$$T = \begin{bmatrix} A & D & E \\ F & B & G \\ H & J & C \end{bmatrix},$$

- How does the locality (A, B, C) influence the global T?
- More deeper results of harder cases beyond physics.

Query: What does Locality mean for $\otimes$

Consider $\mathbf{C}^6 = \mathbf{C}^2 \otimes \mathbf{C}^3$. There is **no natural/canonical** way to put $\mathbf{C}^2$ into $\mathbf{C}^6$ as a linear subspace.

- Thus, we must think of a **DIFFERENT** concept of LOCALITY in the setting of tensor products.

- **Warning:**
  Mathematical ambiguity <=> Physical uncertainty

- **Big Challenge**: What is the down-to-earth meaning of locality? We need to think better (than Einstein and Bell), in order to understand the deep structure theory of the Principle of Locality of Quantum Information?

- Without basic training of mathematical notion of tensor products, anybody will never understand locality.

The Only Way to define *Locality* in the *Quantum World*

❖ Make use of the TRACE function to go through all matrices.

➢ **Query**: Given $T \in M_n \otimes M_k$,
in what manner, should Alice and Bob see T differently as $A \in M_n$, and $B \in M_k$ locally?

➢ **Answer**: -- Apply the trace functions τ to different tensor-product components!

Namely, $id \otimes \tau$ changes $M_n \otimes M_k$ to M_n ,
while $\tau \otimes id$ changes $M_n \otimes M_k$ to M_k .

Concrete computation:

Given $T \in M_2 \otimes M_3 = M_2(M_3) = M_6$

as $T = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}$ with X,Y, Z, and W ∈ M_3.

- Alice will read T as a 2 x 2 matrix $A = \begin{bmatrix} \tau(X) & \tau(Y) \\ \tau(Z) & \tau(W) \end{bmatrix}$

- while Bob will read T as a 3 x 3 matrix $B = X + W$

- Together, they will read T as simple as

$$A \otimes B = \begin{bmatrix} \tau(X)\,(X + W) & \tau(Y)\,(X+W) \\ \tau(Z)\,(X + W) & \tau(W)\,(X + W) \end{bmatrix} \neq T$$

RECAP

- $M_n \otimes M_k$ is very wild because of quantum entanglements.
- Let $\mathbb{S}$ be the special semi-group of $M_n \otimes M_k$, consisting of UNTANGLED elements $\{A \otimes B: A \in M_n, B \in M_k\}$.

- Then each $T \in M_n \otimes M_k$ corresponds to a canonical $S \in \mathbb{S}$ which resembles T most.

- Namely, T of the form $\sum_j A_j \otimes B_j$ --- fully ENTANGLED--- can be simplified (or viewed-down) as UNTANGLED $A \otimes B$, where (A, B) is the pair of locality for T.

- Moreover, the anticipated Principle of Locality says that with extra assumption (such as T is well behaved or disciplined), T will be dominated by the simple S.

Aside

A Metaphysical Elephant

ASIDE

A Metaphysical Elephant

What is ANALYSIS for a phenomenon T?

- The FULL expression

$$T = \sum_j A_j \otimes B_j$$

is incomprehensible to everybody.

- Should seek the help of experts, like Alice and Bob, to provide operators A and B from their perspectives.

- Then, by Principle of Locality, the useful information $A \otimes B$ serves BEST for COMMUNICATION (in order to describe T).

NEW PROBLEMS OF UNKNOWN DEPTH.

- **Setting for quantum Information**:
  Suppose T is a density matrix (i.e. a positive semi-definite matrix of trace 1) in $M_n \otimes M_k$.
  Then locally, there exist a unique pair of trace-1 positive semi-definite matrices $A \in M_n$, $B \in M_k$ such that $A \otimes B$ serves as the locality of T.

- **Major Question**: How to classify T, based on information of (A, B) only?

- **Sample Result:** Assume further that A is rank-1, then $T = A \otimes B$ exactly.

SUMMARY: Two Kinds of Locality

$\oplus$ for human world, and $\otimes$ for quantum world.

Never mix up!!! $\otimes$ is NEVER a generalization of $\oplus$.

- Given two matrices $A \in M_n$ and $B \in M_k$, then

(a) $A \oplus B$ stands for the locality of a big class of matrices $T \in M_{n+k}$ acting on $\mathbf{C}^n \oplus \mathbf{C}^k$;

(b) $A \otimes B$ stands for the locality for a big class of matrices $T \in M_{nk} = M_n \otimes M_k$ acting on $\mathbf{C}^n \otimes \mathbf{C}^k$.

- Conversely, (thanks to the possible Principle of Locality).

(a) Given $T \in M_{n+k}$ acting on $\mathbf{C}^n \oplus \mathbf{C}^k$, then T looks like $A \oplus B$.

(b) Given $T \in M_{nk} = M_n \otimes M_k$ acting on $\mathbf{C}^n \otimes \mathbf{C}^k$, then T looks like $A \otimes B$.

EPILOGUE