Positive Maps in Quantum Information Theory

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Preface

Positive linear functionals and positive linear maps had been playing important roles in the theory of operator algebras, which reflect noncommutative order structures. Such order structures provide basic mathematical frameworks for current quantum information theory. The main purpose of this lecture note is to introduce basic notions like separability/entanglement and Schmidt numbers from quantum information theory in terms of positive maps between matrix algebras.

Basic tools are Choi matrices and duality arising from bilinear pairing between matrices. We begin with concrete examples of positive maps Ad_s which sends x to s^*xs, and define separability/entanglement and Schmidt numbers in terms of Choi matrices. We also use duality to introduce various kinds of positivity, like k-positivity and complete positivity. Positive maps which are not completely positive are indispensable tools to detect entanglement through the duality.

In Chapter 1, we introduce the above notions and exhibit nontrivial examples of positive maps. We also provide a unified argument to recover various known criteria through ampliation. We will focus in Chapter 2 on the issue how positive maps detect entanglement. Through the discussion, exposed faces of the convex cones of all positive maps play important roles. We exhibit three classes of positive maps, by Choi, Woronowicz and Robertson in 1970's and 1980's which generate exposed extreme rays of the convex cone of all positive linear maps.

This is the collection of lecture notes during the fall semester of 2022, at Seoul National University, Seoul, Korea. The author tried to minimize preliminaries, requiring only undergraduate linear algebra. The author is grateful to all the audiences for their feedbacks on the notes. Special thanks are due to Kyung Hoon Han for his careful reading of the drafts. Nevertheless, any faults in this lecture notes are responsibility of the author.

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Chapter 1

Positive Maps and Bi-partite States

In this chapter, we introduce various kinds of positive maps between matrix algebras together with the corresponding notions for the tensor products of matrix algebras. We begin with the most elementary positive maps Ad_s for given matrices s, which send x to s^*xs. These maps give rise to important classes of positive maps according to the ranks of s, by taking convex combinations. Positive maps arising in this way with matrices whose ranks are at most k are called k-superpositive maps, and all of them make the class of completely positive maps which play crucial roles. Another important positive map is given by the transpose, which makes the class of decomposable positive maps together with completely positive maps.

For a given linear map between matrix algebras, we assign a matrix in the tensor product of matrices, or equivalently a block matrix, which is called the Choi matrix. The block matrices corresponding to completely positive maps are precisely positive (semi-definite) matrices, which represent bi-partite states after normalizing by the trace. The class of 1-superpositive maps corresponds to separable states. Nonseparable states are called entanglement, which is now recognized as one of the most important resources in the current quantum information theory.

Another main tool is the bilinear pairing between matrices which gives rise to the bilinear pairing between linear maps through the Choi matrices. The dual notion of k-superpositivity by this bilinear pairing is k-positivity, which has been studied among operator algebraists since Stinespring's representation theorem in the 1950's. The usual positivity of linear maps coincides with 1-positivity arising in this way. Therefore, the positivity of linear maps between matrix algebras is just the dual notion of separability of bi-partite states, and this is why positive maps play important roles to detect entanglement.

After we fix notations together with several preliminaries in Section 1.1, we show in Section 1.2 that the map Ad_s generates an extreme ray of the convex cone of all positive linear maps, and introduce the notion of k-superpositive maps and decomposable maps. In Section 1.3, we show that all positive maps are decomposable in low dimensional cases like maps between 2×2 matrices. In Section 1.4, we define Choi matrices through which we introduce various kinds of bi-partite states, under the name of Schmidt numbers. We define the bilinear pairing between linear maps in Section 1.5 and get the notion of k-positivity which is dual to k-superpositivity. During the discussion, we see that complete positivity of maps corresponds to positivity of block matrices through the Choi matrices. Sections 1.6 and 1.7 will be devoted to introduce various examples which distinguish k-superpositivity and k-positivity of maps together with the corresponding states which include Werner states and isotropic states. We also see concrete examples of indecomposable positive maps. Finally, we introduce the notion of mapping cones in Section 1.8 with which we obtain various characterizations of aforementioned notions in terms of ampliation which is the tensor product with the identity map. Our exposition of topics reverses historical development in parts, and this is why we add historical remarks in Section 1.9.

1.1 Preliminaries

1.1.1 Vectors and matrices

A vector in the vector space $\mathbb{C}^n$ over the complex field is denoted by a *ket* $|v\rangle$, which may be understood as a column vector or an $n \times 1$ matrix;

$$|v\rangle = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{C}^n.$$

The adjoint of ket $|v\rangle$ is denoted by a *bra*$\langle v|$, that is,

$$\langle v| = (\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_n) \in \mathbb{C}^n.$$

For given two vectors $|v\rangle$ and $|w\rangle$ in $\mathbb{C}^n$, we have

$$\langle v|w\rangle = \bar{v}_1 w_1 + \bar{v}_2 w_2 + \cdots + \bar{v}_n w_n,$$

and so $\langle v|w\rangle$ is the standard inner product of $\mathbb{C}^n$ which is linear in the second variable and conjugate-linear in the first variable. So, $\langle v|$ may be considered as the linear functional on $\mathbb{C}^n$ which sends $|w\rangle$ to $\langle v|w\rangle$. On the other hand, we see that

$$|v\rangle\langle w| = \begin{pmatrix} v_1\bar{w}_1 & v_1\bar{w}_2 & \cdots & v_1\bar{w}_n \\ v_2\bar{w}_1 & v_2\bar{w}_2 & \cdots & v_2\bar{w}_n \\ \vdots & \vdots & \ddots & \vdots \\ v_m\bar{w}_1 & v_m\bar{w}_2 & \cdots & v_m\bar{w}_n \end{pmatrix}$$

is an $m \times n$ matrix of rank one, for $|v\rangle \in \mathbb{C}^m$ and $|w\rangle \in \mathbb{C}^n$. This is nothing but

$$|v\rangle\langle w| : |u\rangle \mapsto |v\rangle\langle w|u\rangle = \langle w|u\rangle|v\rangle \in \mathbb{C}^m, \qquad |u\rangle \in \mathbb{C}^n,$$

as a linear map from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the one dimensional range space spanned by $|v\rangle$. The set $M_{m\times n}$ of all $m \times n$ matrices is a vector space over the complex field. When $m = n$, we use the notation M_n, which is usually called the *matrix algebra* since two square matrices with the same size can be multiplied.

The vectors appearing in the standard orthonormal basis $\{e_i : i = 1, 2, \ldots, n\}$ of $\mathbb{C}^n$ are denoted by

$$|i\rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{-th},$$

so we may write $|v\rangle = \sum_{i=1}^n v_i|i\rangle$. If $|i\rangle \in \mathbb{C}^m$ and $|j\rangle \in \mathbb{C}^n$ then

$$|i\rangle\langle j|$$

is the $m\times n$ matrix whose entries are zeros except for (i, j)-entry 1. So, the collection

$$\{|i\rangle\langle j| : i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n\}$$

is the standard *matrix units*. The $m \times n$ matrix $A = [a_{ij}]$ may be written by

$$A = \sum_{i,j} a_{ij}|i\rangle\langle j| = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

When we consider concrete examples, we sometimes use notations $|0\rangle, |1\rangle, \ldots,$ beginning with $|0\rangle$ for orthonormal basis. In this case, we have

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^2, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{C}^2$$

for the two dimensional case.

1.1.2 Positive matrices

A self-adjoint $n \times n$ matrix A is called *positive semi-definite* or just *positive* if

$$\langle x|A|x\rangle \geqslant 0 \tag{1.1}$$

for every $|x\rangle \in \mathbb{C}^n$. Every $|x\rangle \in \mathbb{C}^n$ gives rise to the rank one matrix $|x\rangle\langle x|$, which is positive since

$$\langle y|x\rangle\langle x|y\rangle = |\langle x|y\rangle|^2 \geqslant 0$$

for every $|y\rangle \in \mathbb{C}^n$. If $|x\rangle$ is a unit vector then $|x\rangle\langle x|$ is the projection onto the one dimensional space spanned by $|x\rangle$. By the spectral decomposition, we know that every positive matrix can be written by

$$\sum_\iota |x_\iota\rangle\langle x_\iota|$$

with a finite family $\{|x_\iota\rangle\}$ of vectors. It should be noted that this expression is far from being unique. By another application of the spectral decomposition, we know that a positive matrix is the square of a matrix. We write $A \leqslant B$ for Hermitian matrices A and B if $B - A$ is positive.

For two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ with the same sizes, we define

$$\langle A, B\rangle = \mathrm{Tr}\,(AB^{\mathrm{T}}) = \sum_{i,j} a_{ij} b_{ij},$$

where B^{T} and $\mathrm{Tr}\,(AB^{\mathrm{T}})$ denote the transpose of B and the trace of AB^{T}, respectively. This is a bilinear pairing which is *non-degenerate*, that is, satisfies

$$\langle A, B\rangle = 0 \text{ for every } B \iff A = 0.$$

It should be noted that $\langle A, A\rangle$ may be negative. For an example, we take a self-adjoint

$$A = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix},$$

to get $\langle A, A\rangle = -2 < 0$. We have the following relations

$$\langle A, BC\rangle = \langle AC^{\mathrm{T}}, B\rangle = \langle B^{\mathrm{T}} A, C\rangle,$$

whenever the sizes of matrices are given so that the above relation is meaningful.

We also note that the identity

$$|x\rangle\langle y|^{\mathrm{T}} = \langle y|^{\mathrm{T}} |x\rangle^{\mathrm{T}} = |\bar{y}\rangle\langle \bar{x}|$$

holds with the notation $|\bar{x}\rangle = \sum_i \bar{x}_i |i\rangle$. Therefore, we also have

$$\begin{aligned} \langle A, |\bar{x}\rangle\langle \bar{y}|\rangle &= \mathrm{Tr}\,(A|\bar{x}\rangle\langle \bar{y}|^{\mathrm{T}}) \\ &= \mathrm{Tr}\,(A|y\rangle\langle x|) \\ &= \mathrm{Tr}\,(\langle x|A|y\rangle) = \langle x|A|y\rangle. \end{aligned} \tag{1.2}$$

Especially, we have

$$\langle x|A|x\rangle = \langle A, |\bar{x}\rangle\langle \bar{x}|\rangle$$

for every $A \in M_n$ and $|x\rangle \in \mathbb{C}^n$. Therefore, we see that $A \in M_n$ is positive if and only if $\langle A, B\rangle \geqslant 0$ for every positive B.

We note that every linear functional ϱ on the space M_n is of the form

$$\varrho : A \mapsto \langle A, B_\varrho\rangle, \qquad A \in M_n, \tag{1.3}$$

for an $n \times n$ matrix B_ϱ. Then ϱ is *positive*, that is, sends a positive matrix to a nonnegative real numbers if and only if B_ϱ is positive. Furthermore, it is easily seen that ϱ is *unital*, that is, sends the identity matrix to 1, if and only if B_ϱ is of trace one. A positive matrix ϱ of trace one is called a *density matrix*, or a *state*, which is of the form

$$\varrho = \sum_i p_i |\xi_i\rangle\langle\xi_i|, \tag{1.4}$$

for a probability distribution $\{p_i\}$ with $\sum_i p_i = 1$ and a family $\{|\xi_i\rangle\}$ of unit vectors. We note that the set, denoted by $\mathcal{D}_n$, of all states in M_n is a convex set whose elements are convex combinations of one dimensional projections $\{|\xi\rangle\langle\xi|\}$ with unit vectors $\{|\xi\rangle\}$, which are called *pure states*. Sometimes, a positive matrix itself is called an *(unnormalized) state*.

1.1.3 Tensor products

An $m \times n$ matrix $A = \sum_{i,j} a_{ij}|i\rangle\langle j|$ corresponds to the vector

$$\langle\tilde{A}| = \sum_{i,j} a_{ij}\langle i|\langle j| \in \mathbb{C}^m \otimes \mathbb{C}^n, \tag{1.5}$$

where we usually delete the tensor notation in $\langle i| \otimes \langle j|$ to write as $\langle i|\langle j|$, or even as $\langle ij|$. If we endow $\{\langle i|\langle j|\}$ with the lexicographic order then $\langle\tilde{A}|$ is the concatenation

$$\langle\tilde{A}| = (a_{11}, \dots, a_{1n}, a_{21}, \dots, a_{2n}, \dots, \dots, a_{m1}, \dots, a_{mn}) \in \mathbb{C}^{mn}$$

of row vectors of the matrix A.

The tensor product $A \otimes B$ of two matrices A and B is defined by

$$A \otimes B = \left(\sum_{ij} a_{ij}|i\rangle\langle j|\right) \otimes \left(\sum_k b_{k\ell}|k\rangle\langle\ell|\right) = \sum_{i,j,k,\ell} a_{ij}b_{k\ell}|i\rangle|k\rangle\langle j|\langle\ell|.$$

If we endow $\{|i\rangle|k\rangle\}$ and $\{|j\rangle|\ell\rangle\}$ with the lexicographic orders again, then $A \otimes B$ has the matrix form. For example, when both A and B are 2×2 matrices, we have

$$A \otimes B = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}.$$

In this way, a matrix in $M_m \otimes M_n$ may be identified as an $m \times m$ *block matrix* whose entries are $n \times n$ matrices. So, we may identify $M_m \otimes M_n$ with $M_m(M_n)$. Especially, $I_m \otimes B$ is the $m \times m$ diagonal block matrix whose diagonal block is given by $B \in M_n$, where I_m denotes the $m \times m$ identity matrix. Sometimes, we denote by I for the identity matrix. On the other hand, $A \otimes I_n$ is the $m \times m$ block matrix whose (i, j) block is given by $a_{ij} I_n \in M_n$. Note that

$$\begin{aligned}(A \otimes B)|j\rangle|\ell\rangle &= \sum_{i,k} a_{ij} b_{k\ell} |i\rangle|k\rangle \\ &= \left(\sum_i a_{ij}|i\rangle\right) \otimes \left(\sum_k b_{k\ell}|k\rangle\right) = A|j\rangle \otimes B|\ell\rangle,\end{aligned}$$

and so $A \otimes B$ sends $|j\rangle|\ell\rangle$ to $A|j\rangle \otimes B|\ell\rangle$ as a linear map from $\mathbb{C}^m \otimes \mathbb{C}^n$ into itself.

The following identities are easily checked;

$$\begin{aligned}&(C \otimes D)^{\mathrm{T}} = C^{\mathrm{T}} \otimes D^{\mathrm{T}}, \\ &(A \otimes B)(C \otimes D) = AC \otimes BD, \\ &\mathrm{Tr}\,(A \otimes B) = \mathrm{Tr}\,(A)\mathrm{Tr}\,(B),\end{aligned}$$

whenever they are defined. Suppose that A and C are $m \times n$ matrices, and B, D are $k \times \ell$ matrices. Using the above relations, we also have the identity

$$\langle A \otimes B, C \otimes D\rangle = \langle A, C\rangle\langle B, D\rangle. \tag{1.6}$$

For linear maps $\phi_1 : M_{m_1} \to M_{n_1}$ and $\phi_2 : M_{m_2} \to M_{n_2}$, we define the linear map

$$\phi_1 \otimes \phi_2 : M_{m_1} \otimes M_{m_2} \to M_{n_1} \otimes M_{n_2}$$

as follows: Every element of $M_{m_1} \otimes M_{m_2}$ is given by $A = \sum_{i,j=1}^{m_1} |i\rangle\langle j| \otimes A_{ij}$ with $A_{ij} \in M_{m_2}$ in a unique way. We define

$$(\phi_1 \otimes \phi_2)(A) = \sum_{i,j=1}^{m_1} \phi_1(|i\rangle\langle j|) \otimes \phi_2(A_{ij}) \in M_{n_1} \otimes M_{n_2}.$$

Then we see that $\phi \otimes \psi$ sends $A \otimes B \in M_{m_1} \otimes M_{m_2}$ to $\phi(A) \otimes \psi(B) \in M_{n_1} \otimes M_{n_2}$.

1.1.4 Singular value decomposition

Let A be an arbitrary $m \times n$ matrix. Then A^*A is an $n \times n$ positive matrix with eigenvalues

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r > 0 = \lambda_{r+1} = \cdots = \lambda_n,$$

with $r \leqslant m$ and $r \leqslant n$. Denoting by D the $r \times r$ diagonal matrix with the diagonal entries $\lambda_1, \dots, \lambda_r$, we have

$$V^*(A^*A)V = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

for an $n \times n$ unitary matrix V. We write $V = \begin{pmatrix} V_1 & V_2 \end{pmatrix}$ with the $n \times r$ matrix V_1 and the $n \times (n-r)$ matrix V_2. Then we have

$$V_1^*(A^*A)V_1 = D \in M_r, \qquad V_2^*(A^*A)V_2 = 0 \in M_{n-r}.$$

Define the $m \times r$ matrix U_1 by $U_1 = AV_1D^{-1/2}$ with the obvious meaning for $D^{-1/2}$. Then we have

$$U_1^*U_1 = D^{-1/2}V_1^*A^*AV_1D^{-1/2} = I_r,$$

and so we can take an $m \times (m-r)$ matrix U_2 so that $U = \begin{pmatrix} U_1 & U_2 \end{pmatrix}$ is an $m \times m$ unitary. We have

$$\begin{aligned} U_1^*AV_1 &= U_1^*AV_1D^{-1/2}D^{1/2} = U_1^*U_1D^{1/2} = D^{1/2}, \\ U_2^*AV_1 &= U_2^*AV_1D^{-1/2}D^{1/2} = U_2^*U_1D^{1/2} = 0. \end{aligned}$$

From $V_2^*(A^*A)V_2 = 0$, we also have $AV_2 = 0$, and so we have

$$U^*AV = \begin{pmatrix} U_1^*AV_1 & U_1^*AV_2 \\ U_2^*AV_1 & U_2^*AV_2 \end{pmatrix} = \begin{pmatrix} D^{1/2} & 0 \\ 0 & 0 \end{pmatrix},$$

as $m \times n$ matrices.

Therefore, the $m \times n$ matrix A is written by

$$A = U\Sigma V^*, \tag{1.7}$$

with the matrix Σ given by

$$\Sigma = \sum_{k=1}^{r} \lambda_k^{1/2} |k\rangle\langle k| \in M_{m\times n},$$

and unitary matrices $U \in M_m$ and $V \in M_n$. This is known as the *singular value decomposition* of A, and $\lambda_1^{1/2}, \dots, \lambda_r^{1/2}$ are called the *singular values* of A. By the identity

$$\Sigma = \begin{pmatrix} D^{1/2} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} D^{1/2} & 0 \\ 0 & I_{n-r} \end{pmatrix},$$

we also see that A can be written as

$$A = U \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} W, \tag{1.8}$$

with invertible matrices $U \in M_m$ and $W \in M_n$. This can be also obtained from row and column elementary operations.

References: [11]

1.1.5 Ranks

For a matrix A, we denote by $\operatorname{Im} A$ the range space which is the span of column vectors. The span of the conjugates of row vectors is the orthogonal complement of the kernel space $\ker A$. By the dimension theorem, the span of columns and the span of rows share the same dimension, which is called the *rank* of A and is denoted by $\operatorname{rank} A$. We see that $\ker A^*A = \ker A$ which implies $\operatorname{rank} A = \operatorname{rank} A^*A$, and so the number r in the singular value decomposition (1.7) is the rank of A.

When $A = \sum_{i,j} a_{ij}|i\rangle\langle j|$, the j-th column vector is given by

$$A|j\rangle = \sum_i a_{ij}|i\rangle.$$

Take any basis $\{|\xi_k\rangle : k = 1, 2, \ldots, \operatorname{rank} A\}$ of $\operatorname{Im} A$ and write $A|j\rangle = \sum_k s_{jk}|\xi_k\rangle$, then we have

$$A = AI_n = \sum_{j=1}^{n} A|j\rangle\langle j| = \sum_k |\xi_k\rangle \left(\sum_j s_{jk}\langle j| \right).$$

Therefore, we have

$$A = \sum_{k=1}^{\operatorname{rank} A} |\xi_k\rangle\langle \eta_k| \tag{1.9}$$

with $\langle \eta_k| = \sum_j s_{jk}\langle j|$, and we see that it is possible to express A with the sum of a family of rank one matrices with the cardinality $\operatorname{rank} A$. In the singular value decomposition (1.7), we take column vectors $\{|\xi_i\rangle\}$ and $\{|\eta_j\rangle\}$ of the unitary matrices U and V, respectively. Then we have

$$\begin{aligned} A &= \left(\textstyle\sum_i |\xi_i\rangle\langle i|\right) \left(\textstyle\sum_k \lambda_k^{1/2}|k\rangle\langle k|\right) \left(\textstyle\sum_j |\eta_j\rangle\langle j|\right)^* \\ &= \left(\textstyle\sum_i |\xi_i\rangle\langle i|\right) \left(\textstyle\sum_k \lambda_k^{1/2}|k\rangle\langle k|\right) \left(\textstyle\sum_j |j\rangle\langle \eta_j|\right) = \sum_{k=1}^{\operatorname{rank} A} \lambda_k^{1/2}|\xi_k\rangle\langle \eta_k|. \end{aligned}$$

Therefore, vectors $\{|\xi_k\rangle\}$ and $\{|\eta_k\rangle\}$ in (1.9) can be chosen to be orthogonal.

A vector of the form $|\xi\rangle \otimes |\eta\rangle$ in $\mathbb{C}^m \otimes \mathbb{C}^n$ is called a *product vector* which corresponds to the $m \times n$ matrix $|\xi\rangle\langle \bar{\eta}|$ of rank one under the correspondence (1.5). The *Schmidt rank* of a vector $|\zeta\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$, denoted by $\operatorname{SR}|\zeta\rangle$, is the smallest number of product vectors whose sum is $|\zeta\rangle$. By the correspondence (1.5) and the expression (1.9), we see that the Schmidt rank coincides with the rank of the corresponding matrix.

1.1.6 Boundary of a convex set

Suppose that C is a nonempty convex set in a real vector space $\mathbb{R}^\nu$. A point x of C is called a *relative interior point* or just an *interior point* of C if for each $y \in C$ there

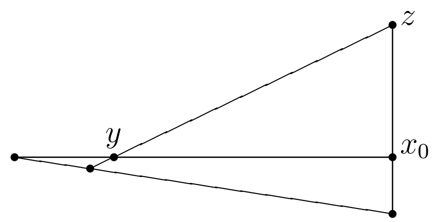

Figure 1.1: If the line segment from an interior point x_0 to y can be extended, then the line segment from any point z to y also can be extended

is $t > 1$ such that $(1-t)y + tx \in C$. This means that the line segment from every $y \in C$ to x may be extended within C. The set of all interior points of C will be denoted by $\operatorname{int} C$, which coincides with the relative interior of C with respect to the affine manifold generated by C. From this, one may see that $\operatorname{int} C$ is nonempty for any nonempty convex set C. A point $y \in C$ which is not an interior point is called a *boundary point*, and the set of all boundary points of C is denoted by ∂C.

We fix an interior point x_0 of a convex set C. Suppose that $y \in C$ satisfies the condition $\sup\{t : (1-t)x_0 + ty \in C\} = 1$. That is, we suppose that the line segment from x_0 to y cannot be extended within C. Then it is clear that y is a boundary point of C. Suppose that y does not satisfy the condition. Then there is $t > 1$ such that $(1-t)x_0 + ty \in C$. For an arbitrary $z \in C$ we can take $s < 0$ such that $(1-s)x_0 + sz \in C$ since x_0 is an interior point. Put $r = \frac{s(t-1)}{t-s}$. Then we see that $r < 0$ and

$$(1-r)y + rz = \frac{1-s}{t-s}[(1-t)x_0 + ty] + \frac{t-1}{t-s}[(1-s)x_0 + sz] \in C.$$

This shows that y is an interior point of C. See Figure 1.1. Therefore, we conclude that y is an interior point of C if and only if the line segment from a single interior point x_0 to y can be extended within C.

A nonempty convex subset F of a convex set C is called a *face* of C if the following property

$$x_0, x_1 \in C,\ 0 < t < 1,\ (1-t)x_0 + tx_1 \in F \implies x_0, x_1 \in F$$

holds. This means that if an interior point of a line segment in C belongs to F then the line segment itself is contained in F. An *extreme point* is nothing but a face consisting of a single point. A face F of a convex set C is called *proper* if it is a proper subset of C. If a point of a face F is an interior point of C then we see that $F = C$. Therefore, a proper face must be contained in the boundary. The intersection of an arbitrary family of faces is again a face whenever it is nonempty, and so every point x of a convex set has the smallest face F containing the point.

We see that x is an interior point of F, and a convex set is completely partitioned into interiors of faces.

Suppose that C is a compact convex set. If $x \in C$ is not an extreme point then it is an interior point of a line segment. We extend this line segment until it meets the boundary, and so we see that x is a convex combination of two boundary points. If one of them is not an extreme point and contained in a proper face then we express it as a convex combination of two boundary points of the face. We continue this process to see that every point of a compact convex set in a finite dimensional space is a convex combination of extreme points. Carathéodory theorem tells us that every point of a d-dimensional convex set is the convex combination of extreme points with the cardinality at most $d + 1$. An important consequence is that the convex hull of a compact set in a finite dimensional space is again compact. It should be noted that this is not the case for infinite dimensional spaces.

References: [99], [127], [74]

1.1.7 convex cones

Recall that a subset of a real vector space is called a *convex cone* if it is closed under summations and scalar multiplications by nonnegative numbers. A point $x_0 \in K$ is said to *generate an extreme ray* of a convex cone K or to be *extremal* in K if the *ray* $\{tx_0 : t \geqslant 0\}$ itself is a face of K. This is the case if and only if $x_0 = x_1 + x_2$ with $x_1, x_2 \in K$ implies that x_1 is a nonnegative scalar multiplication of x_0. In this case, the ray $\{tx_0 : t \geqslant 0\}$ is called an *extreme ray*. Sometimes, the point x_0 itself is called extreme.

The set of all $n \times n$ positive matrices, denoted by M_n^+, is a closed convex cone in the real vector space M_n^{h} of all Hermitian matrices. We note that a positive matrix generates an extreme ray of M_n^+ if and only if it is of rank one. We see that $A \in M_n^+$ is an interior point of M_n^+ if and only if $\operatorname{rank} A = n$, or equivalently A is non-singular. Especially, the identity matrix I_n is an interior point of M_n^+. The ray generated by I_n may be considered to be located at the center of the convex cone M_n^+. In fact, if we take a projection P onto a proper subspace then we have

$$\sup \left\{ t \in \mathbb{R} : (1-t)P + t \cdot \frac{1}{2} I_n \geqslant 0 \right\} = 2.$$

If we extend the line segment from P to $\frac{1}{2}I_n$ until the line meets the boundary of M_n^+, then the line segment meets at the orthogonal complement $I_n - P$, and $\frac{1}{2}I_n$ is located at the center of the line segment.

For a positive matrix A, we consider the range vectors of A and take the convex cone F generated by $\{|\xi\rangle\langle\xi| : |\xi\rangle \in \operatorname{Im} A\}$, then A is an interior point of F. Especially, for any $|\xi\rangle \in \operatorname{Im} A$ there exists $t > 1$ such that $(1-t)|\xi\rangle\langle\xi| + tA$ is positive. The convex cone F is the smallest face containing A, and every face of M_n^+ arises in this way.

We note that a convex cone K itself has no extreme point except zero. The set $\mathcal{D}_n$ of all density matrices is the intersection of the convex cone M_n^+ and the hyperplane determined by the condition that the trace is one. If we take a hyperplane H which does not contain zero but meets an extreme ray of K then the intersection is just a single point which is an extreme point of the convex set $K \cap H$.

1.2 Positive maps

We first note that every positive map between matrix algebras is sitting in the real vector space of all Hermiticity preserving maps. We will show that the map Ad_s defined by $\mathrm{Ad}_s(x) = s^*xs$ generates an extreme ray of the convex cone $\mathbb{P}_1$ of all positive linear maps. Using those maps together with the transpose map, we define the notions of k-superpositivity and decomposability for positive maps.

1.2.1 Positive maps between matrix algebras

The vector space of all linear maps from M_m into M_n will be denoted by $L(M_m, M_n)$ which is of m^2n^2 dimension over the complex field. A linear map $\phi : M_m \to M_n$ is called *Hermiticity preserving* if $\phi(a) \in M_n$ is Hermitian whenever $a \in M_m$ is Hermitian. When ϕ is Hermiticity preserving, we write $a = b + \mathrm{i}c$ with Hermitian matrices b and c, then we have

$$\phi(a^*) = \phi(b - \mathrm{i}c) = \phi(b) - \mathrm{i}\phi(c) = (\phi(b) + \mathrm{i}\phi(c))^* = \phi(a)^*.$$

In short, ϕ satisfies the following relation

$$\phi(a^*) = \phi(a)^*, \qquad a \in M_m. \tag{1.10}$$

If ϕ satisfies the property (1.10) and a is Hermitian then we have $\phi(a)^* = \phi(a^*) = \phi(a)$, and so $\phi(a)$ is Hermitian. Therefore, we see that ϕ is Hermiticity preserving if and only if it satisfies (1.10). The restriction of a Hermiticity preserving map $\phi : M_m \to M_n$ on M_m^{h} is a real linear map from M_m^{h} into M_n^{h}. Conversely, every real linear map $\phi : M_m^{\mathrm{h}} \to M_n^{\mathrm{h}}$ can be extended to the complex linear map $\tilde{\phi} : M_m \to M_n$ by

$$\tilde{\phi}(a + \mathrm{i}b) = \phi(a) + \mathrm{i}\phi(b). \tag{1.11}$$

The set $H(M_m, M_n)$ of all Hermiticity preserving maps is a vector space over the real field with the real dimension m^2n^2.

A linear map $\phi : M_n \to M_m$ is called *positive* if it sends positive matrices to positive matrices. Because every Hermitian matrix is the difference of two positive matrices, it is clear that every positive linear map is Hermiticity preserving. The set of all positive maps from M_m into M_n will be denoted by $\mathbb{P}_1[M_m, M_n]$, which is a closed convex cone in the real vector space $H(M_m, M_n)$.

For a given linear map $\phi : M_m \to M_n$, the *adjoint map* $\phi^* : M_n \to M_m$ is defined by

$$\langle \phi^*(x), y \rangle = \langle x, \phi(y) \rangle, \qquad x \in M_n,\ y \in M_m.$$

It is easy to see that the following relation

$$(\phi \circ \psi)^* = \psi^* \circ \phi^*$$

holds whenever the composition $\phi \circ \psi$ is possible. A linear map $\phi : M_m \to M_n$ is positive if and only if $\langle \phi(x), y \rangle \geqslant 0$ for every positive $x \in M_m$ and positive $y \in M_n$ if and only if $\langle x, \phi^*(y) \rangle \geqslant 0$ for every positive x and y if and only if $\phi^* : M_n \to M_m$ is a positive linear map. In short, we have seen that ϕ is positive if and only if its adjoint map ϕ^* is positive.

1.2.2 Extremal positive maps

For a given $m \times n$ matrix s, we define the linear map $\mathrm{Ad}_s : M_m \to M_n$ by

$$\mathrm{Ad}_s(x) = s^* x s, \qquad x \in M_m.$$

It is clear that Ad_s is a positive map for any matrix s. The map Ad_u is called a *congruence map* when u is a unitary. It is easy to see that the following identities

$$\mathrm{Ad}_{st} = \mathrm{Ad}_t \circ \mathrm{Ad}_s, \qquad (\mathrm{Ad}_s)^* = \mathrm{Ad}_{s^{\mathrm{T}}}$$

hold for matrices s and t, whenever st is defined. It is also clear that if $s \in M_n$ is nonsingular then $\mathrm{Ad}_s : M_n \to M_n$ is an order isomorphism. Recall that a bijective map $\phi : M_n \to M_n$ is called an *order isomorphism* when both ϕ and ϕ^{-1} are positive, that is, ϕ satisfies

$$s \leqslant t \iff \phi(s) \leqslant \phi(t). \tag{1.12}$$

A linear map which generates an extreme ray of the convex cone $\mathbb{P}_1$ is called an *extremal positive* map. It is clear that ϕ is extremal positive if and only if so is ϕ^*. If ϕ is extremal positive and σ is a linear order isomorphism then both $\sigma \circ \phi$ and $\phi \circ \sigma$ are also extremal whenever they are defined, because both $\phi \mapsto \sigma \circ \phi$ and $\phi \mapsto \phi \circ \sigma$ define affine isomorphisms between $\mathbb{P}_1[M_m, M_n]$. We will show that the map Ad_s is an extremal positive map. To do this, we need the following:

Proposition 1.2.1 *Suppose that $a \in M_n$ commutes every $b \in M_n$. Then a is a scalar multiple of the identity matrix.*

Proof. Write $a = \sum_{i,j} a_{ij}|i\rangle\langle j|$. Then $|k\rangle\langle\ell|a = a|k\rangle\langle\ell|$ implies

$$\langle\ell|a|k\rangle = \langle k|k\rangle\langle\ell|a|k\rangle = \langle k|a|k\rangle\langle\ell|k\rangle = 0,$$

when $k \neq \ell$, and so a must be a diagonal matrix. We also have

$$\langle\ell|a|\ell\rangle = \langle k|k\rangle\langle\ell|a|\ell\rangle = \langle k|a|k\rangle\langle\ell|\ell\rangle = \langle k|a|k\rangle,$$

for each k, ℓ. □

The identity map $\mathrm{id} = \mathrm{Ad}_I$ with the identity matrix I is the key case for the extremeness of the maps Ad_s. Sometimes, we denote by id_n for the identity map on M_n.

Lemma 1.2.2 *The identity map $\mathrm{id}_n = \mathrm{Ad}_{I_n}$ is extremal in $\mathbb{P}_1[M_n, M_n]$.*

Proof. Suppose that both ϕ and $\mathrm{id}_n - \phi$ are positive. Then for every unit vector $|\xi\rangle \in \mathbb{C}^n$, we have $0 \leqslant \phi(|\xi\rangle\langle\xi|) \leqslant |\xi\rangle\langle\xi|$, and so there exists λ_ξ with $0 \leqslant \lambda_\xi \leqslant 1$ such that

$$\phi(|\xi\rangle\langle\xi|) = \lambda_\xi|\xi\rangle\langle\xi|,$$

because $|\xi\rangle\langle\xi|$ generates an extreme ray of the convex cone M_n^+. We have to show that $\lambda_\xi = \lambda_\eta$ for any unit vectors $|\xi\rangle$ and $|\eta\rangle$. We note that $\phi(p) \leqslant p$ for every projection in M_n^{h}, which implies $(I_n - p)\phi(p) = 0$. We replace p by $I_n - p$, to get $p\phi(I_n - p) = 0$. Therefore, we have

$$0 = (I_n - p)\phi(p) - p\phi(I_n - p) = \phi(p) - p\phi(I_n).$$

Taking the adjoint, we also have

$$0 = (\phi(p) - p\phi(I_n))^* = \phi(p) - \phi(I_n)p,$$

and conclude that $\phi(I_n)$ commutes with every projection. Since every matrix is the linear combination of projections, we see that $\phi(I_n) = \lambda I_n$ by Proposition 1.2.1. Taking an orthonormal basis $\{|\xi_i\rangle\}$, we have

$$\sum_i \lambda_{\xi_i}|\xi_i\rangle\langle\xi_i| = \sum_i \phi(|\xi_i\rangle\langle\xi_i|) = \phi(I_n) = \lambda I_n = \sum_i \lambda|\xi_i\rangle\langle\xi_i|.$$

Putting $|\xi_i\rangle$ at the right-side of the above identity, we have $\lambda_{\xi_i} = \lambda$, and we conclude that $\phi = \lambda \cdot \mathrm{id}_n$. This shows that id_n is an extremal positive map. □

Lemma 1.2.3 *Suppose that $m \leqslant n$ and s is an $m \times n$ matrix of the form*

$$s = \begin{pmatrix} I_m & 0 \end{pmatrix} \in M_{m\times n}$$

with the $m \times m$ identity matrix I_m in the left corner. If $\phi : M_\ell \to M_m$ is extremal positive then $\mathrm{Ad}_s \circ\phi : M_\ell \to M_n$ is also extremal.

Proof. Note that $\sigma = \mathrm{Ad}_s$ is of the form:

$$\sigma : a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in M_n.$$

We define $\tau : M_n \to M_m$ by

$$\tau : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a \in M_m.$$

Then we have $\tau \circ \sigma = \mathrm{id}_m$. Suppose that $\sigma \circ \phi = \psi_1 + \psi_2$ with $\psi_1, \psi_2 \in \mathbb{P}_1[M_\ell, M_n]$. Then we have

$$\phi = \tau \circ \sigma \circ \phi = \tau \circ \psi_1 + \tau \circ \psi_2.$$

Since ϕ is extremal, there exists $\lambda \geqslant 0$ such that $\tau(\psi_1(a)) = \lambda\phi(a)$ for each $a \in M_\ell$. Because

$$\psi_1(a) \leqslant \sigma(\phi(a)) = \begin{pmatrix} \phi(a) & 0 \\ 0 & 0 \end{pmatrix}, \qquad a \in M_\ell^+,$$

we have $(\sigma \circ \tau)(\psi_1(a)) = \psi_1(a)$ for $a \in M_\ell$. Therefore, it follows that

$$\psi_1(a) = \sigma(\tau(\psi_1(a))) = \sigma(\lambda\phi(a)) = \lambda\sigma(\phi(a)),$$

for each $a \in M_m$, and so $\sigma \circ \phi = \mathrm{Ad}_s \circ\phi$ is extremal. □

Theorem 1.2.4 *For any $m \times n$ matrix s, the linear map $\mathrm{Ad}_s : M_m \to M_n$ is extremal positive.*

Proof. By singular value decomposition (1.8), we have $s = upqw$ with the $m \times k$ matrix $p = \begin{pmatrix} I_k \\ 0 \end{pmatrix}$ and $k \times n$ matrix $q = \begin{pmatrix} I_k & 0 \end{pmatrix}$ together with invertible matrices u, w. Then we have

$$\mathrm{Ad}_s = \mathrm{Ad}_w \circ \mathrm{Ad}_q \circ \mathrm{Ad}_p \circ \mathrm{Ad}_u .$$

By lemma 1.2.3 and Lemma 1.2.2, we see that $\mathrm{Ad}_{p^{\mathrm{T}}} = \mathrm{Ad}_{p^{\mathrm{T}}} \circ\mathrm{id}$ is extremal, and so $\mathrm{Ad}_p = (\mathrm{Ad}_{p^{\mathrm{T}}})^*$ is also extremal. Therefore, $\mathrm{Ad}_q \circ \mathrm{Ad}_p$ is extremal by Lemma 1.2.3 again. On the other hand, Ad_w and Ad_u are order isomorphisms since w and v are invertible. Therefore, we conclude that Ad_s is extremal. □

It is very difficult in general to find out all the extreme rays of the convex cone $\mathbb{P}_1[M_m, M_n]$, and it is an open problem even for low dimensional cases like $m = n = 3$. But, it is not so difficult to find boundary points of $\mathbb{P}_1[M_m, M_n]$. We see that if $\phi(a)$ is singular for a nonzero $a \in M_m^+$ then ϕ must be a boundary point of $\mathbb{P}_1$. Indeed, the following is immediate from the definition of interior points.

Proposition 1.2.5 *If $\phi \in \operatorname{int} \mathbb{P}_1[M_m, M_n]$ then ϕ sends every nonzero element in M_m^+ to an interior point of M_n^+.*

Proof. Take a nonzero $a \in M_m^+$ and $b \in M_n^+$, and consider the map $\psi : M_m \to M_n$ given by $\psi(x) = \frac{\operatorname{Tr}(x)}{\operatorname{Tr}(a)} b$. Then there exists $t > 1$ such that $(1-t)\psi + t\phi \in \mathbb{P}_1$, and so we have $(1-t)b + t\phi(a) \geqslant 0$. Since $b \in M_n^+$ was arbitrary, we see that $\phi(a)$ is an interior point of M_n^+. □

We will see later that the converse of Proposition 1.2.5 also holds. See Proposition 2.3.3. Therefore, ϕ is on the boundary of $\mathbb{P}_1[M_m, M_n]$ if and only if there is a nonzero $a \in M_m^+$ such that $\phi(a) \in M_n$ is singular. It is easily seen that the positive map $\mathrm{Ad}_s : M_m \to M_n$ satisfies this property for every $s \in M_{m\times n}$. Especially, if $s \in M_{n\times n}$ itself is singular then $\mathrm{Ad}_s(a)$ is singular for every $a \in M_n$.

References: [111], [118]

1.2.3 k-superpositive maps

We denote by $\mathbb{SP}_k$ the convex hull of $\{\mathrm{Ad}_s : \operatorname{rank} s \leqslant k\}$;

$$\mathbb{SP}_k := \operatorname{conv}\{\mathrm{Ad}_s : \operatorname{rank} s \leqslant k\} \subset \mathbb{P}_1,$$

where $\operatorname{conv} S$ denotes the convex hull of S. We also use the notation $\mathbb{SP}_k[M_m, M_n]$ when we need to specify the domain and range. If $k < \ell \leqslant m \wedge n$ and $\operatorname{rank} s = \ell$ then $\mathrm{Ad}_s \notin \mathbb{SP}_k$ since it is extremal, where $m \wedge n$ denotes the the minimum of m and n. Therefore, we have the following chain of strict inclusions

$$\mathbb{SP}_1 \subsetneqq \mathbb{SP}_2 \subsetneqq \cdots \subsetneqq \mathbb{SP}_k \subsetneqq \cdots \subsetneqq \mathbb{SP}_{m\wedge n}$$

of convex cones of positive maps. A positive map in $\mathbb{SP}_k$ is called *k-superpositive*. A 1-superpositive map may be called just *superpositive*.

Proposition 1.2.6 *For each $k = 1, 2, \ldots, m \wedge n$, the convex cone $\mathbb{SP}_k$ is closed in $H(M_m, M_n)$.*

Proof. We consider the linear functional τ on the space $H(M_m, M_n)$ defined by

$$\tau(\phi) = \operatorname{Tr} \phi(I_m), \qquad \phi \in H(M_m, M_n).$$

Then we have $\tau(\mathrm{Ad}_s) = \sum_{ij} |s_{ij}|^2$ for $s = \sum_{ij} s_{ij}|i\rangle\langle j|$. Next, we consider the set

$$C_k := \{s \in M_{m\times n} : \mathrm{rank}\, s \leqslant k,\ \textstyle\sum_{ij} |s_{ij}|^2 = 1\},$$

which is compact. Then the image of C_k under the continuous map $s \mapsto \mathrm{Ad}_s$ is a compact subset of $H(M_m, M_n)$ whose convex hull is $\{\phi \in \mathbb{SP}_k : \tau(\phi) = 1\}$, which is also compact. Therefore, we conclude that the convex cone $\mathbb{SP}_k$ is closed. □

It is clear that the *transpose map* $\mathrm{T} : x \mapsto x^{\mathrm{T}}$ is a positive map between M_n. The transpose map is extremal in $\mathbb{P}_1[M_n, M_n]$, since $\mathrm{T} = \mathrm{T} \circ \mathrm{id}$ and T is an order isomorphism. When $n \geqslant 2$, it is easy to see that T is not a member of the convex cone $\mathbb{SP}_n[M_n, M_n]$, whose extreme rays consist of Ad_s. If we assume that $\mathrm{T} \in \mathbb{SP}_n$ then we have $\mathrm{T} = \mathrm{Ad}_s$ for a matrix $s = \sum_{i,j} s_{ij}|i\rangle\langle j|$. We have

$$\mathrm{Ad}_s(|i\rangle\langle j|) = \sum_{k,\ell,p,q} \bar{s}_{qp} s_{k\ell} |p\rangle\langle q|i\rangle\langle j|k\rangle\langle \ell| = \sum_{p,\ell} \bar{s}_{ip} s_{j\ell} |p\rangle\langle \ell|.$$

Since $\mathrm{T}(|i\rangle\langle j|) = |j\rangle\langle i|$, we have $\bar{s}_{ij} s_{ji} = 1$ for every i, j, especially s has no zero entries. On the other hand, we have $\bar{s}_{ip} s_{j\ell} = 0$ whenever $(p, \ell) \neq (j, i)$, which is absurd.

Since the composition of positive maps is positive, we see that

$$\mathbb{SP}^k := \{\mathrm{T} \circ \phi : \phi \in \mathbb{SP}_k\}$$

is a convex cone in $\mathbb{P}_1$, and we have another chain of strict inclusions

$$\mathbb{SP}^1 \subsetneq \mathbb{SP}^2 \subsetneq \cdots \subsetneq \mathbb{SP}^k \subsetneq \cdots \subsetneq \mathbb{SP}^{m\wedge n} \tag{1.13}$$

of convex cones in $\mathbb{P}_1$. By the identity

$$\mathrm{T} \circ \mathrm{Ad}_s(a) = (s^* a s)^{\mathrm{T}} = s^{\mathrm{T}} a^{\mathrm{T}} \bar{s} = \mathrm{Ad}_{\bar{s}}(a^{\mathrm{T}}) = \mathrm{Ad}_{\bar{s}} \circ \mathrm{T}(a),$$

we see that $\phi \in \mathbb{SP}^k$ if and only if $\mathrm{T} \circ \phi \in \mathbb{SP}_k$ if and only if $\phi \circ \mathrm{T} \in \mathbb{SP}_k$.

In the identity $\mathrm{Ad}_{|\xi\rangle\langle\eta|}(a) = |\eta\rangle\langle\xi|a|\xi\rangle\langle\eta|$, we see that $\langle\xi|a|\xi\rangle$ is a scalar and $\mathrm{T}(|\eta\rangle\langle\eta|) = |\bar{\eta}\rangle\langle\bar{\eta}|$. Therefore, we have $\mathrm{T} \circ \mathrm{Ad}_{|\xi\rangle\langle\eta|}(a) = \mathrm{Ad}_{|\xi\rangle\langle\bar{\eta}|}(a)$, and so get the identity

$$\mathrm{T} \circ \mathrm{Ad}_{|\xi\rangle\langle\eta|} = \mathrm{Ad}_{|\xi\rangle\langle\bar{\eta}|}\,. \tag{1.14}$$

Hence, it follows that two convex cones $\mathbb{SP}_1$ and $\mathbb{SP}^1$ coincide, and we have the following inclusion:

$$\mathbb{SP}_1 \subset \mathbb{SP}_{m\wedge n} \cap \mathbb{SP}^{m\wedge n}. \tag{1.15}$$

Proposition 1.2.7 *Suppose that $s \in M_{m\times n}$ is nonzero. Then $\mathrm{Ad}_s \in \mathbb{SP}^{m\wedge n}$ if and only if s is of rank one.*

Proof. It remains to prove the 'only if' part. Suppose that Ad_s belongs to the convex cone $\mathbb{SP}^{m\wedge n}$. Since Ad_s generates an extreme ray of $\mathbb{P}_1$, it also generates an extreme ray of the smaller cone $\mathbb{SP}^{m\wedge n}$, and so $\mathrm{Ad}_s = \mathrm{Ad}_t \circ \mathrm{T}$ for an $m \times n$ matrix t, which implies that the identity

$$s^*|\xi\rangle\langle\eta|s = t^*|\bar{\eta}\rangle\langle\bar{\xi}|t$$

holds for every $|\xi\rangle$ and $|\eta\rangle$ in $\mathbb{C}^m$. We fix a vector $|\eta\rangle$ so that $t^*|\bar{\eta}\rangle \neq 0$ and $\langle\eta|s \neq 0$. For arbitrary nonzero $|\xi\rangle$, the left side is a rank one matrix with the range vector $s^*|\xi\rangle$. On the other hand, the right side is a rank one matrix with the range vector $t^*|\bar{\eta}\rangle$, and so $s^*|\xi\rangle$ is a scalar multiple of $t^*|\bar{\eta}\rangle$ for arbitrary $|\xi\rangle \in \mathbb{C}^m$. Therefore s^* is of rank one. □

Proposition 1.2.7 tells us that an extreme ray of $\mathbb{SP}_{m\wedge n}$ belongs to $\mathbb{SP}^{m\wedge n}$ then it actually belongs to $\mathbb{SP}_1 = \mathbb{SP}^1$. Note that this does not imply that $\mathbb{SP}_{m\wedge n} \cap \mathbb{SP}^{m\wedge n}$ coincides with $\mathbb{SP}_1$. The convex hull of two convex cones K_1 and K_2 is nothing but

$$K_1 + K_2 = \{x_1 + x_2 : x_i \in K_i\}.$$

Now, we have concrete examples of positive maps in $\mathbb{SP}_{m\wedge n} + \mathbb{SP}^{m\wedge n}$. Those maps in this class, which are called *decomposable*, can be expressed by the combinations of Ad_s and T. One question arises naturally: Does every positive map arise in this way?

References: [3], [106], [108]

1.3 Positive maps between 2×2 matrices

In this section, we show that all positive maps between 2×2 matrices can be expressed with Ad_s and the transpose map. The main step is to realize the congruence map Ad_U by a 2×2 special unitary matrix U as a rotation in the three dimensional real vector space.

1.3.1 Congruence maps and orthogonal transformations

In order to show that every positive map from M_2 into itself is decomposable, we first look at the 4-dimensional real vector space M_2^{h} whose elements may be written by

$$\varrho = \begin{pmatrix} t+z & x-\mathrm{i}y \\ x+\mathrm{i}y & t-z \end{pmatrix} = tI_2 + x\sigma_x + y\sigma_y + z\sigma_z,$$

with real numbers t, x, y, z and *Pauli matrices*

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.16}$$

The following four matrices

$$\left\{\tfrac{1}{\sqrt{2}}I_2,\ \tfrac{1}{\sqrt{2}}\sigma_x,\ \tfrac{1}{\sqrt{2}}\sigma_y,\ \tfrac{1}{\sqrt{2}}\sigma_z\right\}$$

make an orthonormal basis of $M_2^{\rm h}$ with respect to the inner product given by

$$(a,b) = \operatorname{Tr}(ab^*) = \sum a_{ij}\bar{b}_{ij}, \tag{1.17}$$

for $a, b \in M_2^{\rm h}$. We see that $\varrho \in M_2^+$ if and only if $t \geqslant |z|$ and $t^2 - z^2 \geqslant x^2 + y^2$ if and only if $t \geqslant 0$ and $x^2 + y^2 + z^2 \leqslant t^2$ holds. Therefore, we see that $\varrho \in \mathcal{D}_2$ if and only if

$$x^2 + y^2 + z^2 \leqslant t^2, \qquad t = \frac{1}{2} \tag{1.18}$$

holds, and so $\mathcal{D}_2$ is the three dimensional ball, called the *Bloch ball*.

We recall that the matrix $\frac{1}{2}I_2$ is located at the center of the convex cone M_2^+, and consider the distance from $\varrho \in \mathcal{D}_2$ satisfying (1.18) to

$$\varrho_* := \frac{1}{2}I_2 \in \mathcal{D}_2.$$

The distance $\|x - y\|_{\rm HS}$ arising from the inner product (1.17) is called the *Hilbert–Schmidt distance*. We have

$$\|\varrho - \varrho_*\|_{\rm HS}^2 = \left\| \begin{pmatrix} z & x - {\rm i}y \\ x + {\rm i}y & -z \end{pmatrix} \right\|_{\rm HS}^2 = 2(x^2 + y^2 + z^2).$$

Therefore, we see that $\varrho \in \mathcal{D}_2$ if and only if t, x, y and z satisfy the relation (1.18) if and only if the inequality

$$\|\varrho - \varrho_*\|_{\rm HS} \leqslant \frac{1}{\sqrt{2}}$$

holds, as it was expected. We also see that ϱ is on the boundary, that is, satisfies the equality $\|\varrho_* - \varrho\|_{\rm HS} = \frac{1}{\sqrt{2}}$ if and only if $\varrho = |\xi\rangle\langle\xi|$ is of rank one if and only if ϱ generates an extreme ray of the convex cone M_2^+. The 2-dimensional sphere which is the boundary of $\mathcal{D}_2$ is called the *Bloch sphere*. We also note that each boundary point of $\mathcal{D}_2$ corresponds to the range vector $|\xi\rangle \in \mathbb{C}^2$ up to scalar multiplications, which corresponds to a point of the complex projective space $\mathbb{CP}^1$.

We say that $\phi : M_m \to M_n$ is *trace preserving* if $\operatorname{Tr}(\phi(x)) = \operatorname{Tr}(x)$ for every $x \in M_m$. By the identity $\operatorname{Tr} a = \langle I, a\rangle$, we see that ϕ is trace preserving if and only if ϕ^* is unital. A trace preserving unital positive map between M_2 has an interesting geometric interpretation. We begin with a Hermiticity preserving map ϕ on $M_2^{\rm h}$, which is the orthogonal sum of $\mathbb{R}I_2$ and the subspace

$$Z := \operatorname{span}\{\sigma_x, \sigma_y, \sigma_z\}$$

of $M_2^{\rm h}$. Suppose that ϕ is unital and trace preserving. Then both $\mathbb{R}I_2$ and Z are invariant under ϕ, and $\phi|_Z$ is a linear transform in the three dimensional space Z over the real field.

We examine what happens for Ad_U with a 2×2 special unitary

$$U = \begin{pmatrix} \alpha & -\bar\beta \\ \beta & \bar\alpha \end{pmatrix}, \qquad |\alpha|^2 + |\beta|^2 = 1.$$

It is clear that the congruence map Ad_U is unital and trace preserving. We have

$$\mathrm{Ad}_U(\sigma_x) = \begin{pmatrix} \bar\alpha & \bar\beta \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & -\bar\beta \\ \beta & \bar\alpha \end{pmatrix} = \begin{pmatrix} 2\mathrm{Re}\,(\alpha\bar\beta) & \bar\alpha^2 - \bar\beta^2 \\ \alpha^2 - \beta^2 & -2\mathrm{Re}\,(\alpha\bar\beta) \end{pmatrix}$$

We choose real variables a, b, c, d with $\alpha = a + \mathrm{i}b$, $\beta = c + \mathrm{i}d$ and $a^2 + b^2 + c^2 + d^2 = 1$, then we have

$$\begin{aligned} \alpha^2 - \beta^2 &= (1 - 2b^2 - 2c^2) + \mathrm{i}(2ab - 2cd), \\ 2\alpha\bar\beta &= (2ac + 2bd) + \mathrm{i}(-2ad + 2bc), \end{aligned}$$

and so, it follows that

$$\mathrm{Ad}_U(\sigma_x) = (1 - 2b^2 - 2c^2)\sigma_x + (2ab - 2cd)\sigma_y + (2ac + 2bd)\sigma_z.$$

Similarly, we also have

$$\begin{aligned} \mathrm{Ad}_U(\sigma_y) &= (-2ab - 2cd)\sigma_x + (1 - 2b^2 - 2d^2)\sigma_y + (2ad - 2bc)\sigma_z, \\ \mathrm{Ad}_U(\sigma_z) &= (-2ac + 2bd)\sigma_x + (-2ad - 2bc)\sigma_y + (1 - 2c^2 - 2d^2)\sigma_z. \end{aligned}$$

Therefore, the map ϕ determines a linear transform between three dimensional space Z spanned by Pauli matrices, which corresponds to the following 3×3 matrix

$$\begin{pmatrix} 1 - 2b^2 - 2c^2 & -2ab - 2cd & -2ac + 2bd \\ 2ab - 2cd & 1 - 2b^2 - 2d^2 & -2ad - 2bc \\ 2ac + 2bd & 2ad - 2bc & 1 - 2c^2 - 2d^2 \end{pmatrix}.$$

This is an orthogonal matrix with the determinant 1, and sends $(d, -c, b)^{\rm T}$ to itself. Therefore, it represents the rotation in $\mathbb{R}^3$ around the direction $(d, -c, b)^{\rm T} \in \mathbb{R}^3$. It rotates by the angles 2θ satisfying $\cos\theta = a$ and $\sin^2\theta = b^2 + c^2 + d^2$, and every rotation in $\mathbb{R}^3$ arises in this way.

Note that the transpose map $\mathrm{T} : M_2^{\rm h} \to M_2^{\rm h}$ is also unital and trace preserving. Since $\mathrm{T}(\sigma_x) = \sigma_x$, $\mathrm{T}(\sigma_y) = -\sigma_y$ and $\mathrm{T}(\sigma_z) = \sigma_z$, we see that the trace map induces the reflection

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is an orthogonal matrix with the determinant -1. We recall that every orthogonal matrix which is not a rotation can be expressed by the composition of a rotation and the above reflection. We also recall that a linear map ϕ between $\mathbb{R}^n$ is an *isometry*, that is, $\|\phi(x)\| = \|x\|$ for every x, if and only if ϕ preserves the inner product if and only if it is represented by an orthogonal matrix. Therefore, we have the following:

Proposition 1.3.1 *Let $\phi : M_2 \to M_2$ is unital and trace preserving. If $\phi|_Z$ is an isometry then $\phi = \mathrm{Ad}_U$ or $\phi = \mathrm{Ad}_U \circ \mathrm{T}$ for a unitary matrix U.*

References: [88], [8], [6], [7]

1.3.2 Decomposability of positive maps

Now, suppose that ϕ is positive as well as unital and trace preserving. Then ϕ sends $\mathcal{D}_2$ into itself, that is, we have

$$\varrho \in \mathcal{D}_2 \implies \phi(\varrho) \in \mathcal{D}_2.$$

For a given $\varrho \in Z$, we have $\varrho + \varrho_* \in \mathcal{D}_2$ if and only if $\|\varrho\|_{\mathrm{HS}} \leqslant \frac{1}{\sqrt{2}}$. By the relation $\phi(\varrho + \varrho_*) = \phi(\varrho) + \varrho_*$, we have

$$\varrho \in Z,\ \|\varrho\|_{\mathrm{HS}} \leqslant \frac{1}{\sqrt{2}} \implies \|\phi(\varrho)\|_{\mathrm{HS}} \leqslant \frac{1}{\sqrt{2}}.$$

When we identity $(Z, \|\ \|_{\mathrm{HS}})$ with the normed space $(\mathbb{R}^3, \|\ \|)$, we see that $\phi|_Z$ sends the unit ball in $\mathbb{R}^3$ into itself. Such a linear map is called a *contraction*. Then, the map $\phi|_Z$ belongs to the convex set $\mathcal{C}_3$ of all real linear contractions between $\mathbb{R}^3$, and so, $\phi|_Z$ is a convex combination of extreme points of $\mathcal{C}_3$. It is easy to see that $\phi \in \mathcal{C}_3$ is an extreme point of $\mathcal{C}_3$ if and only if ϕ is an isometry.

Proposition 1.3.2 *Let $\mathcal{C}_n$ be the convex set of all contractions between $\mathbb{R}^n$. A linear map $\phi \in \mathcal{C}_n$ is an extreme point of $\mathcal{C}_n$ if and only if ϕ is an isometry.*

Proof. By the singular value decomposition, ϕ is the composition of isometries and a diagonal map $\delta \in \mathcal{C}_n$. If ϕ is extreme in $\mathcal{C}_n$ then δ is also extreme in $\mathcal{C}_n$. It is evident that a diagonal map which is extreme in $\mathcal{C}_n$ is an isometry. For the converse, we suppose that ϕ is an isometry of $\mathbb{R}^n$ and $\phi = (1-t)\phi_0 + t\phi_1$ with $\phi_0, \phi_1 \in \mathcal{C}_n$. When $\|x\| = 1$, we have $(1-t)\phi_0(x) + t\phi_1(x) = \phi(x)$ is an extreme point of the unit ball. Therefore, it follows that $\phi_0(x) = \phi_1(x) = \phi(x)$ for every x with $\|x\| = 1$, and we conclude that $\phi = \phi_0 = \phi_1$. □

By Proposition 1.3.1 and Proposition 1.3.2, we conclude that every unital trace preserving positive map between 2×2 matrices is decomposable. For general cases, we begin with a map in the interior of $\mathbb{P}_1$. We recall that if $\phi \in \operatorname{int} \mathbb{P}_1[M_m, M_n]$ then

ϕ sends every nonzero element in M_m^+ to an interior point of M_n^+ by Proposition 1.2.5. In short, $\phi(a)$ is nonsingular whenever a is nonzero. It is easy to see that ϕ is an interior point of $\mathbb{P}_1$ if and only if ϕ^* is an interior point of $\mathbb{P}_1$.

Proposition 1.3.3 *Let ϕ be an interior point of the convex cone $\mathbb{P}_1[M_n, M_n]$. Then there exist nonsingular $a, b \in M_n^+$ such that the map $\tilde{\phi} := \mathrm{Ad}_a \circ\phi \circ \mathrm{Ad}_b$ is positive, unital and trace preserving.*

Proof. We note that $\tilde{\phi}$ is unital if and only if $a\phi(b^2)a = I$ if and only if the identity

$$\phi(b^2)^{-1} = a^2 \tag{1.19}$$

holds, and $\tilde{\phi}$ is trace preserving if and only if $\mathrm{Ad}_{b^{\mathrm{T}}} \circ\phi^* \circ \mathrm{Ad}_{a^{\mathrm{T}}}$ is unital if and only if

$$\phi^*(a^{2\mathrm{T}})^{-1} = b^{2\mathrm{T}}, \tag{1.20}$$

where $a^{2\mathrm{T}} = (a^2)^{\mathrm{T}} = (a^{\mathrm{T}})^2$ has the obvious meaning. We define the map f from $\mathcal{D}_n$ into itself by

$$f(s) = \frac{1}{\mathrm{Tr}\,\phi[\phi^*(s^{\mathrm{T}})^{-\mathrm{T}}]^{-1}}\phi[\phi^*(s^{\mathrm{T}})^{-\mathrm{T}}]^{-1}, \qquad s \in \mathcal{D}_n,$$

which is a continuous self map on the compact convex set $\mathcal{D}_n$. By Brower's fixed point theorem, there exists $s_0 \in \mathcal{D}_n$ such that $f(s_0) = s_0$. We take $a, b \in M_n^+$ so that

$$a^2 = s_0, \qquad b^2 = \phi^*(a^{2\mathrm{T}})^{-\mathrm{T}}.$$

The second identity tells us that a and b satisfy (1.20), and so $\tilde{\phi}$ is trace preserving. Furthermore, we have

$$a^2 = s_0 = \frac{1}{t}\phi[\phi^*(a^{2\mathrm{T}})^{-\mathrm{T}}]^{-1} = \frac{1}{t}\phi(b^2)^{-1}$$

with $t = \mathrm{Tr}\,\phi[\phi^*(a^{2\mathrm{T}})^{-\mathrm{T}}]^{-1}$, which implies $I_n = ta\phi(b^2)a = t\tilde{\phi}(I_n)$. Because $\tilde{\phi}$ is trace preserving, we have $t = 1$. Therefore, we conclude that $\tilde{\phi}$ is unital by (1.19).
□

Therefore, we conclude that every interior point ϕ of the convex cone $\mathbb{P}_1[M_2, M_2]$ is decomposable. By the exactly same argument as in Proposition 1.2.6, it is easily seen that the convex cone $\mathbb{SP}_{m\wedge n} + \mathbb{SP}^{m\wedge n}$ is also closed. Therefore, we have the following:

Theorem 1.3.4 *Every positive map in $\mathbb{P}_1[M_2, M_2]$ is decomposable.*

From this, we see that every extremal positive map between 2×2 matrices is of the form Ad_s or $\mathrm{Ad}_s \circ\mathrm{T}$ for a 2×2 matrix s. It is known [129] that every positive map in $\mathbb{P}_1[M_2, M_3]$ and $\mathbb{P}_1[M_3, M_2]$ is also decomposable. We will see in Section 1.5 that this is not the case for $\mathbb{P}_1[M_3, M_3]$. Concrete examples of indecomposable positive linear maps will be given in Section 1.6.

References: [111], [129], [6], [7]

1.4 Choi matrices and separable states

For a given linear map ϕ from M_m into M_n, we associate the Choi matrix C_ϕ in the tensor product $M_m \otimes M_n$ which plays a central role throughout the topics. With this isomorphism $\phi \mapsto \mathrm{C}_\phi$, 1-superpositive maps correspond to separable states, and composition with the transpose map corresponds to taking the partial transpose.

1.4.1 Choi matrices

We note that a linear map $\phi : M_m \to M_n$ is completely determined by $\phi(|i\rangle\langle j|)$ for $i, j = 1, 2, \ldots, m$, because $\{|i\rangle\langle j| : i, j = 1, \ldots, m\}$ is a basis of M_m. We define the *Choi matrix* C_ϕ of a linear map $\phi \in L(M_m, M_n)$ by

$$\mathrm{C}_\phi = \sum_{i,j=1}^{m} |i\rangle\langle j| \otimes \phi(|i\rangle\langle j|) \in M_m \otimes M_n.$$

Then it is clear that

$$\phi \mapsto \mathrm{C}_\phi : L(M_m, M_n) \to M_m \otimes M_n$$

is a linear isomorphism, which is called the *Jamiołkowski–Choi isomorphism*. By the relation $\sum_{i,j} \langle a, |i\rangle\langle j| \rangle \phi(|i\rangle\langle j|) = \phi(a)$, we have

$$\langle a \otimes b, \mathrm{C}_\phi \rangle = \sum_{i,j} \langle a, |i\rangle\langle j| \rangle \langle b, \phi(|i\rangle\langle j|) \rangle = \langle b, \phi(a) \rangle \tag{1.21}$$

for $a \in M_m$ and $b \in M_n$. The identity (1.21) may be used as the definition of the Choi matrix, because it determines the entries of the Choi matrix when $a = |i\rangle\langle j|$ and $b = |k\rangle\langle \ell|$.

If ϕ maps from M_m into M_n then ϕ^* interchanges the domain and the range, and so C_{ϕ^*} belongs to $M_n \otimes M_m$. For given $a \in M_m$ and $b \in M_n$, we have

$$\langle b \otimes a, \mathrm{C}_{\phi^*} \rangle = \langle a, \phi^*(b) \rangle = \langle \phi(a), b \rangle = \langle a \otimes b, \mathrm{C}_\phi \rangle. \tag{1.22}$$

Therefore, $\mathrm{C}_{\phi^*} \in M_n \otimes M_m$ is the flip of $\mathrm{C}_\phi \in M_m \otimes M_n$, where the *flip* operation is defined by $x \otimes y \mapsto y \otimes x$.

Now, we look for properties of the Choi matrices which correspond to Hermiticity preserving maps and positive maps.

Proposition 1.4.1 *A linear map $\phi : M_m \to M_n$ is Hermiticity preserving if and only if C_ϕ is Hermitian.*

Proof. By the definition of Choi matrix, we see that C_ϕ is Hermitian if and only if the following

$$\phi(|i\rangle\langle j|)^* = \phi(|j\rangle\langle i|), \qquad i,j = 1,2,\dots,m \tag{1.23}$$

holds. Recall that ϕ is Hermiticity preserving if and only if $\phi(a)^* = \phi(a^*)$ for every $a \in M_m$, which implies (1.23). Conversely, the relation (1.23) implies

$$\phi(a)^* = \phi\left(\sum a_{ij}|i\rangle\langle j|\right)^* = \sum \bar{a}_{ij}\phi(|i\rangle\langle j|)^* = \sum \bar{a}_{ij}\phi(|j\rangle\langle i|) = \phi(a^*),$$

for $a = \sum a_{ij}|i\rangle\langle j| \in M_m$. □

A matrix $\varrho \in M_m \otimes M_n$ is called *block-positive* when

$$\langle\zeta|\varrho|\zeta\rangle \geqslant 0$$

holds for every product vector $|\zeta\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$. Compare with the definition (1.1) of positive matrices. The convex cone of all block-positive matrices is denoted by $\mathcal{BP}_1[M_m \otimes M_n]$ or just by $\mathcal{BP}_1$.

Proposition 1.4.2 *A linear map $\phi : M_m \to M_n$ is positive if and only if C_ϕ is block-positive in $M_m \otimes M_n$.*

Proof. We note that ϕ is positive if and only if $\phi(|\xi\rangle\langle\xi|)$ is positive in M_n for every $|\xi\rangle \in \mathbb{C}^m$ if and only if

$$\langle|\eta\rangle\langle\eta|, \phi(|\xi\rangle\langle\xi|)\rangle = \langle|\xi\rangle\langle\xi| \otimes |\eta\rangle\langle\eta|, \mathrm{C}_\phi\rangle = \langle|\xi\rangle|\eta\rangle\langle\xi|\langle\eta|, \mathrm{C}_\phi\rangle = \langle\bar{\xi}|\langle\bar{\eta}|\mathrm{C}_\phi|\bar{\xi}\rangle|\bar{\eta}\rangle$$

is nonnegative for every $|\xi\rangle \in \mathbb{C}^m$ and $|\eta\rangle \in \mathbb{C}^n$, where we used (1.2). □

For an $m \times n$ matrix $s = \sum_{i,j} s_{ij}|i\rangle\langle j|$, we have

$$\begin{aligned}
\mathrm{C}_{\mathrm{Ad}_s} &= \textstyle\sum_{p,q} |p\rangle\langle q| \otimes \left(\sum_{k,\ell} \bar{s}_{\ell k}|k\rangle\langle\ell|\right) |p\rangle\langle q| \left(\sum_{i,j} s_{ij}|i\rangle\langle j|\right) \\
&= \left(\textstyle\sum_{k,\ell} \bar{s}_{\ell k} \sum_p \langle\ell|p\rangle|p\rangle|k\rangle\right) \left(\sum_{i,j} s_{ij} \sum_q \langle q|i\rangle\langle q|\langle j|\right) \\
&= \left(\textstyle\sum_{k,\ell} \bar{s}_{\ell k}|\ell\rangle|k\rangle\right) \left(\sum_{i,j} s_{ij}\langle i|\langle j|\right).
\end{aligned}$$

Therefore, we see that

$$\mathrm{C}_{\mathrm{Ad}_s} = |\tilde{s}\rangle\langle\tilde{s}|, \tag{1.24}$$

under the correspondence between $s \in M_{m\times n}$ and the vector $|\tilde{s}\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ given by (1.5). It is worthwhile to write down the Choi matrix of the map Ad_s when $s = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We have $\langle\tilde{s}| = (a,b,c,d)$, and

$$\mathrm{C}_{\mathrm{Ad}_s} = |\tilde{s}\rangle\langle\tilde{s}| = \begin{pmatrix} |a|^2 & \bar{a}b & \bar{a}c & \bar{a}d \\ \bar{b}a & |b|^2 & \bar{b}c & \bar{b}d \\ \bar{c}a & \bar{c}b & |c|^2 & \bar{c}d \\ \bar{d}a & \bar{d}b & \bar{d}c & |d|^2 \end{pmatrix}.$$

We also have $\langle \widetilde{s^{\mathrm{T}}}| = (a, c, b, d)$, and

$$\mathrm{C}_{(\mathrm{Ad}_s)^*} = \mathrm{C}_{\mathrm{Ad}_{s^{\mathrm{T}}}} = \begin{pmatrix} |a|^2 & \bar{a}c & \bar{a}b & \bar{a}d \\ \bar{c}a & |c|^2 & \bar{c}b & \bar{c}d \\ \bar{b}a & \bar{b}c & |b|^2 & \bar{b}d \\ \bar{d}a & \bar{d}c & \bar{d}b & |d|^2 \end{pmatrix},$$

which is the flip of $\mathrm{C}_{\mathrm{Ad}_s}$. On the other hand, we have $\langle \widetilde{s^*}| = (\bar{a}, \bar{c}, \bar{b}, \bar{d})$, and so we see that $\mathrm{C}_{\mathrm{Ad}_{s^*}}$ is the conjugation of the flip of $\mathrm{C}_{\mathrm{Ad}_s}$.

References: : [32], [67], [21],

1.4.2 Schmidt numbers and entanglement

The identity (1.24) tells us that the Choi matrix of the map Ad_s is of rank one $mn \times mn$ positive matrix onto the vector in $\mathbb{C}^m \otimes \mathbb{C}^n$ whose Schmidt rank coincides with the rank of the matrix s. We define

$$\begin{aligned} \mathcal{S}_k[M_m \otimes M_n] :&= \{\mathrm{C}_\phi \in M_m \otimes M_n : \phi \in \mathbb{SP}_k[M_m, M_n]\} \\ &= \mathrm{conv}\,\{|\zeta\rangle\langle\zeta| \in M_m \otimes M_n : |\zeta\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n,\ \mathrm{SR}\,|\zeta\rangle \leqslant k\}, \end{aligned}$$

for $k = 1, 2, \dots, m \wedge n$. Then we see that $\varrho \in \mathcal{S}_k$ if and only if ϱ is an (unnormalized) state of the form

$$\varrho = \sum_i |\zeta_i\rangle\langle\zeta_i|$$

with $\mathrm{SR}\,|\zeta_i\rangle \leqslant k$. Furthermore, a linear map ϕ belongs to $\mathbb{SP}_{m\wedge n}$ if and only if C_ϕ is positive. Because a positive matrix generates an extreme ray of the cone $(M_m \otimes M_n)^+$ if and only if it is of rank one, we see that Ad_s generates an extreme ray of the cone $\mathbb{SP}_{m\wedge n}$. The point of Theorem 1.2.4 is that Ad_s generates an extreme ray of the much larger convex cone $\mathbb{P}_1$. We may summarize in the following diagram:

$$\begin{array}{ccccccccc} L(M_m, M_n): & \mathbb{SP}_1 & \subsetneqq & \mathbb{SP}_k & \subsetneqq & \mathbb{SP}_{m\wedge n} & \subsetneqq & \mathbb{P}_1 \\ \downarrow \mathrm{JC} & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M_m \otimes M_n: & \mathcal{S}_1 & \subsetneqq & \mathcal{S}_k & \subsetneqq & (M_m \otimes M_n)^+ & \subsetneqq & \mathcal{BP}_1 \end{array} \tag{1.25}$$

A state $\varrho \in \mathcal{S}_k \backslash \mathcal{S}_{k-1}$ is called to have *Schmidt number* k. A state of Schmidt number one is also called *separable*, and a state which is not separable is called *entangled*. Recall that a vector in $\mathbb{C}^m \otimes \mathbb{C}^n$ with Schmidt rank one is called a product vector. Then, a state in $(M_m \otimes M_n)^+$ is separable if and only if it is the convex combination of rank one projections onto product vectors. In other words, a state ϱ is separable if and only if there exists a finite family of product vectors $\{|\xi_i\rangle|\eta_i\rangle\}$ satisfying

$$\varrho = \sum_i |\xi_i\rangle|\eta_i\rangle\langle\xi_i|\langle\eta_i|. \tag{1.26}$$

If we restrict ourselves to the case of normalized states with trace one, then we have

$$\varrho = \sum_i p_i|\xi_i\rangle|\eta_i\rangle\langle\xi_i|\langle\eta_i|$$

with unit vectors $|\xi_i\rangle$, $|\eta_i\rangle$ and probability distribution $\{p_i\}$. A state in the tensor product $M_m \otimes M_n$ is called a *bi-partite state*, or more precisely an $m \otimes n$ *state* to emphasize the size of matrices.

We denote by $M_m^+ \otimes M_n^+$ the convex cone generated by $a \otimes b$ with $a \in M_m^+$ and $b \in M_n^+$;

$$M_m^+ \otimes M_n^+ := \text{conv}\,\{a \otimes b \in M_m \otimes M_n : a \in M_m^+,\ b \in M_n^+\}.$$

Since $|\xi\rangle|\eta\rangle\langle\xi|\langle\eta| = |\xi\rangle\langle\xi| \otimes |\eta\rangle\langle\eta|$ belongs to $M_m^+ \otimes M_n^+$, we have $\mathcal{S}_1 \subset M_m^+ \otimes M_n^+$. Conversely, if $a = \sum_p |\xi_p\rangle\langle\xi_p| \in M_m^+$ and $b = \sum_q |\eta_q\rangle\langle\eta_q| \in M_n^+$ then we have

$$a \otimes b = \sum_{p,q} |\xi_p\rangle|\eta_q\rangle\langle\xi_p|\langle\eta_q|$$

is separable. Therefore, we have

$$\mathcal{S}_1 = M_m^+ \otimes M_n^+.$$

In short, a bi-partite state is separable if and only if it is the sum of *product states* of the forms $a \otimes b$ with positive $a \in M_m^+$ and $b \in M_n^+$. So, entanglement consists of the difference

$$(M_m \otimes M_n)^+ \backslash M_m^+ \otimes M_n^+.$$

In the case of the space $C(X)$ of all continuous functions on a compact Hausdorff space X, the algebraic tensor product $C(X) \otimes C(Y)$ consists of finite sums of functions of separable variables, that is, functions of the form $f(x)g(y)$. By *Stone–Weierstrass theorem*, we see that $C(X) \otimes C(Y)$ is dense in $C(X \times Y)$. We write $P := C(X)^+ \otimes C(Y)^+$. Then $P - P := \{f - g : f, g \in P\}$ is a unital subalgebra of $C(X \times Y, \mathbb{R})$ which separates points in $X \times Y$. Applying the real version of Stone–Weierstrass theorem, we see that $P - P$ is dense in $C(X \times Y, \mathbb{R})$. Therefore, we conclude that $P = C(X)^+ \otimes C(Y)^+$ is dense in $C(X \times Y)^+$, and there exists no entanglement in function spaces.

If a state $\varrho \in M_m \otimes M_n$ is of rank one, then it is easy to find its Schmidt number by the Schmidt rank of its range vector. For example,

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.27}$$

is of Schmidt number one, or equivalently separable, because the range vector

$$(|0\rangle + |1\rangle)|0\rangle = (1, 0, 1, 0)^{\mathrm{T}} \in \mathbb{C}^2 \otimes \mathbb{C}^2$$

corresponds to the 2×2 matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

which is of rank one. On the other hand,

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \tag{1.28}$$

is of Schmidt number two, and so entangled, because the range vector $(1, 0, 0, 1)^{\mathrm{T}}$ corresponds to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ which is of rank two.

Note that the matrix in (1.28) is the Choi matrix of the identity map on M_2. The Choi matrix

$$\mathrm{C}_{\mathrm{id}} = \sum_{i,j=1}^{n} |i\rangle\langle j| \otimes |i\rangle\langle j| = |\omega\rangle\langle\omega|$$

of the identity map on M_n with

$$|\omega\rangle = \sum_{i=1}^{n} |i\rangle|i\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$$

plays an important role. It is a bi-partite state with the maximal Schmidt number. In general, a pure state $\varrho = |\zeta\rangle\zeta|$ is called *maximally entangled* if $|\zeta\rangle = \sum |\xi_i\rangle \otimes |\eta_i\rangle$ with orthonomal bases $\{|\xi_i\rangle\}$ and $\{|\eta_i\rangle\}$. On the other hand, the identity matrix in $M_m \otimes M_n$ is the Choi matrix of the *trace map*

$$\mathrm{Tr} : X \mapsto \mathrm{Tr}\,(X) I_n, \qquad X \in M_m.$$

The trace map may be considered to be located at the center of the convex cone $\mathbb{P}_1$ of all positive maps, by the location of the identity matrix in $M_m \otimes M_n$. In fact, we will see in Section 2.3 that the converse of Proposition 1.2.5 holds, and so the trace map is an interior point of the convex cone $\mathbb{P}_1[M_m, M_n]$. On the other hand, the identity map between M_n is located at a corner of the convex cone $\mathbb{P}_1[M_n, M_n]$, since it generates an extreme ray of $\mathbb{P}_1$.

For given complex numbers α, β with modulus one, we take a product vector

$$|\zeta\rangle = (1, \alpha\bar{\beta})^{\mathrm{T}} \otimes (1, \beta)^{\mathrm{T}} = (1, \beta, \alpha\bar{\beta}, \alpha)^{\mathrm{T}} \in \mathbb{C}^2 \otimes \mathbb{C}^2,$$

to get the pure separable state

$$|\zeta\rangle\langle\zeta| = \begin{pmatrix} 1 & \bar\beta & \bar\alpha\beta & \bar\alpha \\ \beta & 1 & \bar\alpha\beta^2 & \bar\alpha\beta \\ \alpha\bar\beta & \alpha\bar\beta^2 & 1 & \bar\beta \\ \alpha & \alpha\bar\beta & \beta & 1 \end{pmatrix}.$$

Taking $\beta = 1, \mathrm{i}, -1, -\mathrm{i}$, and averaging the corresponding states, we get the following separable state

$$\begin{pmatrix} 1 & \cdot & \cdot & \bar\alpha \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \alpha & \cdot & \cdot & 1 \end{pmatrix} \in M_2 \otimes M_2. \tag{1.29}$$

For double indices (i,k) and (j,ℓ) from $\{1,\dots,m\}\times\{1,\dots,n\}$ with $(i,k) \neq (j,\ell)$, we consider the bi-partite state $\varrho_{(i,k),(j,\ell)}(\alpha)$ in $M_m \otimes M_n$ defined by

$$\varrho_{(i,k),(j,\ell)}(\alpha) = I_{mn} + \bar\alpha|ik\rangle\langle j\ell| + \alpha|j\ell\rangle\langle ik|,$$

for a complex number α with modulus one. It is clear that $\varrho_{(i,k),(j,\ell)}(\alpha)$ is positive. If $i = j$ then the entries α and $\bar\alpha$ appear in the i-the diagonal block, and so it is separable. This is also the case when $k = \ell$. In order to deal with the case $i \neq j$ and $k \neq \ell$, we consider the product vector

$$|\zeta\rangle = (|i\rangle + \alpha\bar\beta|j\rangle) \otimes (|k\rangle + \beta|\ell\rangle) = |ik\rangle + \beta|i\ell\rangle + \alpha\bar\beta|jk\rangle + \alpha|j\ell\rangle. \tag{1.30}$$

By the exactly same way to get the separable state (1.29), we see that $\varrho_{(i,k),(j,\ell)}(\alpha)$ is separable whenever $|\alpha| = 1$, by adding diagonal entries. Now, it is easily seen that

$$\varrho_{(i,k),(j,\ell)}(+1) - I_{mn}, \qquad \varrho_{(i,k),(j,\ell)}(+\mathrm{i}) - I_{mn}, \qquad (I_{mn} + |ik\rangle\langle ik|) - I_{mn}$$

make a linearly independent family with the cardinality $(mn)^2$. Therefore, we conclude that $\mathcal{S}_1[M_m \otimes M_n]$ has nonzero volume in $(M_m \otimes M_n)^+$.

References: [128], [34], [124], [40], [106]

1.4.3 Partial transposes

The Choi matrix of the transpose map is given by

$$\mathrm{C_T} = \sum_{i,j=1}^{n} |i\rangle\langle j| \otimes T(|i\rangle\langle j|) = \sum_{i,j=1}^{n} |i\rangle\langle j| \otimes |j\rangle\langle i|.$$

When $n = 2$, its matrix form is given by

$$\mathrm{C_T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which is not positive.

We see that $C_T \in M_n(M_n)$ is the block-wise transpose of C_{id} as an $n \times n$ matrix with entries from M_n. Every $\varrho \in M_m \otimes M_n$ is uniquely expressed by

$$\varrho = \sum_{i,j=1}^{m} |i\rangle\langle j| \otimes \varrho_{ij}$$

with $\varrho_{ij} \in M_n$. We define the *partial transpose* ϱ^Γ by

$$\varrho^\Gamma = \sum_{i,j=1}^{m} |j\rangle\langle i| \otimes \varrho_{ij} = \sum_{i,j=1}^{m} |i\rangle\langle j| \otimes \varrho_{ji}.$$

We note that C_T is the partial transpose of C_{id}. In general, we have

$$C_{\phi\circ T} = \sum_{i,j} |i\rangle\langle j| \otimes \phi(|j\rangle\langle i|) = (C_\phi)^\Gamma. \tag{1.31}$$

For a given linear map $\phi : M_m \to M_n$, we define the linear map $\bar{\phi} : M_m \to M_n$ by

$$\bar{\phi}(x) = \overline{\phi(\bar{x})}, \qquad x \in M_m,$$

where $\bar{x}$ is the matrix whose entries are the conjugates of the corresponding entries of x. Then, we have

$$C_{\bar{\phi}} = \overline{C_\phi}.$$

If ϕ is Hermiticity preserving then we have

$$\phi(x^T)^T = \overline{\phi(x^T)^*} = \overline{\phi(x^{T*})} = \overline{\phi(\bar{x})} = \bar{\phi}(x).$$

In short, we have

$$T \circ \phi \circ T = \bar{\phi},$$

for a Hermiticity preserving map ϕ. Therefore, we also have

$$C_{T\circ\phi} = C_{\bar{\phi}\circ T} = C_{\bar{\phi}}^\Gamma = \overline{C_\phi^\Gamma}, \tag{1.32}$$

for a Hermiticity preserving map ϕ. We also note the following identity

$$(a \otimes b)^\Gamma = a^T \otimes b$$

holds. If a and b are Hermitian, then we have

$$\overline{(a \otimes b)^\Gamma} = \overline{a^T \otimes b} = a^* \otimes (b^*)^T = a \otimes b^T.$$

Because $\mathcal{S}_1 = M_m^+ \otimes M_n^+$ and the transpose of a positive matrix is again positive, we have the following:

Theorem 1.4.3 *The partial transpose of a separable state is positive.*

A positive matrix is called *of positive partial transpose* or just *PPT* if its partial transpose is positive. Theorem 1.4.3, which is called the *PPT criterion* for separability says that every separable state is of PPT. It is also can be seen from the inclusion relation (1.15). On the other hand, Proposition 1.2.7 tells us that if a rank one state in $M_m \otimes M_n$ is of PPT then it should be separable. More generally, it is known [63] that if a PPT state in $M_m \otimes M_n$ has rank at most $\max\{m, n\}$ then it is separable.

Note that the matrix in (1.27) is of PPT, but the matrix C_{id} in (1.28) is not of PPT, from which we may infer that C_{id} is not separable. In fact, we know that the positive map $\mathrm{id} = \mathrm{Ad}_I$ is not 1-superpositive, because $\operatorname{rank} I > 1$. The convex cone of all $m \otimes n$ PPT state will be denoted by $\mathcal{PPT}[M_m \otimes M_n]$ or just by $\mathcal{PPT}$. Then we have $\mathcal{S}_1 \subset \mathcal{PPT} \subset \mathcal{S}_{m \wedge n} = (M_m \otimes M_n)^+$ in the diagram (1.25). The converse of Theorem 1.4.3 does not hold in general.

References: [24], [94], [63]

1.4.4 Entanglement with positive partial transpose

For given positive matrices a and b, it is easy to see that $\ker(a+b) \subset \ker a$ by the identity $\langle \xi | (a+b) | \xi \rangle = \langle \xi | a | \xi \rangle + \langle \xi | b | \xi \rangle$. Therefore, we have $\operatorname{Im} a \subset \operatorname{Im}(a+b)$ for $a, b \in M_n^+$, and so we also have

$$\operatorname{Im} a + \operatorname{Im} b = \operatorname{Im}(a+b), \qquad a, b \in M_n^+.$$

This simple fact is useful to show that a given state is not separable, because the expression $\varrho = \sum |\zeta_i\rangle\langle\zeta_i|$ gives rise to the restriction $|\zeta_i\rangle \in \operatorname{Im} \varrho$. In fact, we note that the following

$$\operatorname{Im} \varrho = \operatorname{span} \{ |\zeta_i\rangle \}$$

holds for a positive matrix $\varrho = \sum |\zeta_i\rangle\langle\zeta_i|$.

It is easily seen that the following matrix

$$\varrho = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \frac{1}{2} & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 2 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{2} & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \frac{1}{2} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 2 & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \end{pmatrix} \tag{1.33}$$

is a PPT state. In fact, we have $\varrho^\Gamma = \varrho$. We see that the range is the 4-dimensional space which is spanned by

$$\begin{aligned}
&|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle, \\
&\frac{1}{\sqrt{2}}|0\rangle|1\rangle + \sqrt{2}|1\rangle|0\rangle, \quad \frac{1}{\sqrt{2}}|1\rangle|2\rangle + \sqrt{2}|2\rangle|1\rangle, \quad \frac{1}{\sqrt{2}}|2\rangle|0\rangle + \sqrt{2}|0\rangle|2\rangle,
\end{aligned} \tag{1.34}$$

with the corresponding 3×3 matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}. \tag{1.35}$$

If ϱ is separable, then the four dimensional space spanned by these four matrices must contain rank one matrices. In other words, there exist $a, b, c, d \in \mathbb{C}$ such that

$$\begin{pmatrix} a & \frac{b}{\sqrt{2}} & d\sqrt{2} \\ b\sqrt{2} & a & \frac{c}{\sqrt{2}} \\ \frac{d}{\sqrt{2}} & c\sqrt{2} & a \end{pmatrix}$$

is of rank one. Considering various 2×2 submatrices, one see easily that this is not possible. Therefore, we conclude that ϱ is an entangled state.

A nonzero subspace of $\mathbb{C}^m \otimes \mathbb{C}^n$ is called *entangled* when it has no nonzero product vector. A state is entangled whenever its range space is entangled. We have seen that the four dimensional subspace of $\mathbb{C}^3 \otimes \mathbb{C}^3$ spanned by four vectors in (1.34) is entangled.

Even though the range space of a state is not entangled, a PPT state ϱ may be shown to be entangled by considering the range space of the partial transpose ϱ^Γ as well as the range space of ϱ itself. Suppose that ϱ is a separable state with the expression in (1.26) whose range space is spanned by the product vectors $|\xi_i\rangle|\eta_i\rangle$. By the relation

$$\begin{aligned}
|\xi\rangle|\eta\rangle\langle\xi|\langle\eta|^\Gamma &= (|\xi\rangle\langle\xi| \otimes |\eta\rangle\langle\eta|)^\Gamma \\
&= |\xi\rangle\langle\xi|^{\mathrm{T}} \otimes |\eta\rangle\langle\eta| \\
&= |\bar{\xi}\rangle\langle\bar{\xi}| \otimes |\eta\rangle\langle\eta| = |\bar{\xi}\rangle|\eta\rangle\langle\bar{\xi}|\langle\eta|,
\end{aligned}$$

we have the following identity

$$\varrho^\Gamma = \sum_i |\bar{\xi}_i\rangle|\eta_i\rangle\langle\bar{\xi}_i|\langle\eta_i|,$$

for $\varrho = \sum_i |\xi_i\rangle|\eta_i\rangle\langle\xi_i|\langle\eta_i|$. If the partial transpose ϱ^Γ is positive then we see that the range space ϱ^Γ is spanned by product vectors $|\bar{\xi}_i\rangle|\eta_i\rangle$. Therefore, we have the following necessary condition for separability.

Theorem 1.4.4 *Suppose that ϱ is a PPT state. If ϱ is separable then there exists a finite family $\{|\xi_i\rangle|\eta_i\rangle\}$ of product vectors satisfying*

$$\operatorname{Im}\varrho = \operatorname{span}\{|\xi_i\rangle|\eta_i\rangle\}, \qquad \operatorname{Im}\varrho^\Gamma = \operatorname{span}\{|\bar{\xi}_i\rangle|\eta_i\rangle\}.$$

The vector $|\bar{\xi}\rangle|\eta\rangle$ is called the *partial conjugate* of the product vector $|\xi\rangle|\eta\rangle$. Theorem 1.4.4 is called the *range criterion* for separability which is especially useful for PPT states whose range is not full. For an example, we consider the following PPT state ϱ_p defined by

$$\varrho_p = \begin{pmatrix} p & \cdot & \cdot & \cdot & p & \cdot & \cdot & \cdot & p \\ \cdot & p^2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ p & \cdot & \cdot & \cdot & p & \cdot & \cdot & \cdot & p \\ \cdot & \cdot & \cdot & \cdot & \cdot & p^2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p^2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ p & \cdot & \cdot & \cdot & p & \cdot & \cdot & \cdot & p \end{pmatrix}, \tag{1.36}$$

for a positive number $p > 0$. The partial transpose is given by

$$\varrho_p^\Gamma = \begin{pmatrix} p & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & p^2 & \cdot & p & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & p & \cdot & \cdot \\ \cdot & p & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & p & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & p^2 & \cdot & p & \cdot \\ \cdot & \cdot & p & \cdot & \cdot & \cdot & p^2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & p & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & p \end{pmatrix}.$$

We note that ranks of ϱ_p and ϱ_p^Γ are 7 and 6, respectively. We also note that the kernels of ϱ_p and ϱ_p^Γ are spanned by

$$\begin{pmatrix} 1 \\ \cdot \\ \cdot \\ \cdot \\ -1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}, \quad \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ -1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \cdot \\ -1 \\ \cdot \\ p \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}, \quad \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ -1 \\ \cdot \\ p \\ \cdot \end{pmatrix}, \quad \begin{pmatrix} \cdot \\ \cdot \\ p \\ \cdot \\ \cdot \\ \cdot \\ -1 \\ \cdot \\ \cdot \end{pmatrix},$$

respectively.

If ϱ_p is separable then there exists a product vector $|\xi\rangle|\eta\rangle \in \operatorname{Im}\varrho$ such that $|\bar{\xi}\rangle|\eta\rangle \in \operatorname{Im}\varrho^\Gamma$. We write $|\xi\rangle = (x_1, x_2, x_3)^{\mathrm{T}} \in \mathbb{C}^3$ and $|\eta\rangle = (y_1, y_2, y_3)^{\mathrm{T}} \in \mathbb{C}^3$ then we have

$$|\xi\rangle|\eta\rangle = (x_1y_1, x_1y_2, x_1y_3, x_2y_1, x_2y_2, x_2y_3, x_3y_1, x_3y_2, x_3y_3)^{\mathrm{T}} \in \mathbb{C}^3 \otimes \mathbb{C}^3.$$

Therefore, we have the following system of equations

$$\begin{aligned} x_1y_1 &= x_2y_2, \\ x_2y_2 &= x_3y_3, \\ \bar{x}_1y_2 &= p\bar{x}_2y_1, \\ \bar{x}_2y_3 &= p\bar{x}_3y_2, \\ \bar{x}_3y_1 &= p\bar{x}_1y_3 \end{aligned} \tag{1.37}$$

We note that this system of equations has four unknowns up to scalar multiplications and five equations, and so we may expect that there is no nontrivial solution like $|\xi\rangle|\eta\rangle \neq 0$. In fact, one can easily see that (1.37) has a nontrivial solution only when $p = 1$. Therefore, we conclude that ϱ_p is a PPT entangled states for $p \neq 1$.

References: [129], [24], [113], [62], [83], [68], [69]

1.5 Duality and completely positive maps

We define in this section a bilinear pairing between Hermiticity preserving linear maps through Choi matrices. The dual object of k-superpositivity is nothing but k-positivity which had been studied since 1950's. Through the discussion, we naturally get the correspondence between complete positivity of linear maps and positivity of their Choi matrices.

1.5.1 Dual cones of convex cones

For finite dimensional vector spaces X and Y, we recall that every linear map $\phi \in L(X, Y^*)$ corresponds to the linear functional $L_\phi \in (X \otimes Y)^*$ given by

$$L_\phi(x \otimes y) = \phi(x)(y), \qquad x \in X,\ y \in Y, \tag{1.38}$$

where X^* denotes the *dual space* of X consisting of all linear functionals on X. In case of matrix algebras, we employ the identification (1.3) between M_n and its dual M_n^*. If $\phi : M_m \to M_n^*$ then the Choi matrix C_ϕ plays the role of a linear functional on $M_m \otimes M_n$ by the relation (1.21):

$$\langle a \otimes b, \mathrm{C}_\phi\rangle_{M_m \otimes M_n} = \langle b, \phi(a)\rangle_{M_n}, \qquad a \in M_m,\ b \in M_n,$$

which is nothing but the reformulation of (1.38) in terms of bilinear pairing between matrices. Therefore, it is natural to define the bilinear pairing between $M_m \otimes M_n$ and $L(M_m, M_n)$ by

$$\langle a \otimes b, \phi \rangle_1 := \langle b, \phi(a) \rangle, \qquad a \in M_m,\ b \in M_n,\ \phi \in L(M_m, M_n). \tag{1.39}$$

Then we have $\langle \varrho, \phi \rangle_1 = \langle \varrho, \mathrm{C}_\phi \rangle$ by (1.21) for $\varrho \in M_m \otimes M_n$ and $\phi \in L(M_m, M_n)$. Because every $\varrho \in M_m \otimes M_n$ can be written by $\varrho = \mathrm{C}_\psi$ for $\psi \in L(M_m, M_n)$, it is also natural to define the bilinear pairing

$$\langle \psi, \phi \rangle_2 = \langle \mathrm{C}_\psi, \mathrm{C}_\phi \rangle, \qquad \phi, \psi \in L(M_m, M_n) \tag{1.40}$$

between mapping spaces. If $\varrho = \mathrm{C}_\psi$, then we have

$$\langle \varrho, \phi \rangle_1 = \langle \varrho, \mathrm{C}_\phi \rangle = \langle \mathrm{C}_\psi, \mathrm{C}_\phi \rangle = \langle \psi, \phi \rangle_2,$$

and so we do not distinguish the bilinear pairings $\langle\,\cdot\,,\cdot\,\rangle$, $\langle\,\cdot\,,\cdot\,\rangle_1$ and $\langle\,\cdot\,,\cdot\,\rangle_2$, and use the same notation $\langle\,\cdot\,,\cdot\,\rangle$ for them.

The following identity

$$\langle \phi, \psi \rangle = \langle \phi^*, \psi^* \rangle \tag{1.41}$$

is immediate, because $\mathrm{C}_{\phi^*} \in M_n \otimes M_m$ is the flip of $\mathrm{C}_\phi \in M_m \otimes M_n$ by (1.22). For given linear maps $\phi \in L(M_A, M_B)$, $\psi \in L(M_B, M_C)$ and $\sigma \in L(M_A, M_C)$, we also have

$$\langle \psi \circ \phi, \sigma \rangle = \sum_{i,j} \langle \psi(\phi(e_{ij})), \sigma(e_{ij}) \rangle = \sum_{i,j} \langle \phi(e_{ij}), \psi^*(\sigma(e_{ij})) \rangle = \langle \phi, \psi^* \circ \sigma \rangle,$$

which also implies $\langle \phi, \psi^* \circ \sigma \rangle = \langle \sigma^* \circ \psi, \phi^* \rangle = \langle \psi, \sigma \circ \phi^* \rangle$ by (1.41). Therefore, we have

$$\langle \psi \circ \phi, \sigma \rangle = \langle \phi, \psi^* \circ \sigma \rangle = \langle \psi, \sigma \circ \phi^* \rangle. \tag{1.42}$$

Suppose that X and Y are finite-dimensional real spaces which are dual to each other through a *non-degenerate* bilinear pairing $\langle\ ,\ \rangle$, that is, $x \in X$ satisfies $\langle x, y \rangle = 0$ for every $y \in Y$ implies $x = 0$, and same for $y \in Y$. To be precise, we define the linear map $\sigma_X : X \to Y^*$; for a given $x \in X$, we define $\sigma_X(x) \in Y^*$ as the linear functional which sends y to $\langle x, y \rangle$. Then non-degeneracy condition tells us that σ_X is injective, and so we have $\dim X \leqslant \dim Y$. By the similar map $\sigma_Y : Y \to X^*$, we have $\dim X = \dim Y$, and both σ_X and σ_Y are linear isomorphisms. Throughout this note, we always assume that a bilinear pairing is non-degenerate.

For a given subset C of X, we define the *dual cone* C° in Y by

$$C^\circ = \{ y \in Y : \langle x, y \rangle \geqslant 0 \text{ for each } x \in C \},$$

and the dual cone $D^\circ \subset X$ similarly for a subset D of Y. If C is a closed convex cone in a finite dimensional real vector space X and $x_0 \notin C$, then it is known that there exists a linear functional f on X such that $f(x) \geqslant 0$ for every $x \in C$ and $f(x_0) < 0$. From this, we have the following:

Proposition 1.5.1 *For any subset C of X, the bidual cone $C^{\circ\circ}$ is the smallest closed convex cone containing C.*

Proof. We denote by $\mathcal{X}$ temporarily the smallest closed convex cone containing C, then it is clear that $\mathcal{X} \subset C^{\circ\circ}$. Suppose that $x_0 \notin \mathcal{X}$. Then there exists $y \in Y$ such that $\langle x, y\rangle \geqslant 0$ for every $x \in \mathcal{X}$ but $\langle x_0, y\rangle < 0$. Since $C \subset \mathcal{X}$, we have $y \in C^\circ$. But the relation $\langle x_0, y\rangle < 0$ tells us that $x_0 \notin C^{\circ\circ}$. Therefore, we conclude that $C^{\circ\circ} = \mathcal{X}$. □

References: [99], [127], [34], [108], [107], [36]

1.5.2 k-positive maps and completely positive maps

By Proposition 1.5.1, we see that every closed convex cone C of X is the dual cone of $C^\circ \subset Y$, that is $C^{\circ\circ} = C$. This means that C is determined by the intersection of 'half-spaces' $\{x \in X : \langle x, y\rangle \geqslant 0\}$ given by $y \in C^\circ$. We are going to determine the dual cone of the convex cone $\mathbb{SP}_k$ with respect to the pairing (1.40). This is same as the dual cone of $\mathcal{S}_k$ with respect to the pairing (1.39).

For any $|\zeta\rangle = \sum_{i=1}^k |x_i\rangle|y_i\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ with $\operatorname{SR}|\zeta\rangle \leqslant k$, we have

$$|\zeta\rangle\langle\zeta| = \sum_{i,j=1}^k |x_i\rangle\langle x_j| \otimes |y_i\rangle\langle y_j|,$$

and so, we also have

$$\begin{aligned}\langle|\zeta\rangle\langle\zeta|, \mathrm{C}_\phi\rangle &= \sum_{i,j=1}^k \langle|y_i\rangle\langle y_j|, \phi(|x_i\rangle\langle x_j|)\rangle \\ &= \left\langle \sum_{i,j=1}^k |i\rangle\langle j| \otimes |y_i\rangle\langle y_j|, \sum_{i,j=1}^k |i\rangle\langle j| \otimes \phi(|x_i\rangle\langle x_j|) \right\rangle.\end{aligned}$$

Putting

$$|\xi\rangle = \sum_{i=1}^k |i\rangle|x_i\rangle \in \mathbb{C}^k \otimes \mathbb{C}^m, \qquad |\eta\rangle = \sum_{i=1}^k |i\rangle|y_i\rangle \in \mathbb{C}^k \otimes \mathbb{C}^n,$$

we see that

$$\langle|\zeta\rangle\langle\zeta|, \mathrm{C}_\phi\rangle = \langle|\eta\rangle\langle\eta|, (\mathrm{id}_k \otimes \phi)(|\xi\rangle\langle\xi|)\rangle. \tag{1.43}$$

Therefore, we have the following:

Theorem 1.5.2 *A map* $\phi : M_m \to M_n$ *belongs to* $\mathbb{SP}_k^\circ$ *if and only if the map* $\mathrm{id}_k \otimes \phi$ *from* $M_k \otimes M_m$ *into* $M_k \otimes M_n$ *is positive.*

We say that ϕ is *k-positive* if the map $\mathrm{id}_k \otimes \phi$ is positive, and *completely positive* if it is k-positive for every $k = 1, 2, \ldots$. The convex cone of all k-positive maps (respectively completely positive maps) from M_m into M_n is denoted by $\mathbb{P}_k[M_m, M_n]$ (respectively $\mathbb{CP}[M_m, M_n]$).

Theorem 1.5.3 *For a linear map* $\phi : M_m \to M_n$*, the following are equivalent:*

(i) ϕ *is completely positive,*

(ii) $\mathrm{id}_{m \wedge n} \otimes \phi$ *is positive,*

(iii) ϕ *belongs to* $\mathbb{SP}_{m \wedge n}$*,*

(iv) C_ϕ *is positive.*

Proof. Since the convex cone $(M_m \otimes M_n)^+$ is self-dual, the equivalences between (ii), (iii) and (iv) follows from Theorem 1.5.2. By the relation

$$(\mathrm{id}_k \otimes \mathrm{Ad}_s)(a \otimes b) = a \otimes (s^* b s) = (I_k \otimes s)^*(a \otimes b)(I_k \otimes s),$$

we see that every map in $\mathbb{SP}_{m \wedge n}$ is completely positive. □

For $a = \sum_{i,j} a_{ij} |i\rangle\langle j|$ and $b = \sum_{i,j} b_{ij} |i\rangle\langle j|$ in M_n, the *Schur product* $a \circ b$ is given by $a \circ b = \sum_{i,j} a_{ij} b_{ij} |i\rangle\langle j|$. For a positive $a \in M_n^+$, the map

$$\Sigma_a : x \mapsto a \circ x, \qquad x \in M_n \tag{1.44}$$

is a completely positive map whose Choi matrix is given by

$$\mathrm{C}_{\Sigma_a} = \sum_{i,j} a_{ij} |i\rangle\langle j| \otimes |i\rangle\langle j| = \sum_{i,j} a_{ij} |i\rangle |i\rangle\langle j| \langle j|,$$

which is positive. Considering the partial transpose $\mathrm{C}_{\Sigma_a}^\Gamma$, we see that C_{Σ_a} is separable if and only if it is diagonal by Theorem 1.4.3.

By the statement (iii) of Theorem 1.5.3, every completely positive map ϕ is of the form

$$\phi = \sum_{i \in I} \mathrm{Ad}_{s_i}$$

for a finite family $\{s_i : i \in I\}$ of $m \times n$ matrices. This is called a *Kraus decomposition* of ϕ. This decomposition is obtained from a decomposition of $\mathrm{C}_\phi = \sum |\xi_i\rangle\langle \xi_i|$ into the sum of rank one positive matrices $|\xi_i\rangle\langle \xi_i|$, and so it is far from being unique.

If we take $\{|\xi_i\rangle\}$ to be linearly independent then we can also take a decomposition $\phi = \sum \mathrm{Ad}_{s_i}$ so that $\{s_i\}$ is linearly independent $m \times n$ matrices.

A state $\varrho \in M_m \otimes M_n$ is called *k-block-positive* if $\langle\zeta|\varrho|\zeta\rangle \geqslant 0$ for every vector $|\zeta\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ with $\mathrm{SR}\,|\zeta\rangle \leqslant k$. Since $\langle|\zeta\rangle\langle\zeta|, \mathrm{C}_\phi\rangle = \langle\bar{\zeta}|\mathrm{C}_\phi|\bar{\zeta}\rangle$, we also see that ϕ is k-positive if and only if C_ϕ is k-block-positive. The convex cone of all k-block-positive matrices in $M_m \otimes M_n$ is denoted by $\mathcal{BP}_k[M_m \otimes M_n]$. We complement the diagram (1.25) as follows:

$$\begin{array}{ccccccccccc} L(M_m, M_n): & \mathbb{SP}_1 & \subsetneq & \mathbb{SP}_k & \subsetneq & \mathbb{CP} & \subsetneq & \mathbb{P}_k & \subsetneq & \mathbb{P}_1 \\ \downarrow \mathrm{JC} & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M_m \otimes M_n: & \mathcal{S}_1 & \subsetneq & \mathcal{S}_k & \subsetneq & (M_m \otimes M_n)^+ & \subsetneq & \mathcal{BP}_k & \subsetneq & \mathcal{BP}_1 \end{array} \tag{1.45}$$

In this diagram, $\mathbb{SP}_k$ and $\mathbb{P}_k$ are dual to each others with respect to the pairing (1.40). On the other hand, the duality relation between $\mathcal{S}_k$ and $\mathbb{P}_k$ with respect to the pair (1.39) may be stated as follows:

Theorem 1.5.4 *For $k = 1, 2, \ldots, m \wedge n$, we have the following:*

(i) *A linear map $\phi : M_m \to M_n$ is k-positive if and only if $\langle\varrho, \phi\rangle \geqslant 0$ for every state ϱ with Schmidt number $\leqslant k$,*

(ii) *A state $\varrho \in M_m \otimes M_n$ is of Schmidt number $\leqslant k$ if and only if $\langle\varrho, \phi\rangle \geqslant 0$ for every k-positive map $\phi : M_m \to M_n$.*

Compositions of k-positive maps by the transpose map give rise another important class of positive maps. We denote by T_k the transpose map on M_k.

Proposition 1.5.5 *For a linear map $\phi : M_m \to M_n$, the following are equivalent:*

(i) $\phi \circ \mathrm{T}_m$ *is k-positive,*

(ii) $\mathrm{T}_n \circ \phi$ *is k-positive,*

(iii) $\mathrm{T}_k \otimes \phi$ *is positive,*

(iv) C_ϕ^Γ *is k-block-positive.*

Proof. We note that ϱ is k-block-positive if and only if $\bar{\varrho}$ is k-block-positive, and so the equivalence between (i), (ii) and (iv) follows from the identities (1.31) and (1.32). On the other hand, the identity

$$\mathrm{id}_k \otimes (\mathrm{T}_n \circ \phi) = (\mathrm{T}_k \otimes \mathrm{T}_n) \circ (\mathrm{T}_k \otimes \phi)$$

tells us that (ii) and (iii) are equivalent, because both $\mathrm{T}_k \otimes \mathrm{T}_m$ and its inverse are positive maps. □

We say that ϕ is *k-copositive* when ϕ satisfies the conditions in Proposition 1.5.5, and *completely copositive* when it is k-copositive for every $k = 1, 2, \ldots$. The convex cone of all completely copositive maps will be denoted by $\mathbb{CCP}$, or $\mathbb{CCP}[M_m, M_n]$ to specify the domain and the range. Then we have

$$\mathbb{CCP}[M_m, M_n] = \mathbb{SP}^{m\wedge n},$$

because $\mathbb{CP} = \mathbb{SP}_{m\wedge n}$. We also see that ϕ is completely copositive if and only if $\phi \circ \mathrm{T}_m$ is completely positive if and only if C_ϕ^Γ is positive. We say that $\varrho \in M_m \otimes M_n$ is *copositive* when ϱ^Γ is positive.

References: [72], [73], [21], [66], [34], [124], [108],

1.5.3 Faces for completely positive maps

Recall that every face of the convex cone $(M_m \otimes M_n)^+$ is determined by a subspace of $\mathbb{C}^m \otimes \mathbb{C}^n$; for a subspace V, the set of all positive matrices whose range spaces are contained in V is a face of $(M_m \otimes M_n)^+$, and every face is in this form. Therefore, every face of the convex cone $\mathbb{CP}[M_m, M_n]$ is also determined by a subspace of the space $M_{m\times n}$ of all $m \times n$ matrices.

Proposition 1.5.6 *For a given subspace V of $M_{m\times n}$, the set*

$$F_V := \mathrm{conv}\,\{\mathrm{Ad}_{s_i} : s_i \in V\} \tag{1.46}$$

is a face of $\mathbb{CP}[M_m, M_n]$, and every face arises in this way.

For a given $p \in M_n^+$, we denote by $\mathbb{CP}[M_m, M_n; p]$ the convex set of all completely positive maps ϕ with $\phi(I) = p$. We note that $\phi = \sum \mathrm{Ad}_{s_i}$ belongs to $\mathbb{CP}[M_m, M_n; p]$ if and only if $\sum s_i^* s_i = p$. For a given completely positive map $\phi = \sum_{i\in I} \mathrm{Ad}_{s_i}$ in $\mathbb{CP}[M_m, M_n; p]$, we take the subspace $V = \mathrm{span}\,\{s_i : i \in I\}$. Then every map in F_V is of the form $\psi = \sum_k \mathrm{Ad}_{t_k}$ with $t_k \in V$. Write $t_k = \sum_i \alpha_{ik} s_i$. Then we have

$$\begin{aligned}\psi(x) = \sum_k t_k^* x t_k &= \sum_k \left(\sum_i \bar{\alpha}_{ik} s_i^*\right) x \left(\sum_j \alpha_{jk} s_j\right)\\ &= \sum_{i,j} \left(\sum_k \bar{\alpha}_{ik}\alpha_{jk}\right) s_i^* x s_j.\end{aligned}$$

We write $\beta_{ij} = \sum_k \bar{\alpha}_{ik}\alpha_{jk}$. Then we see that $\psi \in \mathbb{CP}[M_m, M_n; p]$ if and only if $\sum_{i,j} \beta_{ij} s_i^* s_j = p$, and $\phi(I) = \psi(I)$ if and only if $\sum_{i,j} (\beta_{ij} - \delta_{ij}) s_i^* s_j = 0$, where $\delta_{ij} = 1$

for $i = j$, and $\delta_{ij} = 0$ for $i \neq j$. If $\{s_i^* s_j : i, j \in I\}$ is linearly independent then we see that $\psi = \phi$, that is, the the intersection of F_V and $\mathbb{CP}[M_m, M_n; p]$ consists of a single point $\{\phi\}$. Therefore, we conclude that ϕ is an extreme point of the convex set $\mathbb{CP}[M_m, M_n; p]$ whenever $\{s_i^* s_j\}$ is linearly independent. The converse is also true. We will use the following lemma whose simple proof is omitted.

Lemma 1.5.7 *Suppose that* $\{|v_i\rangle : i = 1, \dots, n\}$ *is linearly independent in* $\mathbb{C}^n$. *Then* $\{|v_i\rangle\langle v_j| : i, j = 1, \dots, n\}$ *is linearly independent in* M_n.

Theorem 1.5.8 *Suppose that* $p \in M_n^+$ *and* $\phi = \sum_{i \in I} \mathrm{Ad}_{s_i}$ *with linearly independent family* $\{s_i\}$ *of* $m \times n$ *matrices with* $\sum_{i \in I} s_i^* s_i = p$. *Then* ϕ *is an extreme point of the convex set* $\mathbb{CP}[M_m, M_n; p]$ *if and only if* $\{s_i^* s_j : i, j \in I\}$ *is linearly independent.*

Proof. Suppose that $\phi = \sum \mathrm{Ad}_{s_i}$ is an extreme point of the convex set $\mathbb{CP}[M_m, M_n; p]$ and $\sum_{i,j} \alpha_{ij} s_i^* s_j = 0$. We have to show that $\alpha_{ij} = 0$. We first consider the case when $\alpha = [\alpha_{ij}]$ is a Hermitian matrix. We may assume that $-I \leqslant [\alpha_{ij}] \leqslant I$ by taking a scalar multiplication. Define

$$\phi_\pm(x) = \sum_i \mathrm{Ad}_{s_i}(x) \pm \sum_{i,j} \alpha_{ij} s_i^* x s_j, \qquad x \in M_m.$$

If we write $\delta_{ij} + \alpha_{ij} = \beta_{ij}$, then we have $\phi_+(x) = \sum_{i,j} \beta_{ij} s_i^* x s_j$. Writing $[\beta_{ij}] = \gamma^* \gamma$, we have

$$\phi_+(x) = \sum_{i,j} \sum_k \bar{\gamma}_{ki} \gamma_{kj} s_i^* x s_j = \sum_k t_k^* x t_k,$$

with $t_k = \sum_j \gamma_{kj} s_j$. Therefore, we see that ϕ_+ is completely positive. By the same way, we also see that ϕ_- is also completely positive. The relation $\phi = \frac{1}{2}(\phi_+ + \phi_-)$ tells us that $\phi = \phi_+ = \phi_-$. Considering the Choi matrices of the map $x \mapsto \sum_{i,j} \alpha_{ij} s_i^* x s_j$, we conclude that $\alpha = [\alpha_{ij}] = 0$ by Lemma 1.5.7. For the general cases, we take conjugate to get $\sum_{i,j} \bar{\alpha}_{ji} s_i^* s_j = 0$. Then we have $\sum_{i,j} (\alpha_{ij} + \bar{\alpha}_{ji}) s_i^* s_j = 0$. Since $\alpha + \alpha^*$ is self-adjoint, we see that $\alpha + \alpha^* = 0$. By the same reasoning, we also have $\mathrm{i}\alpha - \mathrm{i}\alpha^* = 0$. Therefore, we conclude that $\alpha = \frac{1}{2}(\alpha + \alpha^*) + \frac{1}{2\mathrm{i}}(\mathrm{i}\alpha - \mathrm{i}\alpha^*)$ must be zero. □

References: [21], [76]

1.5.4 Decomposable maps and PPT states

Recall that a map in $\mathbb{SP}_{m \wedge n} + \mathbb{SP}^{m \wedge n}$ is called decomposable. Then ϕ is decomposable if and only if it is the sum of a completely positive map and a completely copositive map. The convex cone of all decomposable maps will be denoted by $\mathbb{DEC}[M_m, M_n]$ or just by $\mathbb{DEC}$, and the corresponding convex cone of matrices in $M_m \otimes M_n$ is denoted by $\mathcal{DEC}(M_m \otimes M_n)$ or $\mathcal{DEC}$ whose member is the sum of a positive matrix and a copositive matrix. On the other hand, the linear map corresponding to a PPT matrix will be also called a *PPT map*, and the convex cone of all PPT maps is denoted by $\mathbb{PPT}[M_m, M_n]$ or $\mathbb{PPT}$. A map is a PPT map if and only if it is both completely positive and completely copositive. We may summarize as follows:

$$
\begin{array}{ccccccccccc}
L(M_m, M_n): & \mathbb{SP}_1 & \subset & \mathbb{PPT} & \subset & \mathbb{CP} & \subset & \mathbb{DEC} & \subset & \mathbb{P}_1 \\
\downarrow \text{JC} & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
M_m \otimes M_n: & \mathcal{S}_1 & \subset & \mathcal{PPT} & \subset & (M_m \otimes M_n)^+ & \subset & \mathcal{DEC} & \subset & \mathcal{BP}_1
\end{array} \tag{1.47}
$$

Suppose that C_1 and C_2 are closed convex cones in a finite dimensional real vector space X. Then the identity $(C_1 + C_2)^\circ = C_1^\circ \cap C_2^\circ$ is easily seen. We also have a clear inclusion $(C_1 \cap C_2)^\circ \supset C_1^\circ + C_2^\circ$, from which we have

$$C_1 \cap C_2 \subset (C_1^\circ + C_2^\circ)^\circ = C_1^{\circ\circ} \cap C_2^{\circ\circ} = C_1 \cap C_2.$$

Hence, we have the relations

$$(C_1 + C_2)^\circ = C_1^\circ \cap C_2^\circ, \qquad (C_1 \cap C_2)^\circ = C_1^\circ + C_2^\circ, \tag{1.48}$$

for closed convex cones C_1 and C_2. Therefore, we see that two convex cones $\mathbb{PPT}$ and $\mathbb{DEC}$ are dual to each other with respect to the pairing (1.40). On the other hand, $\mathcal{PPT}$ and $\mathbb{DEC}$ are dual with respect to the pairing (1.39).

Theorem 1.5.9 *A linear map $\phi : M_m \to M_n$ is decomposable if and only if the inequality $\langle \varrho, \phi \rangle \geqslant 0$ holds for every PPT state $\varrho \in M_m \otimes M_n$.*

In view of two diagrams (1.45) and (1.47), it is natural to look for any relations between $\mathcal{PPT}$ and $\mathcal{S}_k$, or equivalently those between $\mathbb{DEC}$ and $\mathbb{P}_k$. We first note that $\mathcal{S}_1[M_m \otimes M_n] = \mathcal{PPT}[M_m \otimes M_n]$ holds if and only if $\mathbb{DEC}[M_m, M_n] = \mathbb{P}_1[M_m, M_n]$ holds. Theorem 1.3.4 tells us the relation $\mathbb{DEC}[M_2, M_2] = \mathbb{P}_1[M_2, M_2]$ holds, by which we also have

$$\mathcal{S}_1[M_2 \otimes M_2] = \mathcal{PPT}[M_2 \otimes M_2].$$

It was also shown in [129] that the following identities

$$\mathbb{DEC}[M_2, M_3] = \mathbb{P}_1[M_2, M_3], \qquad \mathbb{DEC}[M_3, M_2] = \mathbb{P}_1[M_3, M_2]$$

hold which are equivalent to $\mathcal{S}_1[M_2 \otimes M_3] = \mathcal{PPT}[M_2 \otimes M_3]$.

By the example (1.33), we know that the strict inclusion

$$\mathcal{S}_1[M_3 \otimes M_3] \subsetneqq \mathcal{PPT}[M_3 \otimes M_3]$$

holds in $M_3 \otimes M_3$, and so we have

$$\mathbb{DEC}[M_3, M_3] \subsetneqq \mathbb{P}_1[M_3, M_3].$$

We will see a concrete example for this strict inclusion in the next section. It was shown in [18] that the following inclusion

$$\mathbb{P}_2[M_3, M_3] \subset \mathbb{DEC}[M_3, M_3] \tag{1.49}$$

holds for a special class of linear maps, and the equivalent dual inclusion relation

$$\mathcal{S}_2[M_3 \otimes M_3] \subset \mathcal{PPT}[M_3 \otimes M_3]$$

had been conjectured in [100]. The relation (1.49) was proved in [133], and the relation $\mathbb{P}_{n-1}[M_n, M_n] \subset \mathbb{DEC}[M_n, M_n]$ is conjectured in [26].

It is also an interesting question to ask how large Schmidt numbers are attained by PPT states. It is known [17, 65, 14] that PPT states may have arbitrary large Schmidt numbers when we increase the size of matrices. It is equivalent to look for indecomposable k-positive maps for large k. See [10]. In the case of $m \geqslant 3$ and $n = 4$, it is unknown if there exists a PPT state with Schmidt number greater than 2, or equivalently if there exists an indecomposable 2-positive map.

References: [129], [18], [100], [133], [17], [65], [26], [14], [10]

1.6 Nontrivial examples of positive maps

In this section, we exhibit Tomiyama's examples of linear maps which distinguish k-positivity for different k's, by considering the line segment between the trace map and the identity map, which are located at the center and the boundary of the convex cone $\mathbb{P}_1$, respectively. We also give two examples of indecomposable positive maps by Choi and Woronowicz.

1.6.1 Tomiyama's example of k-positive maps

In this section, we will explore examples of positive maps which are not completely positive. By the strict inclusion (1.13), we also have the following strict inclusions of convex cones in the space $H(M_n, M_n)$;

$$\mathbb{CP} = \mathbb{P}_n \subsetneqq \mathbb{P}_{n-1} \subsetneqq \cdots \mathbb{P}_k \subsetneqq \cdots \subsetneqq \mathbb{P}_1.$$

We first look for explicit examples to distinguish the above convex cones.

We recall that the trace map $\frac{1}{n}\mathrm{Tr}_n$ is located in the center of the convex cone $\mathbb{CP}$, and the identity map id_n generates an extreme ray of $\mathbb{P}_1$. We consider the line segment between these two maps to define

$$\phi_\lambda = (1-\lambda)\frac{1}{n}\mathrm{Tr}_n + \lambda\,\mathrm{id}_n, \qquad -\infty < \lambda < +\infty. \tag{1.50}$$

It is clear that $\phi_\lambda \in \mathbb{CP}$ for $\lambda \in [0,1]$, and $\phi_\lambda \in \mathbb{P}_1$ implies $\lambda \leqslant 1$. We have

$$\begin{aligned} \mathrm{C}_{\phi_\lambda} &= \sum_{i,j=1}^{n} |i\rangle\langle j| \otimes \frac{1-\lambda}{n}\delta_{ij} I_n + \lambda \sum_{i,j=1}^{n} |i\rangle\langle j| \otimes |i\rangle\langle j| \\ &= \frac{1-\lambda}{n} I_n \otimes I_n + \lambda|\omega\rangle\langle\omega|, \end{aligned}$$

with $|\omega\rangle = \sum_{i=1}^{n} |i\rangle|i\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$. We take $|\zeta\rangle = \sum_{i=1}^{k} |i\rangle|i\rangle$ with $\mathrm{SR}\,|\zeta\rangle = k$. If $\phi_\lambda \in \mathbb{P}_k$ then we have

$$0 \leqslant \langle \mathrm{C}_{\phi_\lambda}, |\zeta\rangle\langle\zeta|\rangle = \frac{1-\lambda}{n} k + \lambda k^2 = \frac{k}{n}[1 + \lambda(nk-1)],$$

from which we get a necessary condition

$$\frac{-1}{nk-1} \leqslant \lambda \leqslant 1 \tag{1.51}$$

for $\phi_\lambda \in \mathbb{P}_k$. When $\lambda = \frac{-1}{nk-1}$, the map ϕ_λ is the scalar multiplication of the map

$$\tau_{n,k} = k\mathrm{Tr}_n - \mathrm{id}_n, \qquad k = 1, 2, \ldots, n, \tag{1.52}$$

whose Choi matrix is given by

$$\mathrm{C}_{\tau_{n,k}} = kI_n \otimes I_n - |\omega\rangle\langle\omega|.$$

In order to show that (1.51) is also sufficient for $\phi_\lambda \in \mathbb{P}_k$, it suffices to show that $\tau_{n,k}$ is k-positive. For this purpose, we need the following:

Proposition 1.6.1 *For a linear map $\phi : M_m \to M_n$ and $k = 1, 2, \ldots, n$, the following are equivalent:*

(i) *ϕ is k-positive,*

(ii) *$\mathrm{Ad}_p \circ \phi^*$ is completely positive for every rank k projection p in M_m,*

(iii) *$(I_n \otimes p)\mathrm{C}_{\phi^*}(I_n \otimes p) \in M_n \otimes M_m$ is positive for every rank k projection p in M_m.*

(iv) *$(p \otimes I_n)\mathrm{C}_\phi(p \otimes I_n) \in M_m \otimes M_n$ is positive for every rank k projection p in M_m.*

(v) $\sum_{i,j=1}^{k} |\bar{\xi}_i\rangle\langle\bar{\xi}_j| \otimes \phi(|\xi_i\rangle\langle\xi_j|) \in M_m \otimes M_n$ *is positive for every orthornormal family* $\{\xi_i : i = 1, 2, \dots, k\}$ *in* $\mathbb{C}^m$.

(vi) $\sum_{i,j=1}^{k} |i\rangle\langle j| \otimes \phi(|\xi_i\rangle\langle\xi_j|) \in M_k \otimes M_n$ *is positive for every orthornormal family* $\{\xi_i : i = 1, 2, \dots, k\}$ *in* $\mathbb{C}^m$.

Proof. In order to prove (i) $\Longrightarrow$ (ii), we suppose that ϕ is k-positive and $p \in M_m$ is a rank k projection. Take a unitary $u \in M_m$ which sends $\mathbb{C}^k \times \{0\} \subset \mathbb{C}^m$ onto the range of p. Then $\mathrm{Ad}_u \circ \mathrm{Ad}_p \circ \phi^*$ is a k-positive map from M_n into M_k, and so it is completely positive by Theorem 1.5.3. Therefore, we see that $\mathrm{Ad}_p \circ \phi^*$ is completely positive. For the reverse direction, we suppose that (ii) holds, and s is arbitrary rank k matrix in $M_{m\times n}$. Take a projection $p \in M_m$ such that $\operatorname{rank} p \leqslant k$ and $ps = s$, to see that

$$\langle \phi, \mathrm{Ad}_s \rangle = \langle \phi, \mathrm{Ad}_{ps} \rangle = \langle \phi, \mathrm{Ad}_s \circ \mathrm{Ad}_p \rangle = \langle \mathrm{Ad}_{s^{\mathrm{T}}} \circ \phi, \mathrm{Ad}_p \rangle = \langle \mathrm{Ad}_{s^{\mathrm{T}}}, \mathrm{Ad}_p \circ \phi^* \rangle.$$

By the assumption, $\mathrm{Ad}_p \circ \phi^*$ is completely positive, and so we have $\langle \phi, \mathrm{Ad}_s \rangle \geqslant 0$ and conclude that ϕ is k-positive. The equivalence (ii) $\Longleftrightarrow$ (iii) also comes out from Theorem 1.5.3, since the Choi matrix of $\mathrm{Ad}_p \circ \phi^*$ is just $(I_n \otimes p)\mathrm{C}_{\phi^*}(I_n \otimes p)$. Since C_{ϕ^*} is the flip of C_ϕ, we see that (iii) and (iv) are equivalent.

We take arbitrary projection p of rank k, and write $p = \sum_{i=1}^{k} |\bar{\xi}_i\rangle\langle\bar{\xi}_i|$ for orthonormal family $\{|\xi_i\rangle : i = 1, 2, \dots, k\}$ of $\mathbb{C}^m$. Then we have

$$\begin{aligned}
(p \otimes I_n)\mathrm{C}_\phi(p \otimes I_n) &= \sum_{s,t=1}^{m} p|s\rangle\langle t|p \otimes \phi(|s\rangle\langle t|) \\
&= \sum_{s,t=1}^{m} \sum_{i,j=1}^{k} |\bar{\xi}_i\rangle\langle\bar{\xi}_i|s\rangle\langle t|\bar{\xi}_j\rangle\langle\bar{\xi}_j| \otimes \phi(|s\rangle\langle t|) \\
&= \sum_{i,j=1}^{k} |\bar{\xi}_i\rangle\langle\bar{\xi}_j| \otimes \phi\left(\sum_{s,t=1}^{m} \langle\bar{\xi}_i|s\rangle|s\rangle\langle t|\langle t|\bar{\xi}_j\rangle \right) \\
&= \sum_{i,j=1}^{k} |\bar{\xi}_i\rangle\langle\bar{\xi}_j| \otimes \phi(|\xi_i\rangle\langle\xi_j|),
\end{aligned}$$

which shows (iv) $\Longleftrightarrow$ (v). The remaining equivalence between (v) and (vi) is easily seen by considering the isometry from $\mathbb{C}^k$ into $\mathbb{C}^m$ which sends $|i\rangle$ to $|\bar{\xi}_i\rangle$. □

Proposition 1.6.2 *For* $k = 1, 2, \dots, n$, *the map* $\tau_{n,k} : M_n \to M_n$ *is* k*-positive.*

Proof. In order to apply Proposition 1.6.1 (vi), we take an orthonormal family $\{|\xi_i\rangle : i = 1, 2, \dots, k\}$ of vectors in $\mathbb{C}^n$. Then we have $\tau_{n,k}(|\xi_i\rangle\langle\xi_j|) = k\delta_{ij} I_n - |\xi_i\rangle\langle\xi_j|$, and so it follows that

$$\begin{aligned}\sum_{i,j=1}^{k} |i\rangle\langle j| \otimes \phi(|\xi_i\rangle\langle\xi_j|) &= k\sum_{i=1}^{k} |i\rangle\langle i| \otimes I_n - \sum_{i,j=1}^{k} |i\rangle\langle j| \otimes |\xi_i\rangle\langle\xi_j| \\ &= kI_k \otimes I_n - |\xi\rangle\langle\xi|,\end{aligned}$$

with $|\xi\rangle = \sum_{i=1}^{k} |i\rangle|\xi_i\rangle \in \mathbb{C}^k \otimes \mathbb{C}^n$. Now, we conclude that $kI_k \otimes I_n - |\xi\rangle\langle\xi| \geqslant 0$ holds since $\langle\xi|\xi\rangle = k$. ▫

Theorem 1.6.3 *For $k = 1, 2, \dots, n$, the map $\phi_\lambda : M_n \to M_n$ is k-positive if and only if λ satisfies the inequality*

$$\frac{-1}{nk-1} \leqslant \lambda \leqslant 1.$$

We note that ϕ_λ is completely copositive if and only if $(\mathrm{C}_{\phi_\lambda})^\Gamma$ is positive if and only if $\frac{1-\lambda}{n} \geqslant |\lambda|$ if and only if

$$\frac{-1}{n-1} \leqslant \lambda \leqslant \frac{1}{n+1}.$$

Summarizing, we have the following:

Proposition 1.6.4 *The linear ϕ_λ in* (1.50) *satisfies the following:*

(i) ϕ_λ *is positive if and only if* $\frac{-1}{n-1} \leqslant \lambda \leqslant 1$,

(ii) ϕ_λ *is completely positive if and only if* $\frac{-1}{n^2-1} \leqslant \lambda \leqslant 1$,

(iii) ϕ_λ *is completely copositive if and only if* $\frac{-1}{n-1} \leqslant \lambda \leqslant \frac{1}{n+1}$.

Therefore, ϕ_λ is either completely positive or completely copositive whenever it is positive. Especially, there is no indecomposable positive maps among ϕ_λ's.

References: [20], [125], [120], [126]

1.6.2 The Choi map between 3×3 matrices

In the case of $n = 3$, the map $\tau_{3,1} = 2\phi_{-1/2}$ is located at the end point of the interval on which ϕ_λ is positive, and its Choi matrix is given by

$$\mathrm{C}_{\tau_{3,1}} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & -1 \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ -1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

In order to get an example of indecomposable positive map, we adjust the diagonal part of $C_{\tau_{3,1}}$ as follow:

$$\left(\begin{array}{ccc|ccc|ccc}
1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & -1 \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & -1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & 1
\end{array}\right). \tag{1.53}$$

Note that neither the above matrix nor its partial transpose is positive. This is the Choi matrix of the map defined by

$$\phi_{\mathrm{ch}}(x) = \begin{pmatrix} x_{11}+x_{33} & -x_{12} & -x_{13} \\ -x_{21} & x_{22}+x_{11} & -x_{23} \\ -x_{31} & -x_{32} & x_{33}+x_{22} \end{pmatrix} = \psi(x) - x, \tag{1.54}$$

for $x = [x_{ij}] \in M_3$, with

$$\psi(x) = \begin{pmatrix} 2x_{11}+x_{33} & \cdot & \cdot \\ \cdot & 2x_{22}+x_{11} & \cdot \\ \cdot & \cdot & 2x_{33}+x_{22} \end{pmatrix}.$$

In order to show that ϕ_{ch} is positive, we begin with the following:

Proposition 1.6.5 *Let A be an $n \times n$ nonsingular positive matrix, and $|\xi_0\rangle$ a unit vector. Then $A \geqslant |\xi_0\rangle\langle\xi_0|$ if and only if $\langle\xi_0|A^{-1}|\xi_0\rangle \leqslant 1$.*

Proof. We have $A \geqslant |\xi_0\rangle\langle\xi_0|$ if and only if $I \geqslant A^{-1/2}|\xi_0\rangle\langle\xi_0|A^{-1/2}$ if and only if $\|A^{-1/2}|\xi_0\rangle\| \leqslant 1$ if and only if $\langle\xi_0|A^{-1}|\xi_0\rangle \leqslant 1$. □

Now, we see that ϕ_{ch} is positive if and only if $\psi(|\xi_0\rangle\langle\xi_0|) \geqslant |\xi_0\rangle\langle\xi_0|$ for every unit vector $|\xi_0\rangle$ if and only if

$$\langle\xi_0|\psi(|\xi_0\rangle\langle\xi_0|)^{-1}|\xi_0\rangle \leqslant 1$$

for every unit vector $|\xi_0\rangle$ if and only if the inequality

$$\frac{\alpha}{2\alpha+\gamma} + \frac{\beta}{2\beta+\alpha} + \frac{\gamma}{2\gamma+\beta} \leqslant 1 \tag{1.55}$$

holds for all positive real numbers α, β and γ.

Lemma 1.6.6 *The inequality* (1.55) *holds for all positive real numbers α, β and γ.*

Proof. Taking $x = \frac{\gamma}{\alpha}$, $y = \frac{\alpha}{\beta}$ and $z = \frac{\beta}{\gamma}$, it suffices to show the inequality

$$\frac{1}{2+x} + \frac{1}{2+y} + \frac{1}{2+z} \leqslant 1,$$

or equivalently $xy + yz + zx \geqslant 3$ under the constraint $xyz = 1$. This comes out by comparing the arithmetic and geometric means. □

Therefore, we conclude that the map $\phi_{\rm ch}$ is a positive map. One may check that the bilinear pairing of the matrix (1.53) with the PPT matrix in (1.33) is strictly negative, and so we also conclude that $\phi_{\rm ch}$ is not decomposable. This is another way to see that the PPT state in (1.33) is entangled. The map $\phi_{\rm ch}$ in (1.54) is usually called the *Choi map*.

Theorem 1.6.7 *The Choi map $\phi_{\rm ch}$ defined in* (1.54) *is a positive map which is not decomposable.*

The Choi map $\phi_{\rm ch}$ had been extended in higher dimensions in various directions [121, 89, 18, 132, 90, 41]. See also Section 2.5.

References: [22], [25], [23], [121], [89], [18], [132], [90], [41]

1.6.3 The Woronowicz map from 2×2 matrices to 4×4 matrices

We also give an example of an indecomposable positive map from M_2 into M_4. We define the linear map $\phi_{\rm wo}$ from M_2 into M_4 by

$$\phi_{\rm wo} : \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mapsto \begin{pmatrix} 4x-2y-2z+3w & -2x+2z & \cdot & \cdot \\ -2x+2y & 2x & z & \cdot \\ \cdot & y & 2w & -2z-w \\ \cdot & \cdot & -2y-w & 4x+2w \end{pmatrix}. \quad (1.56)$$

Then the map $\phi_{\rm wo}$ sends the rank one matrix

$$p_\alpha = \begin{pmatrix} 1 & \bar\alpha \\ \alpha & |\alpha|^2 \end{pmatrix}$$

to

$$\begin{pmatrix} |\alpha-2|^2+2|\alpha|^2 & -2+2\alpha & \cdot & \cdot \\ -2+2\bar\alpha & 2 & \alpha & \cdot \\ \cdot & \bar\alpha & 2|\alpha|^2 & -2\alpha-|\alpha|^2 \\ \cdot & \cdot & -2\bar\alpha-|\alpha|^2 & 4+2|\alpha|^2 \end{pmatrix}.$$

The $k \times k$ principal submatrix of the left–upper corner has the determinant Δ_k as follows:

$$\begin{aligned} \Delta_1 &= 2|\alpha|^2 + |\alpha-2|^2, \\ \Delta_2 &= 2|\alpha|^2 + 4, \\ \Delta_3 &= |\alpha|^2|\alpha+2|^2, \\ \Delta_4 &= 0. \end{aligned}$$

Therefore, we see that $\phi_{\rm wo}(p_\alpha)$ is positive when $\alpha \neq 0, -2$. For $\alpha = 0$, $\alpha = -2$ and $\alpha = \infty$, the map $\phi_{\rm wo}$ sends

$$p_0 = \begin{pmatrix} 1 & \cdot \\ \cdot & \cdot \end{pmatrix}, \qquad p_{-2} = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \quad \text{and} \quad p_\infty = \begin{pmatrix} \cdot & \cdot \\ \cdot & 1 \end{pmatrix}$$

to the matrices

$$\begin{pmatrix} 4 & -2 & \cdot & \cdot \\ -2 & 2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 4 \end{pmatrix}, \qquad \begin{pmatrix} 24 & -6 & \cdot & \cdot \\ -6 & 2 & -2 & \cdot \\ \cdot & -2 & 8 & \cdot \\ \cdot & \cdot & \cdot & 20 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 2 & -1 \\ \cdot & \cdot & -1 & 2 \end{pmatrix},$$

respectively, which are positive. Therefore, we see that the map $\phi_{\rm wo}$ is a positive map. We note that $\phi(p_\alpha)$ is of rank three for every $\alpha \in \mathbb{C} \cup \{\infty\}$. The map $\phi_{\rm wo}$ is called the *Woronowicz map*. The Choi matrix of $\phi_{\rm wo}$ is given by

$$\mathrm{C}_{\phi_{\rm wo}} = \left(\begin{array}{cccc|cccc} 4 & -2 & \cdot & \cdot & -2 & \cdot & \cdot & \cdot \\ -2 & 2 & \cdot & \cdot & 2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 4 & \cdot & \cdot & -2 & \cdot \\ \hline -2 & 2 & \cdot & \cdot & 3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -2 & \cdot & \cdot & 2 & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 2 \end{array}\right).$$

We will see in Section 2.6 that the map $\phi_{\rm wo}$ in (1.56) is an extremal positive map. If $\phi_{\rm wo}$ is decomposable, then it must be of the form Ad_s or $\mathrm{T} \circ \mathrm{Ad}_s$, and so the Choi matrix is of rank one or its partial transpose is of rank one. We see that neither is the case, and so $\phi_{\rm wo}$ is not decomposable.

References: [130], [49]

1.7 Isotropic states and Werner states

We continue to look for conditions for k-superpositivity of the maps by Tomiyama in the last section, or equivalently, determine the Schmidt numbers of the corresponding Choi matrices and their partial transposes. These are isotropic states and Werner states, respectively.

1.7.1 Isotropic states

We return to the map ϕ_λ defined in (1.50) which is completely positive if and only if $\frac{-1}{n^2-1} \leqslant \lambda \leqslant 1$ by Theorem 1.6.3. In this case, we have the state

$$\varrho_\lambda := \mathrm{C}_{\phi_\lambda} = \frac{1-\lambda}{n} I_n \otimes I_n + \lambda |\omega\rangle\langle\omega| = \frac{1-\lambda}{n} \mathrm{C}_{\rm Tr} + \lambda \mathrm{C}_{\rm id} \tag{1.57}$$

which is called the *isotropic state*. We recall that the vector $|\omega\rangle$ was defined by $|\omega\rangle = \sum_{i=1}^n |ii\rangle$. We have already seen in Proposition 1.6.4 that ϱ_λ^Γ is positive if and only if ϕ_λ is completely copositive if and only if $\frac{-1}{n-1} \leqslant \lambda \leqslant \frac{1}{n+1}$, and so ϱ_λ is a PPT state if and only if

$$\frac{-1}{n^2-1} \leqslant \lambda \leqslant \frac{1}{n+1}$$

holds.

When $\lambda = \frac{-1}{n^2-1}$, one can show that

$$\varrho_\lambda = \frac{n}{n^2-1} I_n \otimes I_n - \frac{1}{n^2-1}|\omega\rangle\langle\omega| \tag{1.58}$$

is separable. In fact, we modify the vector $|\zeta\rangle$ given in (1.30) to define

$$|\zeta_{ik}\rangle = (|i\rangle + \beta|k\rangle) \otimes (|i\rangle - \bar{\beta}|k\rangle) = |ii\rangle - \bar{\beta}|ik\rangle + \beta|ki\rangle - |kk\rangle$$

with a complex number β of modulus one, and put

$$\begin{aligned}\varrho_{ik} &= \frac{1}{4} \sum_{\beta=\pm 1, \pm \mathrm{i}} |\zeta_{ik}\rangle\langle\zeta_{ik}| \\ &= |ii\rangle\langle ii| + |ik\rangle\langle ik| + |ki\rangle\langle ki| + |kk\rangle\langle kk| - |ii\rangle\langle kk| - |kk\rangle\langle ii|\end{aligned}$$

for each ordered pair (i, k) with $i \neq k$ and $i, k = 1, 2, \ldots, n$. Summing up all of them, we get the separable state

$$\begin{aligned}&2(n-1)\sum_{i=1}^n |ii\rangle\langle ii| + 2\sum_{i\neq k} |ik\rangle\langle ik| - 2\left(|\omega\rangle\langle\omega| - \sum_{i=1}^n |ii\rangle\langle ii|\right) \\ =&2n\sum_{i=1}^n |ii\rangle\langle ii| + 2\sum_{i\neq k} |ik\rangle\langle ik| - 2|\omega\rangle\langle\omega|.\end{aligned}$$

Adding the separable state $2(n-1)\sum_{i\neq j}^n |ij\rangle\langle ij|$, we finally get the state ϱ_λ in (1.58) up to scalar multiplication. Therefore, we see that $\varrho_\lambda \in \mathcal{S}_n$ if and only if $\varrho_\lambda \in \mathcal{S}_1$, in case of $\lambda < 0$.

We proceed to determine the Schmidt number of the state ϱ_λ, or equivalently look for the condition on λ for which ϕ_λ is k-superpositive. In order to determine the Schmidt number of the state ϱ_λ, we consider the bilinear pairing with ϕ_μ for $\mu = \frac{-1}{nk-1}$. We compute

$$\begin{aligned}\langle\varrho_\lambda, \varrho_\mu\rangle &= n\left(\frac{1-\lambda}{n} + \lambda\right)\left(\frac{1-\mu}{n} + \mu\right) + (n^2-n)\,\frac{1-\lambda}{n}\,\frac{1-\mu}{n} + (n^2-n)\,\lambda\mu \\ &= \lambda\mu(n^2-1) + 1.\end{aligned}$$

Therefore, $\varrho_\lambda \in \mathcal{S}_k$ implies that $\langle\varrho_\lambda, \varrho_\mu\rangle \geqslant 0$ with $\mu = \frac{-1}{nk-1}$, which holds only when $\lambda \leqslant \frac{nk-1}{n^2-1}$. Therefore, we have the following necessary condition

$$\frac{-1}{n^2-1} \leqslant \lambda \leqslant \frac{nk-1}{n^2-1}. \tag{1.59}$$

for $\phi_\lambda \in \mathcal{S}_k$. Since
$$\frac{nk-1}{n^2-1} = \frac{-1}{n^2-1} + \frac{k}{n}\left(1 - \frac{-1}{n^2-1}\right),$$
we see that $\lambda_k := \frac{nk-1}{n^2-1}$ divides the interval $\left[\frac{-1}{n^2-1}, 1\right]$ into n subintervals with the same lengths.

In order to get sufficient conditions for $\phi_\lambda \in \mathbb{SP}_k$, we will use special kinds of symmetries of the identity map id_n and its Choi matrix C_{id}. We call $U \otimes V$ acting on $\mathbb{C}^m \otimes \mathbb{C}^n$ a local unitary when both U and V are unitaries acting on $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively. It is clear that both $(U \otimes V)|\zeta\rangle$ and $|\zeta\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ share the same Schmidt ranks, and so it follows that both states

$$\mathrm{Ad}_{U\otimes V}(\varrho) = (U^* \otimes V^*)\varrho(U \otimes V)$$

and ϱ have the same Schmidt numbers. For a linear map $\phi : \mathbb{C}^m \to \mathbb{C}^n$, we have

$$\begin{aligned}
\langle a \otimes b, \mathrm{Ad}_{U\otimes V}(\mathrm{C}_\phi)\rangle &= \langle \mathrm{Ad}_{U^{\mathrm{T}}\otimes V^{\mathrm{T}}}(a \otimes b), \mathrm{C}_\phi\rangle \\
&= \langle \mathrm{Ad}_{U^{\mathrm{T}}}(a) \otimes \mathrm{Ad}_{V^{\mathrm{T}}}(b), \mathrm{C}_\phi\rangle \\
&= \langle \mathrm{Ad}_{V^{\mathrm{T}}}(b), \phi \circ \mathrm{Ad}_{U^{\mathrm{T}}}(a)\rangle \\
&= \langle b, \mathrm{Ad}_V \circ \phi \circ \mathrm{Ad}_{U^{\mathrm{T}}}(a)\rangle \\
&= \langle a \otimes b, \mathrm{C}_{\mathrm{Ad}_V \circ\phi\circ \mathrm{Ad}_{U^{\mathrm{T}}}}\rangle,
\end{aligned}$$

and so it follows that
$$\mathrm{Ad}_{U\otimes V}(\mathrm{C}_\phi) = \mathrm{C}_{\mathrm{Ad}_V \circ\phi\circ \mathrm{Ad}_{U^{\mathrm{T}}}}. \tag{1.60}$$
Two $n\times n$ unitaries U and V satisfy $\mathrm{Ad}_{U\otimes V}(\mathrm{C}_{\mathrm{id}}) = \mathrm{C}_{\mathrm{id}}$ if and only if $\mathrm{Ad}_V \circ \mathrm{Ad}_{U^{\mathrm{T}}} = \mathrm{id}$ if and only if $U^{\mathrm{T}}V = \lambda I_n$ with $|\lambda| = 1$. Especially, we have

$$\mathrm{Ad}_{U\otimes\bar{U}}(\mathrm{C}_{\mathrm{id}}) = \mathrm{C}_{\mathrm{id}},$$

for every unitary U.

Now, we look for linear maps $\phi \in L(M_n, M_n)$ satisfying $\mathrm{Ad}_{U\otimes\bar{U}}(\mathrm{C}_\phi) = \mathrm{C}_\phi$ for every unitary U. This holds if and only if $\mathrm{Ad}_{\bar{U}} \circ\phi \circ \mathrm{Ad}_{U^{\mathrm{T}}} = \phi$ if and only if

$$\langle |k\rangle\langle \ell|, \phi(|i\rangle\langle j|)\rangle = \langle U|k\rangle\langle \ell|U^*, \phi(\bar{U}|i\rangle\langle j|U^{\mathrm{T}})\rangle. \tag{1.61}$$

Now, we suppose that ϕ satisfies (1.61) for every unitary U, and proceed to determine the (k, ℓ) entry
$$\langle |k\rangle\langle \ell|, \phi(|i\rangle\langle j|)\rangle$$
of $\phi(|i\rangle\langle j|)$ for those ϕ. We first consider the case when one of i, j, k and ℓ, say i, is different from others. In this case, we take the unitary U satisfying $U|i\rangle = -|i\rangle$ and $U|\iota\rangle = |\iota\rangle$ for $\iota = j, k$ and ℓ, to see that $\langle |k\rangle\langle \ell|, \phi(|i\rangle\langle j|)\rangle = 0$. Therefore, it remains to determine

$$\langle |i\rangle\langle j|, \phi(|i\rangle\langle j|)\rangle, \quad \langle |j\rangle\langle i|, \phi(|i\rangle\langle j|)\rangle, \quad \langle |k\rangle\langle k|, \phi(|i\rangle\langle i|)\rangle, \quad \langle |i\rangle\langle i|, \phi(|i\rangle\langle i|)\rangle,$$

with $i \neq j$ and $i \neq k$. Taking the unitary U with $U|i\rangle = |i\rangle$ and $U|j\rangle = \mathrm{i}|j\rangle$, we see that $\langle |j\rangle\langle i|, \phi(|i\rangle\langle j|)\rangle = 0$. Taking the unitaries permuting $|i\rangle$'s, we also see that the values of

$$\alpha := \langle |k\rangle\langle k|, \phi(|i\rangle\langle i|)\rangle, \qquad \beta := \langle |i\rangle\langle j|, \phi(|i\rangle\langle j|)\rangle$$

are independent on the choices of i, j and i, k, respectively. Now, we take the unitary U with $U|i\rangle = \frac{1}{\sqrt{2}}(|i\rangle + |j\rangle)$ and $U|j\rangle = \frac{1}{\sqrt{2}}(|i\rangle - |j\rangle)$ to see that

$$\begin{aligned} \alpha + \beta &= \langle |i\rangle\langle j|, \phi(|i\rangle\langle j|)\rangle + \langle |j\rangle\langle j|, \phi(|i\rangle\langle i|)\rangle \\ &= \frac{1}{2}(\langle |i\rangle\langle i|, \phi(|i\rangle\langle i|)\rangle + \langle |j\rangle\langle j|, \phi(|j\rangle\langle j|)\rangle) \end{aligned}$$

for $i \neq j$. Taking finally the unitary U with $U|i\rangle = |j\rangle$ and $U|j\rangle = |i\rangle$ for $i \neq j$, we have

$$\langle |i\rangle\langle i|, \phi(|i\rangle\langle i|)\rangle = \langle |j\rangle\langle j|, \phi(|j\rangle\langle j|)\rangle,$$

for $i \neq j$, and so we also have

$$\langle |i\rangle\langle i|, \phi(|i\rangle\langle i|)\rangle = \alpha + \beta$$

for every i. Therefore, we have the following:

Proposition 1.7.1 *For a map $\phi : M_n \to M_n$, the following are equivalent:*

(i) $\mathrm{Ad}_{U\otimes\bar{U}}(\mathrm{C}_\phi) = \mathrm{C}_\phi$ *for every unitary U,*

(ii) $\mathrm{Ad}_{\bar{U}} \circ \phi \circ \mathrm{Ad}_{U^{\mathrm{T}}} = \phi$ *for every unitary U,*

(iii) $\phi = \alpha \operatorname{Tr}_n + \beta \operatorname{id}_n$ *for complex numbers α and β.*

For a given $\phi \in \mathbb{P}_1[M_n, M_n]$, we define the linear map

$$T[\phi] := \int_{U(n)} \mathrm{Ad}_{\bar{U}} \circ \phi \circ \mathrm{Ad}_{U^{\mathrm{T}}} \, \mathrm{d}U,$$

where the integration is taken over the unitary group $U(n)$ of all $n \times n$ unitaries with respect to the Haar measure. More precisely, this map is defined by the relation

$$\langle T[\phi], \psi\rangle = \int_{U(n)} \langle \mathrm{Ad}_{\bar{U}} \circ \phi \circ \mathrm{Ad}_{U^{\mathrm{T}}}, \psi\rangle \, \mathrm{d}U, \qquad \psi \in L[M_n, M_n].$$

Because ϕ is k-superpositive if and only if $\mathrm{Ad}_{\bar{U}} \circ \phi \circ \mathrm{Ad}_{U^{\mathrm{T}}}$ is k-superpositive for every unitary U, we see that if ϕ is k-superpositive then $T[\phi]$ is also k-superpositive. It is also clear that $T[\phi]$ satisfies the condition (ii) of Proposition 1.7.1, and so we have $T[\phi] = \alpha \mathrm{Tr}_n + \beta \mathrm{id}_n$ for complex numbers α and β. In order to determine the coefficients α and β, we first note the relations

$$\begin{aligned}\langle \mathrm{Ad}_{\bar{U}} \circ \phi \circ \mathrm{Ad}_{U^{\mathrm{T}}}, \mathrm{Tr}\,\rangle &= \langle \phi, \mathrm{Ad}_{U^*} \circ \mathrm{Tr}\, \circ \mathrm{Ad}_U \rangle = \langle \phi, \mathrm{Tr}\,\rangle \\ \langle \mathrm{Ad}_{\bar{U}} \circ \phi \circ \mathrm{Ad}_{U^{\mathrm{T}}}, \mathrm{id} \rangle &= \langle \phi, \mathrm{Ad}_{U^*} \circ \mathrm{id} \circ \mathrm{Ad}_U \rangle = \langle \phi, \mathrm{id} \rangle,\end{aligned}$$

from which we have

$$\begin{aligned}\langle \phi, \mathrm{id} \rangle &= \langle T[\phi], \mathrm{id} \rangle = \langle \alpha \mathrm{Tr}\, + \beta \mathrm{id}, \mathrm{id} \rangle = n\alpha + n^2 \beta \\ \langle \phi, \mathrm{Tr}\,\rangle &= \langle T[\phi], \mathrm{Tr}\,\rangle = \langle \alpha \mathrm{Tr}\, + \beta \mathrm{id}, \mathrm{Tr}\,\rangle = n^2 \alpha + n\beta.\end{aligned}$$

Solving this equations, we have

$$\alpha = \frac{1}{n^2 - 1} \langle \phi, \mathrm{Tr}_n - \tfrac{1}{n} \mathrm{id}_n \rangle, \qquad \beta = \frac{1}{n^2 - 1} \langle \phi, \mathrm{id}_n - \tfrac{1}{n} \mathrm{Tr}_n \rangle. \tag{1.62}$$

We summarize as follows:

Proposition 1.7.2 *For a linear map $\phi : M_n \to M_n$, we define complex numbers α and β by* (1.62)*. Then we have $T[\phi] = \alpha \mathrm{Tr}_n + \beta \mathrm{id}_n$.*

We take the positive map ϕ whose Choi matrix is given by $\mathrm{C}_\phi = \frac{n}{k} |\zeta\rangle\langle\zeta|$ with $|\zeta\rangle = \sum_{i=1}^k |ii\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$. Then C_ϕ has Schmidt number k, and ϕ is k-superpositive. We also have

$$\alpha = \frac{n - k}{n^2 - 1}, \qquad \beta = \frac{nk - 1}{n^2 - 1}$$

and $T[\phi] = \phi_{\lambda_k}$ with $\lambda_k = \frac{nk-1}{n^2-1}$. Therefore, we conclude that ϕ_{λ_k} is k-superpositive. We have already seen that ϱ_λ with $\lambda = \frac{-1}{n^2-1}$ is separable, and so it belongs to $\mathcal{S}_k$ for every $k = 1, 2, \ldots, n$. Therefore, we conclude that (1.59) gives rise to the necessary and sufficient condition for $\varrho_\lambda \in \mathcal{S}_k$, and summarize as follows:

Theorem 1.7.3 *The linear map ϕ_λ defined by* (1.50) *is k-superpositive if and only if the isotropic state ϱ_λ in* (1.57) *belongs to $\mathcal{S}_k$ if and only if λ satisfies*

$$\frac{-1}{n^2 - 1} \leqslant \lambda \leqslant \frac{nk - 1}{n^2 - 1}.$$

References: [128], [59], [124], [9], [55]

1.7.2 Werner states

It remains to look for conditions on λ for which $\varrho_\lambda \in \mathcal{S}^k$, or equivalently, $\varrho_\lambda^\Gamma \in \mathcal{S}_k$. We recall that ϱ_λ^Γ is positive if and only if $\frac{-1}{n-1} \leqslant \lambda \leqslant \frac{1}{n+1}$. For those λ's, the state ϱ_λ^Γ is called the *Werner state*, which is given by

$$\varrho_\lambda^\Gamma = (\mathrm{C}_{\phi_\lambda})^\Gamma = \mathrm{C}_{\phi_\lambda \circ \mathrm{T}} = \mathrm{C}_{\psi_\lambda},$$

where the map $\psi_\lambda : M_n \to M_n$ is defined by

$$\psi_\lambda := \phi_\lambda \circ \mathrm{T} = (1-\lambda)\frac{\mathrm{Tr}}{n} + \lambda \mathrm{T}.$$

Note that ψ_λ is positive if and only if ϕ_λ is positive if and only if $\frac{-1}{n-1} \leqslant \lambda \leqslant 1$. We also recall that ϱ_λ^Γ is separable if and only if ϱ_λ is separable if and only if $\frac{-1}{n^2-1} \leqslant \lambda \leqslant \frac{1}{n+1}$ by Theorem 1.7.3.

In order to determine the Schmidt number of ϱ_λ^Γ, we take $|\zeta_{ij}\rangle = |ij\rangle - |ji\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$ with Schmidt rank two. Summing up all of them, we have

$$\sum_{i<j} |\zeta_{ij}\rangle\langle\zeta_{ij}| = \sum_{i \neq j} (|ij\rangle\langle ij| - |ij\rangle\langle ji|) = \mathrm{C}_{\mathrm{Tr}} - \mathrm{C}_{\mathrm{T}},$$

which is the Choi matrix of the map

$$\psi_{-1/(n-1)} = \phi_{-1/(n-1)} \circ \mathrm{T} = \left(1 - \frac{-1}{n-1}\right)\frac{1}{n}\mathrm{Tr}_n + \frac{-1}{n-1}\mathrm{T}_n,$$

up to a scalar multiple.

Theorem 1.7.4 *For $k = 2, 3, \ldots, n$, the Werner state ϱ_λ^Γ belongs to $\mathcal{S}_k$ if and only if ψ_λ is k-superpositive if and only if if and only if the following*

$$\frac{-1}{n-1} \leqslant \lambda \leqslant \frac{1}{n+1}$$

holds.

Proof. Suppose that $\varrho_\lambda^\Gamma \in \mathcal{S}_k$. Then the positivity of ϱ_λ^Γ implies the condition. When $\lambda = \frac{-1}{n-1}$, we have seen that $\varrho_\lambda^\Gamma \in \mathcal{S}_2 \subset \mathcal{S}_k$. For $\lambda = \frac{1}{n+1}$, we know that $\varrho_\lambda^\Gamma \in \mathcal{S}_1 \subset \mathcal{S}_k$. □

Therefore, we see that the state ϱ_λ^Γ has Schmidt number two whenever it is not separable. We exhibit the properties of Werner states corresponding to isotropic states. By the relation (1.60), we have

$$\mathrm{Ad}_{U\otimes U}(\mathrm{C}_{\mathrm{T}}) = \mathrm{C}_{\mathrm{T}}.$$

This property actually characterizes the Werner states as follows:

Proposition 1.7.5 *For a map $\psi : M_n \to M_n$, the following are equivalent:*

(i) $\mathrm{Ad}_{U\otimes U}(\mathrm{C}_\psi) = \mathrm{C}_\psi$ *for every unitary U,*

(ii) $\mathrm{Ad}_U \circ \psi \circ \mathrm{Ad}_{U^{\mathrm{T}}} = \psi$ *for every unitary U,*

(iii) $\psi = \alpha \mathrm{Tr}_n + \beta \mathrm{T}_n$ *for complex numbers α and β.*

Proof. We replace ϕ in Proposition 1.7.1 by $\psi = \mathrm{T} \circ \phi$ to get the condition (iii). From Proposition 1.7.1 (ii), we also have

$$\psi = \mathrm{T} \circ \phi = \mathrm{T} \circ \mathrm{Ad}_{\bar{U}} \circ \phi \circ \mathrm{Ad}_{U^{\mathrm{T}}} = \mathrm{Ad}_U \circ \mathrm{T} \circ \phi \circ \mathrm{Ad}_{U^{\mathrm{T}}} = \mathrm{Ad}_U \circ \psi \circ \mathrm{Ad}_{U^{\mathrm{T}}},$$

which is (ii). This is equivalent to (i) by (1.60). □

For a given $\psi \in \mathbb{P}_1[M_n, M_n]$, we define the linear map

$$W[\psi] := \int_{U(n)} \mathrm{Ad}_U \circ \psi \circ \mathrm{Ad}_{U^{\mathrm{T}}}\, \mathrm{d}U,$$

as the definition of $T[\phi]$. Then we have

$$W[\psi] = \alpha \mathrm{Tr}_n + \beta \mathrm{T}_n,$$

with

$$\alpha = \frac{1}{n^2-1} \langle \psi, \mathrm{Tr}_n - \tfrac{1}{n}\mathrm{T}_n \rangle, \qquad \beta = \frac{1}{n^2-1} \langle \psi, \mathrm{T}_n - \tfrac{1}{n}\mathrm{Tr}_n \rangle$$

by the exactly same method as that of Proposition 1.7.2. We take $|\zeta\rangle = |12\rangle - |21\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$ with $\mathrm{SR}\,|\zeta\rangle = 2$. We also take $\psi \in L(M_n, M_n)$ with $\mathrm{C}_\psi = |\zeta\rangle\langle\zeta|$. Then we have $\langle \psi, \mathrm{T}_n \rangle = -2$ and $\langle \psi, \mathrm{Tr}_n \rangle = 2$, and so, we see that the following linear map

$$\frac{n(n+2)}{2(n-1)} W[\psi] = \frac{1}{n-1} (\mathrm{Tr} - \mathrm{T}) = \psi_{-1/(n-1)}$$

is 2-superpositive, which recovers the Schmidt number of the corresponding Werner states.

Finally, we determine λ for which ϕ_λ is k-copositive, or equivalently, ψ_λ is k-positive, for $k = 2, 3, \ldots, n$. Suppose that $\psi_\lambda \in \mathbb{P}_k$. Then we have

$$0 \leqslant \langle \psi_\lambda, \psi_{-1/(n-1)} \rangle = -(n+1)\lambda + 1,$$

and so $\lambda \leqslant \frac{1}{n+1}$. This implies that ψ_λ is completely positive by Proposition 1.6.4, and so k-positive. Because $\phi_{-1/(n-1)}$ is both 1-copositive and completely copositive, we have the following:

Theorem 1.7.6 *For $k = 2, 3, \ldots, n$, the map ϕ_λ is k-copositive if and only if ψ_λ is k-positive if and only if the following inequality*

$$\frac{-1}{n-1} \leqslant \lambda \leqslant \frac{1}{n+1}$$

holds.

Together with Theorem 1.7.5, we see that 2-superpositivity and 2-positivity coincide for the linear maps ψ_λ's. We summarize in Figure 1.2.

References: [126], [128]

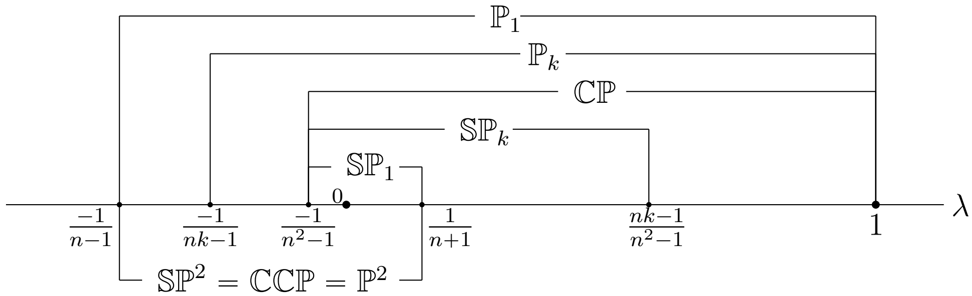

Figure 1.2: Various kinds of positivity of the map $\phi_\lambda = (1-\lambda)\frac{1}{n}\mathrm{Tr}_n + \lambda \mathrm{id}_n$

1.8 Mapping cones and tensor products

Various convex cones of positive maps considered so far are invariant under compositions by completely positive maps. This property implies further characterizations of various kinds of positive maps in terms of composition, ampliation and tensor products. In order to explain them in a unified framework, we exhibit an identity which relates compositions and tensor products of linear maps between matrix algebras.

In this section, we use the notations M_A and M_B for matrix algebras which are acting on the finite dimensional spaces $\mathbb{C}^A$ and $\mathbb{C}^B$, respectively, in order to emphasize the roles of parties in the tensor product. The convex cone $\mathbb{CP}[M_A, M_B]$ will be abbreviated by $\mathbb{CP}_{AB}$, sometimes just by $\mathbb{CP}$; the identity map on M_A will be denoted by id_A. For sets K_1 and K_2 of linear maps, $K_1 \circ K_2$ will denote the set of all $\phi_1 \circ \phi_2$ with $\phi_i \in K_i$ for $i = 1, 2$. We also denote by C_K the set of of all C_ϕ with $\phi \in K$.

1.8.1 Mapping cones of positive maps

A closed convex cone K of positive linear maps in $H(M_A, M_B)$ is called a *right mapping cone* if $K \circ \mathbb{CP}_{AA} \subset K$, and a *left mapping cone* if $\mathbb{CP}_{BB} \circ K \subset K$. A closed convex cone K is also called a *mapping cone* when $\mathbb{CP}_{BB} \circ K \circ \mathbb{CP}_{AA} \subset K$ holds. Since $\mathrm{id}_A \in \mathbb{CP}_{AA}$ and $\mathrm{id}_B \in \mathbb{CP}_{BB}$, it is clear that K is a mapping cone if and only if it is both left and right mapping cones. The following proposition is an immediate consequence of the identity (1.42);

$$\langle \psi \circ \phi, \sigma \rangle = \langle \phi, \psi^* \circ \sigma \rangle = \langle \psi, \sigma \circ \phi^* \rangle.$$

Proposition 1.8.1 *Suppose that K is a closed convex cone satisfying the relation $\mathbb{SP}_1 \subset K \subset \mathbb{P}_1$. Then K is a right mapping cone if and only if K° is a right mapping cone.*

Proof. Note that K is a right mapping cone if and only if $\langle \phi \circ \sigma, \psi \rangle \geqslant 0$ for every $\phi \in K$, $\sigma \in \mathbb{CP}_{AA}$ and $\psi \in K^\circ$. By the identity (1.42), this is the case if and only if $\langle \phi, \psi \circ \sigma^* \rangle \geqslant 0$ for every $\phi \in K$, $\sigma \in \mathbb{CP}_{AA}$ and $\psi \in K^\circ$ if and only if K° is a right mapping cone, since $(\mathbb{CP}_{AA})^* = \mathbb{CP}_{AA}$. □

It is clear that $\mathbb{SP}_k$ is a mapping cone, and so all the convex cones in the mapping space $H(M_A, M_B)$ appearing in the diagrams (1.45) are mapping cones. If K_1 and K_2 are mapping cones then their intersection $K_1 \cap K_2$ and their convex hull $K_1 + K_2$ are also mapping cones. Therefore, all the convex cones of $H(M_A, M_B)$ appearing in (1.47) are also mapping cones. We note that $K \subset K \circ \mathbb{CP}$ holds automatically, and so we have $(K \circ \mathbb{CP})^\circ \subset K^\circ$ in general.

Proposition 1.8.2 *A closed convex cone K of positive maps in $H(M_A, M_B)$ is a left mapping cone if and only if $K^\circ = (\mathbb{CP}_{BB} \circ K)^\circ$, and it is a right mapping cone if and only if $K^\circ = (K \circ \mathbb{CP}_{AA})^\circ$. It is a mapping cone if and only if $K^\circ = (\mathbb{CP}_{BB} \circ K \circ \mathbb{CP}_{AA})^\circ$.*

As another application of the identity (1.42), we see that $\phi \in (K \circ \mathbb{CP})^\circ$ if and only if $\langle \phi, \psi \circ \sigma \rangle \geqslant 0$ for every $\psi \in K$ and $\sigma \in \mathbb{CP}$ if and only if $\langle \psi^* \circ \phi, \sigma \rangle \geqslant 0$ for every $\psi \in K$ and $\sigma \in \mathbb{CP}$ if and only if $\psi^* \circ \phi \in \mathbb{CP}$ for every $\psi \in K$. In short, we have

$$(K \circ \mathbb{CP})^\circ = \{\phi \in H(M_m, M_n) : \psi^* \circ \phi \in \mathbb{CP} \text{ for every } \psi \in K\},$$

which coincide with K° if and only if K is a right mapping cone by Proposition 1.8.2. Therefore, we see that the dual cone of a mapping cone K can be described in terms of composition as well as bilinear pairing, and this is possible only when K is a one-sided mapping cone. We summarize as follows:

Theorem 1.8.3 *For a closed convex cone K of positive maps in the space $H(M_A, M_B)$, the following are equivalent:*

(i) *K is a right mapping cone,*

(ii) *$\phi \in H(M_A, M_B)$ belongs to K° if and only if $\psi^* \circ \phi$ is completely positive for every $\psi \in K$.*

The following are also equivalent:

(iii) *K is a left mapping cone,*

(iv) *$\phi \in H(M_A, M_B)$ belongs to K° if and only if $\phi \circ \psi^*$ is completely positive for every $\psi \in K$.*

References: [114], [116], [106], [107], [117], [36]

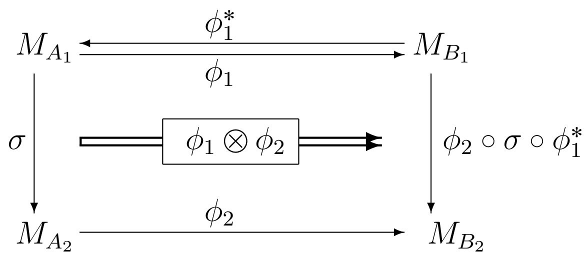

Figure 1.3: The map $\phi_1 \otimes \phi_2$ sends C_σ to $\mathrm{C}_{\phi_2 \circ \sigma \circ \phi_1^*}$.

1.8.2 Tensor products and compositions of linear maps

So far, we have described $(K \circ \mathbb{CP})^\circ$ and $(\mathbb{CP} \circ K)^\circ$ in terms of compositions. We are going to describe them in terms of tensor products of linear maps. To do this, we need an identity which connects compositions and tensor products of linear maps. For given linear maps $\phi_i : M_{A_i} \to M_{B_i}$ with $i = 1, 2$, we note that every element of the domain $M_{A_1} \otimes M_{A_2}$ of the map $\phi_1 \otimes \phi_2$ can be written as the Choi matrix C_σ of a linear map $\sigma : M_{A_1} \to M_{A_2}$, and $\mathrm{C}_{\phi_2 \circ \sigma \circ \phi_1^*}$ belongs to the range $M_{B_1} \otimes M_{B_2}$. See Figure 1.3.

For every $b_1 \in M_{B_1}$ and $b_2 \in M_{B_2}$, we have

$$\begin{aligned}
\langle b_1 \otimes b_2, \mathrm{C}_{\phi_2 \circ \sigma \circ \phi_1^*} \rangle_{B_1 B_2} &= \langle b_2, \phi_2(\sigma(\phi_1^*(b_1))) \rangle_{B_2} \\
&= \langle \sigma^*(\phi_2^*(b_2)), \phi_1^*(b_1) \rangle_{A_1} \\
&= \textstyle\sum_{i,j} \langle \sigma^*(\phi_2^*(b_2)), e_{i,j}^{A_1} \rangle_{A_1} \langle \phi_1^*(b_1), e_{i,j}^{A_1} \rangle_{A_1} \\
&= \textstyle\sum_{i,j} \langle b_2, \phi_2(\sigma(e_{i,j}^{A_1})) \rangle_{B_2} \langle b_1, \phi_1(e_{i,j}^{A_1}) \rangle_{B_1} \\
&= \textstyle\sum_{i,j} \langle b_1 \otimes b_2, \phi_1(e_{i,j}^{A_1}) \otimes \phi_2(\sigma(e_{i,j}^{A_1})) \rangle_{B_1 B_2},
\end{aligned}$$

where $\{e_{i,j}^A\}$ denotes the matrix units of M_A. Therefore, we have

$$\mathrm{C}_{\phi_2 \circ \sigma \circ \phi_1^*} = \textstyle\sum_{i,j} \phi_1(e_{i,j}^{A_1}) \otimes \phi_2(\sigma(e_{i,j}^{A_1})) = \sum_{i,j} (\phi_1 \otimes \phi_2)(e_{i,j}^{A_1} \otimes \sigma(e_{i,j}^{A_1})),$$

which gives rise to the following:

Theorem 1.8.4 *For linear maps $\phi_i : M_{A_i} \to M_{B_i}$ for $i = 1, 2$ and $\sigma : M_{A_1} \to M_{A_2}$, we have the identity*

$$\mathrm{C}_{\phi_2 \circ \sigma \circ \phi_1^*} = (\phi_1 \otimes \phi_2)(\mathrm{C}_\sigma). \tag{1.63}$$

Since $\mathrm{Ad}_{U \otimes V} = \mathrm{Ad}_U \otimes \mathrm{Ad}_V$, we see that the identity (1.60) is, in fact, a special case of (1.63).

References: [80]

1.8.3 Roles of ampliation

We are going to describe the dual cones in terms of ampliation $\mathrm{id}_A \otimes \phi$ or $\phi \otimes \mathrm{id}_B$. To do this, we consider the following diagrams

$$\begin{array}{ccc} M_A & \xrightarrow{\mathrm{id}_A} & M_A \\ \big\downarrow{\scriptstyle \sigma^*} & & \\ M_A & \xrightarrow{\phi} & M_B \end{array} \qquad\qquad \begin{array}{ccc} M_A & \xrightarrow{\phi} & M_B \\ \big\downarrow{\scriptstyle \psi} & & \\ M_B & \xrightarrow{\mathrm{id}_B} & M_B \end{array}$$

which will give us formulae involving $\mathrm{id}_A \otimes \phi$ and $\phi \otimes \mathrm{id}_B$. Indeed, we have

$$\begin{aligned} \langle \phi, \psi \circ \sigma \rangle &= \langle \phi \circ \sigma^*, \psi \rangle = \langle \mathrm{C}_{\phi\circ\sigma^*}, \mathrm{C}_\psi \rangle = \langle (\mathrm{id}_A \otimes \phi)(\mathrm{C}_{\sigma^*}), \mathrm{C}_\psi \rangle, \\ \langle \phi, \sigma \circ \psi \rangle &= \langle \psi \circ \phi^*, \sigma^* \rangle = \langle \mathrm{C}_{\psi\circ\phi^*}, \mathrm{C}_{\sigma^*} \rangle = \langle (\phi \otimes \mathrm{id}_B)(\mathrm{C}_\psi), \mathrm{C}_{\sigma^*} \rangle. \end{aligned}$$

From the first line, we see that $\phi \in (K \circ \mathbb{CP}_{AA})^\circ$ if and only if the ampliation $\mathrm{id}_A \otimes \phi$ sends $(M_A \otimes M_A)^+$ to C_{K°. We also see that $\phi \in (\mathbb{CP}_{BB} \circ K)^\circ$ if and only if $\phi \otimes \mathrm{id}_B$ sends C_K to $(M_B \otimes M_B)^+$ from the second line. Therefore, we have the following:

Theorem 1.8.5 *For a closed convex cone $K \subset H(M_A, M_B)$ of positive maps, the following are equivalent:*

(i) *K is a right mapping cone,*

(ii) *$\phi \in K^\circ$ if and only if $\mathrm{id}_A \otimes \phi$ sends positive matrices to C_{K°.*

The following are also equivalent:

(iii) *K is a left mapping cone,*

(iv) *$\phi \in K^\circ$ if and only if $\phi \otimes \mathrm{id}_B$ sends C_K to positive matrices.*

When K° is given by the mapping cone $\mathbb{SP}_1$, the statement (ii) of Theorem 1.8.5 tells us that $\phi \in \mathbb{SP}_1[M_A, M_B]$ if and only if $\mathrm{id}_A \otimes \phi$ sends every state in $M_A \otimes M_B$ to a separable state. In this context, a 1-superpositive map is also called *entanglement breaking*. On the other hands, the statement (iv) with $K^\circ = \mathbb{DEC}$ and $K^\circ = \mathbb{P}_k$ gives rise to following:

Corollary 1.8.6 *For $\phi : M_A \to M_B$, we have the following:*

(i) *ϕ is decomposable if and only if $\phi \otimes \mathrm{id}_B$ sends PPT states to positive matrices,*

(ii) *ϕ is k-positive if and only if $\phi \otimes \mathrm{id}_B$ sends states with Schmidt numbers $\leqslant k$ to positive matrices.*

Every vector $|\zeta\rangle = \sum_{i=1}^{\ell} |\xi_i\rangle|\eta_i\rangle \in \mathbb{C}^A \otimes \mathbb{C}^A$ has Schmidt number $\leqslant k$ whenever $\ell \leqslant k$. Therefore, we see that ϕ is k-positive if and only if

$$(\phi \otimes \mathrm{id}_A)(|\zeta\rangle\langle\zeta|) = \sum_{i,j=1}^{\ell} \phi(|\xi_i\rangle\langle\xi_j|) \otimes |\eta_i\rangle\langle\eta_j|$$

is positive whenever $\ell \leqslant k$ if and only if $\sum_{i,j=1}^{\ell} |\eta_i\rangle\langle\eta_j| \otimes \phi(|\xi_i\rangle\langle\xi_j|)$ is positive whenever $\ell \leqslant k$. Compare with the conditions (v) and (vi) of Proposition 1.6.1.

The dual cone of a mapping cone can be also explained in terms of Choi matrices of maps in the dual cone. To do this, we consider the following diagrams

$$\begin{array}{ccc} M_A & \xrightarrow{\mathrm{id}_A} & M_A \\ \downarrow{\scriptstyle \phi} & & \\ M_B & \xrightarrow{\psi^*} & M_A \end{array} \qquad\qquad \begin{array}{ccc} M_A & \xrightarrow{\psi} & M_B \\ \downarrow{\scriptstyle \phi} & & \\ M_B & \xrightarrow{\mathrm{id}_B} & M_B \end{array}$$

to get the identities

$$\begin{aligned} \langle \phi, \psi \circ \sigma \rangle &= \langle \psi^* \circ \phi, \sigma \rangle = \langle (\mathrm{id}_A \otimes \psi^*)(\mathrm{C}_\phi), \mathrm{C}_\sigma \rangle, \\ \langle \phi, \sigma \circ \psi \rangle &= \langle \phi \circ \psi^*, \sigma \rangle = \langle (\psi \otimes \mathrm{id}_B)(\mathrm{C}_\phi), \mathrm{C}_\sigma \rangle. \end{aligned}$$

Therefore, we see that $\phi \in (K \circ \mathbb{CP}_{AA})^\circ$ if and only if $\mathrm{id}_A \otimes \psi^*$ sends C_ϕ to a positive matrix for every $\psi \in K$, and $\phi \in (\mathbb{CP}_{BB} \circ K)^\circ$ if and only if $\psi \otimes \mathrm{id}_B$ sends C_ϕ to a positive matrix for every $\psi \in K$.

Theorem 1.8.7 *For a closed convex cone $K \subset H(M_A, M_B)$ of positive maps, the following are equivalent:*

(i) *K is a right mapping cone,*

(ii) *$\phi \in K^\circ$ if and only if $(\mathrm{id}_A \otimes \psi^*)(\mathrm{C}_\phi) \geqslant 0$ for every $\psi \in K$.*

Furthermore, the following are also equivalent:

(iii) *K is a left mapping cone,*

(iv) *$\phi \in K^\circ$ if and only if $(\psi \otimes \mathrm{id}_B)(\mathrm{C}_\phi) \geqslant 0$ for every $\psi \in K$.*

When we take $K = \mathbb{SP}_k^\circ$, this gives us the following characterization of Schmidt numbers of states:

Corollary 1.8.8 *A state $\varrho \in M_A \otimes M_B$ belongs to $\mathcal{S}_k$ if and only if $(\mathrm{id}_A \otimes \psi)(\varrho) \geqslant 0$ for every k-positive map $\psi : M_B \to M_A$ if and only if $(\psi \otimes \mathrm{id}_B)(\varrho) \geqslant 0$ for every k-positive map $\psi : M_A \to M_B$.*

The most important case occurs when $k = 1$; a state ϱ is separable if and only if $(\psi \otimes \mathrm{id}_B)(\varrho) \geqslant 0$ for every positive map $\psi : M_A \to M_B$. If we take $\psi = \mathrm{T}$, the transpose map, then we have the PPT criterion; if ϱ is separable then $(\mathrm{T}\otimes\mathrm{id})(\varrho) \geqslant 0$. We see that $\langle \varrho, \mathrm{T}\rangle \geqslant 0$ if and only if $\langle \varrho^\Gamma, \mathrm{C}_{\mathrm{id}}\rangle \geqslant 0$. This condition tells us that the sum of some coefficients of ϱ^Γ is nonnegative. Therefore, we see that the condition $(\psi \otimes \mathrm{id}_B)(\varrho) \geqslant 0$ is much stronger than $\langle \varrho, \psi\rangle \geqslant 0$ to detect entanglement.

References: [70], [113], [114], [60], [34], [124], [27], [36], [80],

1.8.4 Tensor products of positive maps

The dual cone may be also described in terms of state C_{id} with the maximal Schmidt number. For this purpose, we consider the following diagrams

$$
\begin{array}{ccc}
M_A & \xrightarrow{\ \phi\ } & M_B \\
\big\downarrow{\scriptstyle \mathrm{id}_A} & & \\
M_A & \xrightarrow{\ \psi\ } & M_B
\end{array}
\qquad\qquad
\begin{array}{ccc}
M_B & \xrightarrow{\ \phi^*\ } & M_A \\
\big\downarrow{\scriptstyle \mathrm{id}_B} & & \\
M_B & \xrightarrow{\ \psi^*\ } & M_A
\end{array}
$$

together with the identity

$$
\begin{aligned}
\langle \phi, \sigma \circ \psi\rangle &= \langle \phi^*, \psi^* \circ \sigma^*\rangle = \langle \psi \circ \phi^*, \sigma^*\rangle = \langle (\phi \otimes \psi)(\mathrm{C}_{\mathrm{id}_A}), \mathrm{C}_{\sigma^*}\rangle, \\
\langle \phi, \psi \circ \sigma\rangle &= \langle \psi^* \circ \phi, \sigma\rangle = \langle (\phi^* \otimes \psi^*)(\mathrm{C}_{\mathrm{id}_B}), \mathrm{C}_{\sigma}\rangle,
\end{aligned}
$$

to get the following:

Theorem 1.8.9 *Suppose that K is a closed convex cone of positive maps in $H(M_A, M_B)$. Then the following are equivalent:*

(i) *K is a left mapping cone,*

(ii) *$\phi \in K^\circ$ if and only if $(\phi \otimes \psi)(\mathrm{C}_{\mathrm{id}_A}) \geqslant 0$ for every $\psi \in K$.*

Furthermore, the following are also equivalent:

(iii) *K is a right mapping cone,*

(iv) *$\phi \in K^\circ$ if and only if $(\phi^* \otimes \psi^*)(\mathrm{C}_{\mathrm{id}_B}) \geqslant 0$ for every $\psi \in K$.*

We note that the dual cones are described in terms of images of C_{id} under the tensor product $\phi \otimes \psi$ in Theorem 1.8.9. Finally, we explore properties of $\phi \otimes \psi$ themselves in terms of dual cones. To do this, we put arbitrary τ in the places of id_A to get

$$
\begin{array}{ccc}
M_A & \xrightarrow{\ \phi\ } & M_B \\
\big\downarrow{\scriptstyle \tau} & & \\
M_A & \xrightarrow{\ \psi\ } & M_B
\end{array}
$$

together with the identity

$$\langle \sigma^* \circ \psi \circ \tau, \phi \rangle = \langle \psi \circ \tau, \sigma \circ \phi \rangle = \langle \psi \circ \tau \circ \phi^*, \sigma \rangle = \langle (\phi \otimes \psi)(\mathrm{C}_\tau), \mathrm{C}_\sigma \rangle.$$

Then we see that $\phi \in (\mathbb{CP}_{BB} \circ K \circ \mathbb{CP}_{AA})^\circ$ if and only if $\phi \otimes \psi$ sends positive matrices to positive matrices for every $\psi \in K$, that is, $\phi \otimes \psi$ is a positive map for every $\psi \in K$. Therefore, we have the following:

Theorem 1.8.10 *For a given closed convex cone $K \subset H(M_A, M_B)$ of positive maps, the following are equivalent:*

(i) *K is a mapping cone,*

(ii) *$\phi \in K^\circ$ if and only if $\phi \otimes \psi$ is a positive map for every $\psi \in K$.*

When $K = \mathbb{SP}_k$, Theorem 1.8.10 tells us that $\psi \otimes \phi$ is positive for every k-superpositive map ψ if and only if $\mathrm{id}_k \otimes \phi$ is positive. Noting that id_k is a typical example of $\mathbb{SP}_k$, one may ask if id_k in the definition of k-positivity may be replaced by another k-superpositive map. When we fix a matrix s with rank k, it is easily seen that the map $\mathrm{id}_k \otimes \phi$ is positive if and only if $\mathrm{Ad}_s \otimes \phi$ is positive, using singular value decomposition of s.

We also put two identity maps in the diagram as follows:

$$\begin{array}{ccc} M_A & \xrightarrow{\mathrm{id}_A} & M_A \\ \downarrow{\scriptstyle \mathrm{id}_A} & & \\ M_A & \xrightarrow{\phi} & M_B \end{array}$$

Then we get the identity $(\mathrm{id}_A \otimes \phi)(\mathrm{C}_{\mathrm{id}_A}) = \mathrm{C}_\phi$, and have the following:

Proposition 1.8.11 *A linear map $\phi : M_A \to M_B$ belongs to a convex cone K if and only if $(\mathrm{id}_A \otimes \phi)(\mathrm{C}_{\mathrm{id}_A})$ belongs to C_K.*

References: [117], [80]

1.8.5 Entanglement breaking maps

It is worthwhile to collect applications of results in this section for $K^\circ = \mathbb{SP}_k$ to get equivalent conditions for k-superpositivity of a map ϕ, or equivalently Schmidt numbers of the state C_ϕ.

Corollary 1.8.12 *For a linear map $\phi : M_A \to M_B$, the following are equivalent:*

(i) *ϕ is k-superpositive, that is, $\phi = \sum_i \mathrm{Ad}_{s_i}$ with $\mathrm{rank}\, s_i \leqslant k$,*

(ii) C_ϕ *belongs to* $\mathcal{S}_k$*, that is, has the Schmidt numbers* $\leqslant k$*,*

(iii) $\psi^* \circ \phi$ *is completely positive for every* k*-positive map* $\psi : M_A \to M_B$*,*

(iv) $\phi \circ \psi^*$ *is completely positive for every* k*-positive map* $\psi : M_A \to M_B$*,*

(v) $\mathrm{id}_A \otimes \phi$ *sends every state into* $\mathcal{S}_k$*,*

(vi) $\phi \otimes \mathrm{id}_B$ *sends every matrix in* $\mathcal{BP}_k$ *to a positive matrix,*

(vii) $(\mathrm{id}_A \otimes \psi)(\mathrm{C}_\phi) \geqslant 0$ *for every* k*-positive map* $\psi : M_B \to M_A$*,*

(viii) $(\psi \otimes \mathrm{id}_B)(\mathrm{C}_\phi) \geqslant 0$ *for every* k*-positive map* $\psi : M_A \to M_B$*,*

(ix) $(\phi \otimes \psi)(\mathrm{C}_{\mathrm{id}_A}) \geqslant 0$ *for every* k*-positive map* $\psi : M_A \to M_B$*,*

(x) $(\phi^* \otimes \psi^*)(\mathrm{C}_{\mathrm{id}_B}) \geqslant 0$ *for every* k*-positive map* $\psi : M_A \to M_B$*,*

(xi) $\phi \otimes \psi$ *is positive for every* k*-positive map* $\psi : M_A \to M_B$*.*

(xii) $(\mathrm{id}_A \otimes \phi)(\mathrm{C}_{\mathrm{id}_A})$ *belongs to* $\mathcal{S}_k$*.*

If $s = |\xi\rangle\langle\eta|$ is of rank one, then we have

$$\mathrm{Ad}_s(a) = |\eta\rangle\langle\xi|a|\xi\rangle\langle\eta| = \langle\xi|a|\xi\rangle\, |\eta\rangle\langle\eta| = \langle a, |\bar{\xi}\rangle\langle\bar{\xi}|\rangle\, |\eta\rangle\langle\eta|.$$

Therefore, we see that ϕ is 1-superpositive if and only if ϕ is of the following form

$$\phi(a) = \sum_k \langle a, v_k\rangle u_k \tag{1.64}$$

with positive matrices u_k and v_k. This is called the *Holevo form*.

References: [114], [56], [61], [3], [27], [36], [80]

1.9 Historical remarks

Positive linear maps have played crucial roles in the theory of operator algebras since the Gelfand–Naimark–Segal construction in the 1940's, by which positive linear functionals give rise to $*$-representations of abstract C^*-algebras. This construction was extended for completely positive linear maps into operators in the 1955 paper by Stinespring [110], where a linear map ϕ was defined to be completely positive when $\mathrm{id}_k \otimes \phi$ is positive for every $k = 1, 2, \dots$. Note that a linear functional on a matrix algebra into scalars is completely positive if and only if it is just positive by Theorem 1.5.3. See also [125] for more general situations. Stinespring representation theorem tells us that every completely positive map is of the form $V^*\pi(\,\cdot\,)V$ for a $*$-homomorphism π and a bounded linear map V. See [92].

Størmer defined the notion of decomposability in his paper [111] of 1963 by maps of the form $V^*\pi(\,\cdot\,)V$ with Jordan homomorphisms π which preserve Hermiticity and square of Hermitian elements. It was shown later in [112] that a map is decomposable in this sense if and only if it is the sum of a completely positive map and a completely copositive map. In the paper [111], extremeness of the map Ad_s in Theorem 1.2.4 had been considered in more general situations, and all the extreme points of the convex set $\mathbb{P}_1[M_2, M_2, I]$ of all unital positive maps between 2×2 matrices have been found explicitly, from which Theorem 1.3.4 follows. We followed [118] and [6, 7] for the proofs of Theorem 1.2.4 and Theorem 1.3.4, respectively. The maps Ad_s also play important roles in matrix theory. Suppose that a map $\phi : M_n \to M_n$ satisfies $\phi(0) = 0$ and (1.12). If we suppose further that ϕ is bijective or continuous then it is known [109, 87, 104] that $\phi = \mathrm{Ad}_s$ or $\phi = \mathrm{Ad}_s \circ \mathrm{T}$ for a nonsingular $s \in M_n$.

Choi matrices have been introduced by de Pillis [32] in 1967, where Proposition 1.4.1 was shown. Propositions 1.4.2 and Theorem 1.5.3 were given by Jamiołkowski [67] in 1972 and Choi [21] in 1975, respectively, after whom the one-to-one correspondence $\phi \mapsto \mathrm{C}_\phi$ is now called the Jamiołkowski–Choi isomorphism. Kraus decomposition of completely positive maps were found independently [72, 73]. We note that the matrix $\sum_{i,j} |j\rangle\langle i| \otimes \phi(|i\rangle\langle j|)$ has been considered in [32] and [67], instead of $\mathrm{C}_\phi = \sum_{i,j} |i\rangle\langle j| \otimes \phi(|i\rangle\langle j|)$. Recall that the notions of Hermitcity and 1-block-positivity are invariant under taking partial transposes. Further variants has been considered in [93, 80, 79, 54]. The Choi matrix can be defined for infinite dimensional cases [57, 58, 84, 119, 35, 50, 85, 37] in various situations, and multi-linear maps between matrix algebras [78, 52, 53]. Since Arveson's extension theorem for completely positive maps [4] and Choi's Theorem 1.5.3 on the correspondence between completely positivity of maps and positivity of Choi matrices, it has been widely accepted by operator algebraists that completely positive maps serve as morphisms for operator algebras reflecting noncommutative order structures, and important notions like nuclearity and injectivity have been described in terms of completely positive maps. See the survey article [33]. Theorem 1.5.8 on the extreme points of unital completely positive maps is also taken from [21].

The notion of duality between tensor products and linear maps in (1.39) goes back to the work of Woronowicz [129] in 1976, where he showed that every positive maps between M_2 and M_3 are decomposable. For this purpose, he utilized the duality to get the equivalent claim that for any PPT state $\varrho \in M_2 \otimes M_3$ there exists a product vector $|\xi\rangle|\eta\rangle \in \mathrm{Im}\,\varrho$ such that $|\bar\xi\rangle|\eta\rangle \in \mathrm{Im}\,\varrho^\Gamma$. This is equivalent in principle to the claim that every $2\otimes 3$ PPT state in $M_2 \otimes M_3$ is separable, as it was noticed later by Horodecki's in the paper [60] of 1996, where the duality between

$\mathcal{S}_1$ and $\mathbb{P}_1$ was also given. Theorem 1.5.4 (i) on the duality between k-positivity and states of Schmidt number $\leqslant k$ was obtained in [34]. The duality between linear maps through (1.40) was defined in [107]. Duality between separability of states and positivity of maps can be extended for infinite dimensional cases [115].

The linear map $\tau_{n,n-1}$ in (1.52) with $k = n - 1$, which was given by the 1972 paper of Choi [20], is the first concrete example of a linear map which distinguishes complete positivity and k-positivity. The equivalent condition (iv) of Proposition 1.6.1 to the k-positivity is implicit in [20] and stated in [120] clearly. The condition (vi) of Proposition 1.6.1 together with Theorem 1.6.3 is due to Tomiyama [126]. After the first example of indecomposable positive map in [22], Choi and Lam [25] constructed the Choi map ϕ_{ch} in (1.54), which is known to be extreme [42]. The linear map from M_2 into M_4 given by (1.56) was constructed by Woronowicz [130] in the contexts of non-extendibility which has a close relationship with extremeness of positive maps. There are other examples [122] of indecomposable maps between M_2 and M_4. See also [112] for infinite family of indecomposable positive maps in other dimensions. Further interesting properties of two maps by Choi and Woronowicz will be discussed in the next chapter.

The notion of entanglement had been originated from the work by Einstein, Podolsky and Rosen in the 1930's. It was defined for general mixed states by Werner [128] in 1989 under the name of *Einstein-Podolsky-Rosen correlation*. Separable states, which were defined as convex combinations of product states, were called to be *classically correlated* in the paper. The notion of Schmidt numbers of bi-partite states has been considered in [34] under the name *k-simple vectors* following Woronowicz' terminology [129], together with Theorem 1.5.2. The term Schmidt numbers were introduced in [124], where Corollary 1.8.8 was shown. It was shown in [40] that every state in the unit ball around the identity $I_{mn} \in M_m \otimes M_n$ with respect to the norm $\| \cdot \|_{\text{HS}}$ is separable. It is very hard in general to prove that a given state is entangled, since we have to show that some kinds of expressions are not possible. In fact, it is now known [39] that it is an NP hard problem in general to decide if a given state is separable or entangled. Positive maps play crucial roles in this problem by duality: A state ϱ is entangled if and only if there exists a positive map ϕ such that $\langle \varrho, \phi \rangle < 0$. This will be the main topic of the next chapter. Theorem 1.4.3, which is now called the PPT criterion, was presented by Choi [24] in 1980 together with the example (1.33) of a PPT entangled state, and rediscovered by Peres [94] in 1996. We note that the first example of a PPT entangled state was given by Woronowicz [129] in 1976 among $2 \otimes 4$ states in the context of decomposability of positive maps. For more criteria for separability, we refer to survey articles [38, 64]

The Werner states had been introduced by Werner [128] together with Proposition 1.7.5 and the technique using the integration over the unitary group to show that a given state is separable. It is another problem to look for decomposition into finitely many product states. See [91]. On the other hands, Horodecki's [59] introduced the isotropic states and gave the condition for separability using the corresponding invariant properties in Proposition 1.7.1. Theorem 1.7.3 on the Schmidt numbers of isotropic states was obtained in [124]. The k-copositivity of the map ϕ_λ, or equivalently k-positivity of ψ_λ in Theorem 1.7.6 was obtained in [126] by another application of Proposition 1.6.1.

The notion of superpositive maps has arisen more recently, even though positive maps of the form (1.64) had been considered by Størmer [114] in 1980's. In the paper [61], a map ϕ was called entanglement breaking when ϕ satisfies the the condition (v) of Corollary 1.8.12 with $k = 1$, and it was shown that conditions (i), (iii), (iv) and (xii) are equivalent to (v) in Corollary 1.8.12 together with the Holevo form [56] in (1.64). On the other hand, a map was called superpositive in [3] when its Choi matrix is separable. See [5] for the characteristic property of unital completely positive maps which is not entanglement breaking. Equivalent conditions (ii), (iii) and (v) of Corollary 1.8.12 for k-superpositive maps were given in [27]. We follow [106, 108] for the definition of k-superpositivity. We note that Theorem 1.5.2 on the dual of k-superpositivity recovers the definition of k-positivity.

Mapping cones were introduced by Størmer [114] in the 1980's in more general contexts to study the extension problem of positive maps. It was known [107, 117] that the dual cones of mapping cones can be described in terms of compositions and tensor products; $\phi \in K^\circ$ if and only if $\phi \circ \psi$ is completely positive for every $\psi \in K$ if and only if $\psi \otimes \phi$ is positive for every $\psi \in K$ if and only if $(\psi \otimes \phi)(\mathrm{C}_{\mathrm{id}}) \geqslant 0$ for every for every $\psi \in K$. One-sided mapping cone was introduced quite recently in [36] to see that some of the above equivalences hold only when K is a one-sided mapping cone. Results in Section 1.8, which were taken from [36] and [80], recover many known results in a systematic way. For examples, statements (i) and (ii) of Corollary 1.8.6 were given in [70, 113] and [34], respectively. On the other hand, Corollary 1.8.8 with $k = 1$ gives rise to the Horodecki's separability criterion [60]; a state ϱ is separable if and only if $(\psi \otimes \mathrm{id}_B)(\varrho) \geqslant 0$ for every positive map $\psi : M_A \to M_B$. The analogous characterization of states in $\mathcal{S}_k$ in terms of k-positive maps is due to [124]. See also [115] for an infinite dimensional analogue for Horodecki's separability criterion.

Chapter 2

Detecting Entanglement by Positive Maps

We will focus in this chapter on the duality between positive maps and separable states: A bi-partite state $\varrho \in M_m \otimes M_n$ is separable if and only if $\langle \varrho, \phi \rangle \geqslant 0$ holds for every positive map ϕ from M_m into M_n. It is clear that this is the case if and only if the inequality holds for every positive map which generates an extreme ray of the convex cone $\mathbb{P}_1$ of all positive maps. But, it is not so easy to find extremal positive maps, because we do not know facial structures of the convex cone $\mathbb{P}_1$. In fact, it is enough to check the inequality for positive maps which generate exposed rays of $\mathbb{P}_1$, by Straszewicz's Theorem which tells us that the set of exposed points of a compact convex set is dense among all extreme points.

We will also consider the question how much entanglement a given positive map may detect. A positive map will be called optimal when it detects a maximal set of entanglement. It turns out that the notion of optimality involves facial structures of $\mathbb{P}_1$ again; a positive map is optimal if and only if the smallest face containing it has no completely positive map. We give a sufficient condition on positive maps which detect maximal set of entanglement, in terms of exposed faces. This condition, called the spanning property, is much easier to check, and this explains why we are interested in exposed faces. We also note that the boundary of a convex cone consists of maximal faces which are always exposed.

A face is exposed with respect to the bilinear pairing if and only if it is a dual face, which is determined by a hyperplane given by an element of the dual cone. In order to detect entangled states with positive partial transposes, we need indecomposable positive maps, and so it is important to find indecomposable exposed positive maps, in this contexts. We will exhibit such positive maps in low dimensional cases.

We will see in Section 2.1 that a face F is exposed in the convex if and only if it coincides with the bidual face F''. We also see that every face for completely positive maps turns out to be exposed, and this is also the case for PPT states. Especially, we show that the map Ad_s is exposed in the convex cone $\mathbb{P}_1$ consisting of all positive maps. In Section 2.2, we show that the Choi map ϕ_{ch} has very special properties; the smallest face containing ϕ_{ch} is not exposed; the smallest exposed face ϕ''_{ch} containing ϕ_{ch} has a nontrivial intersection with completely positive maps, but has the trivial intersection with completely copositive maps. We will show in Section 2.3 that a face is maximal if and only if its dual face is minimal among exposed faces. With this, we can understand relative locations of convex cones $\mathbb{P}_k$ of all k-positive maps. Section 2.4 will be devoted to explain optimality and spanning properties of positive maps, and explain them in terms of faces and exposed faces, respectively.

In the remaining part, we will exhibit several examples of indecomposable positive maps in low dimensions. In Section 2.5, we exhibit parameterized examples of indecomposable positive maps between 3×3 matrices, which are variants of the Choi map. We will find positive maps among them which distinguish various kinds of optimality. We also provide in Section 2.6 a sufficient condition for exposedness of a positive map which is not so difficult to check, and we show that the Woronowicz map from M_2 into M_4 is exposed. We will work with parameterized examples. Finally, we exhibit in Section 2.7 an example of indecomposable exposed positive map between 4×4 matrices, which was constructed by Robertson.

2.1 Exposed faces

A face is exposed if and only if it is a dual face. We will see that every face for completely positive maps is exposed. This is also the case for PPT states. We also show that the map Ad_s generates an exposed ray of the convex cone of all positive maps.

2.1.1 Dual faces

Suppose that X and Y are finite dimensional vector spaces with a non-degenerate bilinear pairing between them. For a subset F of a closed convex cone C of X, we define the subset F' of C° by

$$F' = \{y \in C^\circ : \langle x, y \rangle = 0 \text{ for each } x \in F\} \subset C^\circ \subset Y.$$

It is then clear that F' is a face of C°, which is said to be the *dual face* of F. For an arbitrary subset F of X, we have $F \subset F''$, from which we also have $F''' \subset F'$. Therefore, we have $F' = (F'')'$ in general, and we see that every dual face is the dual face of a face. If F is a face with an interior point x_0 then it is also easy to see that

$$F' = \{y \in C^\circ : \langle x_0, y \rangle = 0\}.$$

When $\{x_0\}$ is a singleton, $\{x_0\}'$ is denoted just by x_0'.

We say that a face F of a closed convex cone C in X is an *exposed face* if it is a dual face of a subset in Y. This means that the face is determined by the hyperplane $\{x \in X : \langle x, y_0 \rangle = 0\}$ in X given by a point $y_0 \in C^\circ$. In other words, we have

$$F = C \cap \{x \in X : \langle x, y_0 \rangle = 0\}.$$

Proposition 2.1.1 *For a subset F of a convex cone C in X, we have the following:*

(i) *F'' is the smallest exposed face containing F,*

(ii) *F is an exposed face if and only if $F = F''$.*

Proof. Suppose that an exposed face S' satisfies the relation $F \subset S' \subset F''$. Then we have $F''' \subset S'' \subset F'$, and so $F' = S''$ and $S' = F''$. This proves (i). The second statement (ii) follows form (i). □

We recall that every face of the closed convex cone $\mathbb{CP}[M_m, M_n]$ is of the form

$$F_V = \operatorname{conv}\{\operatorname{Ad}_s : s \in V\}$$

for a subspace V of $M_{m\times n}$. By the relation

$$\langle \operatorname{Ad}_s, \operatorname{Ad}_t \rangle = \langle |\tilde{s}\rangle\langle\tilde{s}|, |\tilde{t}\rangle\langle\tilde{t}| \rangle = \langle \bar{\tilde{t}}|\tilde{s}\rangle\langle\tilde{s}|\bar{\tilde{t}}\rangle = |\langle\tilde{s}|\bar{\tilde{t}}\rangle|^2 = |\langle s, t\rangle|^2, \tag{2.1}$$

we see that $(F_V)'' = F_V$, and we have the following:

Proposition 2.1.2 *Every face of the convex cone $\mathbb{CP}$ is exposed.*

We proceed to show that every face of the convex cone $\mathcal{PPT}$ of all PPT states is also exposed. We begin with more general situations. Recall that we have the relations (1.48);

$$(C_1 + C_2)^\circ = C_1^\circ \cap C_2^\circ, \qquad (C_1 \cap C_2)^\circ = C_1^\circ + C_2^\circ,$$

for closed convex cones C_1 and C_2. Suppose that F_i is a subset of C_i for $i = 1, 2$. Then it is easy to see that the following identity

$$(F_1 + F_2)' = F_1' \cap F_2' \tag{2.2}$$

holds, where $(F_1 + F_2)'$ is a face of the convex cone $(C_1 + C_2)^\circ = C_1^\circ \cap C_2^\circ$ and F_i' is a face of C_i° for $i = 1, 2$.

Now, we suppose that F_i is a face of a convex set C_i for $i = 1, 2$. Then it is clear that $F_1 \cap F_2$ is a face of the convex set $C_1 \cap C_2$ whenever it is not empty. Conversely, suppose that F is a face of $C_1 \cap C_2$. Take an interior point x of F. We also take the face F_i of C_i in which x is an interior point for $i = 1, 2$. Then x is also an interior point of $F_1 \cap F_2$, which is a face of $C_1 \cap C_2$. Because x is an interior point of both faces F and $F_1 \cap F_2$ of $C_1 \cap C_2$, we conclude that $F = F_1 \cap F_2$. In short, we have

$$F = F_1 \cap F_2, \qquad \operatorname{int} F \subset \operatorname{int} F_1, \qquad \operatorname{int} F \subset \operatorname{int} F_2. \tag{2.3}$$

It is clear that faces F_i of C_i satisfying (2.3) are uniquely determined.

Proposition 2.1.3 *Suppose that C_i is a closed convex cone in a finite dimensional space, for $i = 1, 2$. For every face F of $C_1 \cap C_2$, there exist unique faces F_1 and F_2 of C_1 and C_2, respectively, satisfying* (2.3)*. In this case, we have*

$$F' = F_1' + F_2', \qquad F'' = F_1'' \cap F_2''.$$

Proof. The inclusion $F_1' + F_2' \subset F'$ follows from $F_i' \subset F'$. For the reverse inclusion, take $y \in F'$. Since $y \in (C_1 \cap C_2)^\circ = C_1^\circ + C_2^\circ$, we may write $y = y_1 + y_2$ with $y_i \in C_i^\circ$ for $i = 1, 2$. If we take an interior point x of $F_1 \cap F_2$, then $x \in \operatorname{int} F_i \subset C_i$ by (2.3), and $\langle x, y_i \rangle \geqslant 0$ for $i = 1, 2$. By the relation

$$0 = \langle x, y \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle,$$

we see that $\langle x, y_i \rangle = 0$ and $y_i \in F_i'$, since x is an interior point of F_i. Therefore, we have $y \in F_1' + F_2'$. This proves the first identity. The second identity follows from (2.2). □

Because every face of $\mathbb{CP}$ is of the form $F_V = \{\mathrm{Ad}_s : s \in V\}''$ for a subspace V of $M_{m\times n}$, we see that every face of $\mathbb{CCP}$ is of the form

$$F^V := \{\mathrm{Ad}_s \circ \mathrm{T} : s \in V\}''.$$

Therefore, every face of $\mathbb{PPT} = \mathbb{CP} \cap \mathbb{CCP}$ must be of the form

$$F_V \cap F^W, \tag{2.4}$$

for a pair (V, W) of subspaces of $M_{m\times n}$. Note that the intersection of two exposed faces is exposed again by (2.2). By Proposition 2.1.2 and Proposition 2.1.3, we have the following:

Proposition 2.1.4 *Every face of the convex cone $\mathbb{PPT}$ is exposed.*

Every face of $(M_m \otimes M_n)^+$ is also of the form

$$F_V = \{\varrho \in (M_m \otimes M_n)^+ : \operatorname{Im} \varrho \subset V\} \tag{2.5}$$

for a subspace of $\mathbb{C}^m \otimes \mathbb{C}^n$, with the same notation as in (1.46), and so every face of the convex cone $\mathcal{PPT}$ of all PPT states is also of the form

$$F_V \cap F^W,$$

for a pair (V, W) of subspaces $\mathbb{C}^m \otimes \mathbb{C}^n$, with

$$F^W = \{\varrho^\Gamma : \varrho \in F_W\}. \tag{2.6}$$

It is not so easy in general to determine if $F_V \cap F^W$ is nontrivial or not. If we denote by V the subspace of $\mathbb{C}^3 \otimes \mathbb{C}^3$ spanned by vectors in (1.34), then we see that the PPT state ϱ defined by (1.33) belongs to the face $F_V \cap F^V$.

As for the convex cone $\mathbb{P}_1$ of all positive maps, not every face is exposed even in the low dimensional cases like $\mathbb{P}_1[M_2, M_2] = \mathbb{DEC}[M_2, M_2]$. See [13]. It is not known whether every face of the convex cone $\mathcal{S}_1$ is exposed or not.

References: [76], [34], [13], [43]

2.1.2 Exposed positive maps

A ray $\{tx_0 : t \geqslant 0\}$ of a closed convex cone C generated by $x_0 \in C$ is called an *exposed ray* if it is an exposed face. In this case, we say that x_0 is *exposed* in C. We proceed to show that an extreme ray Ad_s is exposed in $\mathbb{P}_1$ as well as in $\mathbb{CP}$ for every $s \in M_{m\times n}$. Recall that the dual cone of $\mathbb{P}_1$ is $\mathbb{SP}_1$, and so $\phi \in \mathbb{P}_1$ belongs to $(\mathrm{Ad}_\sigma)''$ if and only if the following

$$s \in M_{m\times n},\ \operatorname{rank} s = 1,\ \langle \mathrm{Ad}_\sigma, \mathrm{Ad}_s\rangle = 0 \implies \langle \phi, \mathrm{Ad}_s\rangle = 0 \tag{2.7}$$

holds. In order to show that Ad_σ generates an exposed ray of the convex cone $\mathbb{P}_1$, we have to show that the condition (2.7) implies that ϕ is a scalar multiple of Ad_σ by Proposition 2.1.1 (ii).

In the following discussion, we identify $M_m \otimes M_n$ with $M_m(M_n) = M_{mn}$, and the entries of $\mathrm{C}_\phi \in M_{mn}$ will be denoted by $c_{(i,k),(j,\ell)}$ with $i, j = 1, \ldots, m$ and $k, \ell = 1, \ldots, n$, where (i, k)'s and (j, ℓ)'s are endowed with the lexicographic orders. Therefore,

$$c_{(i,k),(j,\ell)} = \langle |ik\rangle\langle j\ell|, \mathrm{C}_\phi\rangle = \langle |i\rangle\langle j| \otimes |k\rangle\langle \ell|, \mathrm{C}_\phi\rangle = \langle |k\rangle\langle \ell|, \phi(|i\rangle\langle j|)\rangle$$

is the (k, ℓ) entry of the (i, j) block of $\mathrm{C}_\phi \in M_m(M_n)$.

By singular value decomposition, every $m \times n$ matrix is of the form $u\sigma v^*$ with invertible matrices $u \in M_m$, $v \in M_n$ and

$$\sigma = \sum_{i=1}^{r} |i\rangle\langle i| \in M_{m\times n}. \tag{2.8}$$

Note that $\mathrm{Ad}_{u\sigma v*} = \mathrm{Ad}_{v*} \circ \mathrm{Ad}_\sigma \circ \mathrm{Ad}_u$, and

$$\phi \mapsto \mathrm{Ad}_{v*} \circ \phi \circ \mathrm{Ad}_u$$

is an affine isomorphism between $H(M_m, M_n)$. Therefore, we see that $\mathrm{Ad}_{u\sigma v*}$ is exposed in $\mathbb{P}_1[M_m, M_n]$ if and only if Ad_σ is exposed in $\mathbb{P}_1[M_m, M_n]$. We note that the entries of $\mathrm{C}_{\mathrm{Ad}_\sigma}$ are 0 or 1, and the $(i,k),(j,\ell)$ entry is 1 if and only if $i = k$ and $j = \ell$ with $i, j = 1, \dots, r$.

From now on, we will prove that Ad_σ is exposed in $\mathbb{P}_1[M_m, M_n]$ when σ is given by (2.8). For this purpose, we suppose that $\phi \in (\mathrm{Ad}_\sigma)''$. We first consider the $m \times n$ matrix $s = |i\rangle\langle k|$ in (2.7), to see that the diagonal entries of C_ϕ are given by

$$c_{(i,k),(i,k)} = 0, \qquad (i,k) \in \{1, \dots, m\} \times \{1, \dots n\} \backslash \{(1,1), \dots, (r,r)\}. \tag{2.9}$$

When J is a subset of $\{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$, we denote by A_J the principal submatrix of A by taking (i,k) rows and columns for $(i,k) \in J$. In order to determine off-diagonal entries $c_{(i,k),(j,\ell)}$ of C_ϕ with $(i,k) \neq (j,\ell)$, we consider

$$J = \{(i,k), (i,\ell), (j,k), (j,\ell)\}, \tag{2.10}$$

and the principal submatrix of $[\mathrm{C}_\phi]_J$ of C_ϕ. If $i = j$ or $k = \ell$ then $[\mathrm{C}_\phi]_J$ is a positive 2×2 matrix and so $c_{(i,k),(j,\ell)} = 0$ by considering the diagonal entries in (2.9). Otherwise, $[\mathrm{C}_\phi]_J$ is a block matrix in $M_2 \otimes M_2 = M_2(M_2)$. We proceed to show that this is a block-positive matrix.

Suppose that $m = 2, 3, \dots$ and $i, j = 1, 2, \dots, m$ with $i \neq j$. We consider the linear map $\lambda^m_{i,j} : M_2 \to M_m$ defined by

$$\lambda^m_{i,j} : \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto a_{11}|i\rangle\langle i| + a_{12}|i\rangle\langle j| + a_{21}|j\rangle\langle i| + a_{22}|j\rangle\langle j| \in M_m.$$

Then it is easily seen that the adjoint map $(\lambda^m_{k,\ell})^* : M_m \to M_2$ is given by

$$(\lambda^m_{k,\ell})^* : \sum_{i,j=1}^m a_{ij}|i\rangle\langle j| \mapsto \begin{pmatrix} a_{kk} & a_{k\ell} \\ a_{\ell k} & a_{\ell\ell} \end{pmatrix} \in M_2.$$

If $\phi : M_m \to M_n$ then $(\lambda^n_{k,\ell})^* \circ \phi \circ \lambda^m_{i,j}$ is a map from M_2 into M_2, whose Choi matrix is given by just $[\mathrm{C}_\phi]_J$ with J in (2.10). Therefore, we see that if C_ϕ is block-positive then $[\mathrm{C}_\phi]_J$ is also block-positive in $M_2 \otimes M_2$.

Lemma 2.1.5 *Suppose that $\varrho \in \mathcal{BP}_1[M_2 \otimes M_2]$ has at most one nonzero diagonal entry. Then all the other entries of ϱ are zero.*

Proof. We first consider the case when all the diagonal entries are zero except for the left upper corner. Then by the definition of block-positivity, the 2×2 principal submatrices $\varrho_{\{(1,1),(1,2)\}}$, $\varrho_{\{(2,1),(2,2)\}}$, $\varrho_{\{(1,1),(2,1)\}}$ and $\varrho_{\{(1,2),(2,2)\}}$ are positive. Therefore, we see that ϱ is of the form

$$\mathrm{C}_\phi = \begin{pmatrix} a & \cdot & \cdot & \alpha \\ \cdot & \cdot & \beta & \cdot \\ \cdot & \bar{\beta} & \cdot & \cdot \\ \bar{\alpha} & \cdot & \cdot & \cdot \end{pmatrix} \in M_2(M_2),$$

with $a \geqslant 0$ and $\alpha, \beta \in \mathbb{C}$. Take $\langle\xi| = (1, x, y, xy) \in \mathbb{C}^2 \otimes \mathbb{C}^2$ whose Schmidt rank is one. Then we have

$$0 \leqslant \langle \varrho, |\xi\rangle\langle\xi| \rangle = a + 2\mathrm{Re}\left[y(x\alpha + \bar{x}\beta)\right]$$

for every $y \in \mathbb{C}$. This implies that $x\alpha + \bar{x}\beta = 0$ for every $x \in \mathbb{C}$, and so it follows that $\alpha = \beta = 0$. The remaining cases can be done in the same way. □

Therefore, we see that $c_{(i,k),(j,\ell)} \neq 0$ only when J contains at least two nonzero diagonal entries of $[\mathrm{C}_\phi]_J$. By diagonal entries given in (2.9), this happens only for $c_{(i,i),(j,j)}$ and $c_{(i,j),(j,i)}$ with $i, j = 1, 2, \ldots, r$ and $i \neq j$. In this case, J is given by

$$J = \{(i,i), (i,j), (j,i), (j,j)\}$$

and the corresponding principal submatrices of $\mathrm{C}_{\mathrm{Ad}_\sigma}$ and C_ϕ are of the forms

$$\mathrm{C}_{\mathrm{id}_2} = \begin{pmatrix} 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & 1 \end{pmatrix} \qquad \text{and} \qquad \varrho_{a,b,\alpha,\beta} = \begin{pmatrix} a & \cdot & \cdot & \alpha \\ \cdot & \cdot & \beta & \cdot \\ \cdot & \bar{\beta} & \cdot & \cdot \\ \bar{\alpha} & \cdot & \cdot & b \end{pmatrix},$$

respectively, with $a, b \geqslant 0$ and $\alpha, \beta \in \mathbb{C}$. They are Choi matrices of the maps $(\lambda^n_{i,j})^* \circ \mathrm{Ad}_\sigma \circ \lambda^m_{i,j}$ and $(\lambda^n_{i,j})^* \circ \phi \circ \lambda^m_{i,j}$, respectively.

We proceed to show that $(\lambda^n_{k,\ell})^* \circ \phi \circ \lambda^m_{i,j}$ belongs to $((\lambda^n_{k,\ell})^* \circ \mathrm{Ad}_\sigma \circ \lambda^m_{i,j})''$ in $\mathbb{P}_1[M_2, M_2]$. To do this, we suppose that $s \in M_{2\times 2}$ is of rank one matrix satisfying the relation $\langle (\lambda^n_{k,\ell})^* \circ \mathrm{Ad}_\sigma \circ \lambda^m_{i,j}, \mathrm{Ad}_s \rangle = 0$. Then we have $\langle \mathrm{Ad}_\sigma, \lambda^n_{k,\ell} \circ \mathrm{Ad}_s \circ (\lambda^m_{i,j})^* \rangle = 0$. Since $\lambda^n_{k,\ell}$ is completely positive, we see that $\lambda^n_{k,\ell} \circ \mathrm{Ad}_s \circ (\lambda^m_{i,j})^*$ belongs to $\mathbb{SP}_1$. In fact, it is easily seen that $\lambda^n_{k,\ell} \circ \mathrm{Ad}_s \circ (\lambda^m_{i,j})^* = \mathrm{Ad}_{\hat{s}}$, with

$$\hat{s} = s_{11}|i\rangle\langle k| + s_{12}|i\rangle\langle \ell| + s_{21}|j\rangle\langle k| + s_{22}|j\rangle\langle \ell| \in M_{m\times n},$$

which is of rank one. Therefore, we have

$$0 = \langle \lambda^n_{k,\ell} \circ \mathrm{Ad}_s \circ (\lambda^m_{i,j})^*, \phi \rangle = \langle \mathrm{Ad}_s, (\lambda^n_{k,\ell})^* \circ \phi \circ \lambda^m_{i,j} \rangle,$$

to see that $(\lambda^n_{k,\ell})^* \circ \phi \circ \lambda^m_{i,j}$ belongs to $((\lambda^n_{k,\ell})^* \circ \mathrm{Ad}_\sigma \circ \lambda^m_{i,j})''$ in $\mathbb{P}_1[M_2, M_2]$. The following two lemmas will show that the Choi matrix of $(\lambda^n_{i,j})^* \circ \phi \circ \lambda^m_{i,j}$ must be a scalar multiple of $\mathrm{C}_{\mathrm{id}_2}$.

Lemma 2.1.6 *Suppose that $a, b \geqslant 0$ and $\alpha, \beta \in \mathbb{C}$. Then $\varrho_{a,b,\alpha,\beta} \in M_2(M_2)$ is block-positive if and only if $|\alpha| + |\beta| \leqslant \sqrt{ab}$.*

Proof. Suppose that $\varrho_{a,b,\alpha,\beta}$ is block-positive, and take $\langle\xi| = (p, pe^{\mathrm{i}\tau}, qe^{\mathrm{i}(\theta-\tau)}, qe^{\mathrm{i}\theta})$ in $\mathbb{C}^2 \otimes \mathbb{C}^2$ with real numbers p, q. Note that $\mathrm{SR}\,\langle\xi| = 1$. Then we have

$$0 \leqslant \langle \varrho_{a,b,\alpha,\beta}, |\xi\rangle\langle\xi| \rangle = p^2 a + q^2 b + 2pq\mathrm{Re}\left[e^{\mathrm{i}\theta}(\alpha + \beta e^{-2\mathrm{i}\tau})\right]$$

for every $p, q \in \mathbb{R}$. Therefore, we have

$$\left|\mathrm{Re}\left[e^{\mathrm{i}\theta}(\alpha + \beta e^{-2\mathrm{i}\tau})\right]\right|^2 \leqslant ab$$

for every θ and τ, which implies the condition. For the converse, we put

$$a_1 = \frac{|\alpha|a}{|\alpha| + |\beta|}, \quad a_2 = \frac{|\beta|a}{|\alpha| + |\beta|}, \quad b_1 = \frac{|\alpha|b}{|\alpha| + |\beta|}, \quad b_2 = \frac{|\beta|b}{|\alpha| + |\beta|}.$$

Then we see that $\varrho_{a,b,\alpha,\beta} = \varrho_{a_1,b_1,\alpha,0} + \varrho_{a_2,b_2,0,\beta}$ is the Choi matrix of a decomposable map. □

Lemma 2.1.7 *The identity map id_2 on M_2 is exposed in $\mathbb{P}_1[M_2, M_2]$.*

Proof. Suppose that $\phi \in \mathrm{id}_2''$. Taking $s = |i\rangle\langle j| \in M_{2\times 2}$ with $i \neq j$ in (2.7), we see that C_ϕ must be of the form $\varrho_{a,b,\alpha,\beta}$. We also take $s = \begin{pmatrix} 1 & e^{\mathrm{i}\theta} \\ -e^{-\mathrm{i}\theta} & -1 \end{pmatrix}$ in (2.7), to see that

$$0 = \langle |\tilde{s}\rangle\langle\tilde{s}|, \varrho_{a,b,\alpha,\beta} \rangle = a + b - 2\mathrm{Re}\,(\alpha + \beta e^{-2\mathrm{i}\theta}),$$

for every θ. Therefore, we have $\beta = 0$, and we also have

$$0 = a + b - 2\mathrm{Re}\,\alpha \geqslant a + b - 2|\alpha| \geqslant 2\sqrt{ab} - 2|\alpha| \geqslant 0,$$

by Lemma 2.1.6. This happens only when $a = b = \alpha$. □

Now, we return to the discussion before Lemma 2.1.6, to see that the Choi matrix of $(\lambda^n_{i,j})^* \circ \phi \circ \lambda^m_{i,j}$ must be a scalar multiple of $\mathrm{C}_{\mathrm{id}_2}$, and it is clear that all the choices i, j share a common scalar multiple. Therefore, we see that C_ϕ is a scalar multiple of $\mathrm{C}_{\mathrm{Ad}_\sigma}$, and conclude that Ad_σ is exposed. Using singular value decomposition $s = u\sigma v^*$, we have the following theorem which provides another proof of Theorem 1.2.4, because an exposed positive map must be extremal.

Theorem 2.1.8 *For every $s \in M_{m\times n}$, the map Ad_s is exposed in $\mathbb{P}_1[M_m, M_n]$.*

References: [134], [86], [53], [81]

2.2 The Choi map revisited

In this section, we consider the bidual of the Choi map $\phi_{\rm ch}$, which is the smallest exposed face of $\mathbb{P}_1$ containing $\phi_{\rm ch}$. Especially, we determine completely positive maps and decomposable maps which belong to this bidual.

2.2.1 The Choi map and completely positive maps

In order to calculate the dual face of a positive map with respect to the duality between $\mathbb{SP}_1$ and $\mathbb{P}_1$, we take an extreme ray Ad_s in $\mathbb{SP}_1$ with $s = |\xi\rangle\langle\eta| \in M_{m\times n}$. If we write $|\xi\rangle = \sum_{i=1}^m \xi_i|i\rangle$ and $|\eta\rangle = \sum_{j=1}^n \eta_j|j\rangle$ then we have $s = \sum_{i,j}\xi_i\bar\eta_j|i\rangle\langle j|$, and so we have $\langle\tilde s| = \sum_{i,j}\xi_i\bar\eta_j\langle i|\langle j|$ and

$$\begin{aligned}
\mathrm{C}_{\mathrm{Ad}_{|\xi\rangle\langle\eta|}} = |\tilde s\rangle\langle\tilde s| &= \sum_{i,j,k,\ell}\bar\xi_i\eta_j\xi_k\bar\eta_\ell|i\rangle|j\rangle\langle k|\langle\ell| \\
&= \left(\sum_{i,k}\bar\xi_i\xi_k|i\rangle\langle k|\right)\otimes\left(\sum_{j,\ell}\eta_j\bar\eta_\ell|j\rangle\langle\ell|\right) = |\bar\xi\rangle\langle\bar\xi|\otimes|\eta\rangle\langle\eta|.
\end{aligned}$$

Therefore, we have

$$\langle\mathrm{Ad}_s,\phi\rangle = \langle|\bar\xi\rangle\langle\bar\xi|\otimes|\eta\rangle\langle\eta|,\phi\rangle = \langle|\eta\rangle\langle\eta|,\phi(|\bar\xi\rangle\langle\bar\xi|)\rangle = \langle\bar\eta|\phi(|\bar\xi\rangle\langle\bar\xi|)|\bar\eta\rangle. \qquad (2.11)$$

We first find all $|\bar\xi\rangle$'s such that $\phi(|\bar\xi\rangle\langle\bar\xi|)$ is singular, and find the kernel $|\bar\eta\rangle$ of $\phi(|\bar\xi\rangle\langle\bar\xi|)$, to get extreme rays Ad_s in the dual face ϕ' in $\mathbb{SP}_1$.

We apply the above method to the Choi map $\phi_{\rm ch}: M_3 \to M_3$ defined in (1.54). For given $|\bar\xi\rangle = (x,y,z)^{\rm T} \in \mathbb{C}^3$, we calculate to get

$$\det\phi_{\rm ch}(|\bar\xi\rangle\langle\bar\xi|) = |x|^4|y|^2 + |y|^4|z|^2 + |z|^4|x|^2 - 3|xyz|^2.$$

This is nonnegative by arithmetic-geometric inequality, and this may give rise to another proof for positivity of the Choi map. It vanishes when and only when two of variables are zero or $|x| = |y| = |z|$. Therefore, $\phi_{\rm ch}(|\bar\xi\rangle\langle\bar\xi|)$ is singular if and only if $|\bar\xi\rangle$ is one of the following:

$$|\bar\xi_1\rangle = (1,0,0)^{\rm T},\quad |\bar\xi_2\rangle = (0,1,0)^{\rm T},\quad |\bar\xi_3\rangle = (0,0,1)^{\rm T},\quad |\bar\xi_4\rangle = (e^{{\rm i}a}, e^{{\rm i}b}, e^{{\rm i}c})^{\rm T},$$

and the corresponding kernel vectors of $\phi_{\rm ch}(|\bar\xi\rangle\langle\bar\xi|)$ are given by

$$|\bar\eta_1\rangle = (0,0,1)^{\rm T},\quad |\bar\eta_2\rangle = (1,0,0)^{\rm T},\quad |\bar\eta_3\rangle = (0,1,0)^{\rm T},\quad |\bar\eta_4\rangle = (e^{{\rm i}a}, e^{{\rm i}b}, e^{{\rm i}c})^{\rm T},$$

respectively, up to scalar multiplications. Then the dual face $\phi'_{\rm ch}$ is the convex cone generated by superpositive maps Ad_s's with following 3×3 rank one matrices $s = |\xi\rangle\langle\eta|$:

$$\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix},\quad s_{\alpha,\beta,\gamma} = \begin{pmatrix}1&\alpha&\bar\gamma\\\bar\alpha&1&\beta\\\gamma&\bar\beta&1\end{pmatrix},$$

where $\alpha\beta\gamma = 1$ with $|\alpha| = |\beta| = |\gamma| = 1$. We denote

$$S = \{e_{21}, e_{32}, e_{13}\} \cup \{s_{\alpha,\beta,\gamma} : \alpha\beta\gamma = 1,\ |\alpha| = |\beta| = |\gamma| = 1\}, \tag{2.12}$$

with matrix units. Then the convex cone ϕ'_{ch} is generated by $\{\mathrm{Ad}_s : s \in S\}$.

We note that matrices in S belong to the 7-dimensional subspace

$$D = \{[a_{ij}] \in M_3 : a_{11} = a_{22} = a_{33}\}. \tag{2.13}$$

We also see that the following four matrices

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

together with e_{21}, e_{32}, e_{13} are linearly independent rank one matrices belonging to D. Therefore, we conclude that S spans the 7-dimensional space D defined by (2.13). We note that all maps in F_D, which is a face of $\mathbb{CP}$, do not belong to $\mathbb{SP}_1$. We also note $\phi'_{\text{ch}} \subsetneqq F_D \cap \mathbb{SP}_1$, because there are rank one matrices in D which do not belong to S.

We proceed to show that the smallest face $F_{\phi_{\text{ch}}}$ of $\mathbb{P}_1$ containing ϕ_{ch} is strictly smaller than the bidual face ϕ''_{ch} of the Choi map $\phi_{\text{ch}} \in \mathbb{P}_1[M_3, M_3]$. From this, we will conclude that $F_{\phi_{\text{ch}}}$ is not exposed face. We see that $\mathrm{Ad}_s \in \phi''_{\text{ch}}$ if and only if $\langle s, t\rangle = 0$ for every $t \in S$ if and only if $\langle s, t\rangle = 0$ for every $t \in D$ if and only if s belongs to the 2-dimensional subspace

$$E = \{s \in M_{3\times 3} : s \text{ is diagonal},\ \mathrm{Tr}\, s = 0\}.$$

Therefore, we conclude that $\phi''_{\text{ch}} \cap \mathbb{CP} = F_E$. We consider three matrices

$$s_1 = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}, \qquad s_2 = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & -1 \end{pmatrix} \qquad s_3 = \begin{pmatrix} -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}$$

in E, and take

$$\phi_0 = \mathrm{Ad}_{s_1} + \mathrm{Ad}_{s_2} + \mathrm{Ad}_{s_3} \in \phi''_{\text{ch}}. \tag{2.14}$$

We put $\phi_t = (1-t)\phi_0 + t\phi_{\text{ch}}$ then we have

$$\phi_t(x) = \begin{pmatrix} (3-t)x_{11} + tx_{33} & \cdot & \cdot \\ \cdot & (3-t)x_{22} + tx_{11} & \cdot \\ \cdot & \cdot & (3-t)x_{33} + tx_{22} \end{pmatrix} - [x_{ij}],$$

for $x = [x_{ij}] \in M_3$. By Proposition 1.6.5, we see that ϕ_t is a positive map if and only if $0 \leqslant t \leqslant 3$ and

$$\frac{\alpha}{(3-t)\alpha + t\gamma} + \frac{\beta}{(3-t)\beta + t\alpha} + \frac{\gamma}{(3-t)\gamma + t\beta} \leqslant 1$$

for every positive α, β and γ. We take $\beta = \frac{1}{\alpha}$ and $\gamma \to 0$, to get a necessary condition

$$(2-t)t\alpha^2 + (3-t)(1-t) \geqslant 0, \qquad \alpha > 0$$

for positivity of ϕ_t. Therefore, we see that ϕ_t is positive only when $0 \leqslant t \leqslant 1$. This tells us that $\phi_{\rm ch}$ is on the boundary of the convex cone $\phi''_{\rm ch}$, and the smallest face $F_{\phi_{\rm ch}}$ determined by $\phi_{\rm ch}$ is strictly smaller than $\phi''_{\rm ch}$. Especially, we conclude that the face $F_{\phi_{\rm ch}}$ of $\mathbb{P}_1$ is not exposed. One can also show that $F_{\phi_{\rm ch}}$ contains no completely positive map. In fact, it is known [42] that the Choi map $\phi_{\rm ch}$ generates an extreme ray of $\mathbb{P}_1$. We summarize as follows:

Theorem 2.2.1 *We define the subset S of $M_{3\times 3}$ by* (2.12)*, and put*

$$D = \operatorname{span} S, \qquad E = \{t \in M_{3\times 3} : \langle s, t\rangle = 0 \text{ for every } s \in S\}.$$

The smallest exposed face $\phi''_{\rm ch}$ and the smallest face $F_{\phi_{\rm ch}}$ of $\mathbb{P}_1$ containing the Choi map $\phi_{\rm ch}$ have the following properties;

(i) $\operatorname{rank} s = 1$ *and* $\operatorname{Ad}_s \in \phi'_{\rm ch}$ *if and only if* $s \in S$*,*

(ii) $\phi'_{\rm ch}$ *is generated by* $\{\operatorname{Ad}_s : s \in S\}$*, and* $\phi'_{\rm ch} \subsetneqq F_D \cap \mathbb{SP}_1$*,*

(iii) $F_{\phi_{\rm ch}} \subsetneqq \phi''_{\rm ch}$ *and* $\phi''_{\rm ch} \cap \mathbb{CP} = F_E$*.*

(iv) $F_{\phi_{\rm ch}}$ *has no nontrivial completely positive map.*

Proof. It remains to prove (iv). If a completely positive map belongs to $F_{\phi_{\rm ch}}$, then there is $s \in E$ such that $\operatorname{Ad}_s \in F_{\phi_{\rm ch}}$. Since $\phi_{\rm ch}$ is an interior point of $F_{\phi_{\rm ch}}$, there is $\lambda > 1$ such that $(1-\lambda)\operatorname{Ad}_s + \lambda\phi_{\rm ch}$ is a positive map. Multiplying a scalar, we may assume that $\phi_{\rm ch} - \operatorname{Ad}_s$ is a positive map. Comparing the 2×2 principal submatrices of the inequality $\phi_{\rm ch}(|x\rangle\langle x|) \geqslant \operatorname{Ad}_s(|x\rangle\langle x|)$, we have

$$\begin{pmatrix} |x_1|^2 + |x_3|^2 & -x_1\bar{x}_2 \\ -x_2\bar{x}_1 & |x_2|^2 + |x_1|^2 \end{pmatrix} \geqslant \begin{pmatrix} |s_1|^2|x_1|^2 & -\bar{s}_1 s_2 x_1\bar{x}_2 \\ -\bar{s}_2 s_1 x_2\bar{x}_1 & |s_2|^2|x_2|^2 \end{pmatrix}$$

for every $x_1, x_2, x_3 \in C$, where s_1, s_2, s_3 are diagonal entries of $s \in M_3$. Taking $x_3 = 0$, we have

$$(1-|s_1|^2)((1-|s_2|^2)|x_2|^2 + |x_1|^2) \geqslant |x_2|^2|1-\bar{s}_1 s_2|^2$$

for every $x_1, x_2 \in \mathbb{C}$. Taking $x_1 = 0$, we have $(1-|s_1|^2)(1-|s_2|^2) \geqslant |1-\bar{s}_1 s_2|^2$, and $|s_1 - s_2| = 0$. By the same way, we have $s_2 = s_3$, and $s = 0$ since $s \in E$. □

References: [25], [19], [29], [42]

2.2.2 The Choi map and decomposable maps

In the above discussion, we have seen that the smallest exposed face $\phi''_{\rm ch}$ meets nontrivial completely positive maps by F_E. We look for decomposable maps in $\phi''_{\rm ch}$. We note that if an extreme ray $\mathrm{T}\circ\mathrm{Ad}_s$ belongs to $\phi''_{\rm ch}$ then $\langle \mathrm{T}\circ\mathrm{Ad}_s, \mathrm{Ad}_t\rangle = 0$ for every $t \in S$, or equivalently $\langle \mathrm{Ad}_s, \mathrm{T}\circ\mathrm{Ad}_t\rangle = 0$ for every $t \in S$. We also note that $\mathrm{T}\circ\mathrm{Ad}_{|\xi\rangle\langle\eta|} = \mathrm{Ad}_{|\xi\rangle\langle\bar\eta|}$. We will see that $\{|\xi\rangle\langle\bar\eta| : |\xi\rangle\langle\eta| \in S\}$ spans the whole space, which implies that there exists no nonzero $\mathrm{T}\circ\mathrm{Ad}_s$ in $\phi''_{\rm ch}$.

For this purpose, we take

$$|\xi_1\rangle = \begin{pmatrix}1\\1\\1\end{pmatrix},\ |\xi_2\rangle = \begin{pmatrix}1\\-1\\1\end{pmatrix},\ |\xi_3\rangle = \begin{pmatrix}1\\1\\-1\end{pmatrix},\ |\xi_4\rangle = \begin{pmatrix}1\\-\mathrm{i}\\1\end{pmatrix},\ |\xi_5\rangle = \begin{pmatrix}1\\1\\-\mathrm{i}\end{pmatrix},\ |\xi_6\rangle = \begin{pmatrix}1\\-\mathrm{i}\\-\mathrm{i}\end{pmatrix}$$

and $|\eta_i\rangle = |\xi_i\rangle$ for $i = 1,2,3,4,5,6$. We also define

$$|\xi_7\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix},\ |\xi_8\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix},\ |\xi_9\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix},\ |\eta_7\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix},\ |\eta_8\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix},\ |\eta_9\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}.$$

Then we have $|\xi_i\rangle\langle\eta_i| \in S$ for $i = 1,2,\dots,9$. We show that the set $\{|\xi_i\rangle\langle\bar\eta_i| : i = 1,2,\dots 9\}$ is linearly independent. Suppose that $B = \sum_{i=1}^9 a_i|\xi_i\rangle\langle\bar\eta_i| = 0$, and look at the entries of the matrix $B = [B_{ij}]$. Then we have

$$\begin{aligned}
B_{11} &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 0,\\
B_{12} &= a_1 - a_2 + a_3 - \mathrm{i}a_4 + a_5 - \mathrm{i}a_6 = 0,\\
B_{22} &= a_1 + a_2 + a_3 - a_4 + a_5 - a_6 = 0,\\
B_{23} &= a_1 - a_2 - a_3 - \mathrm{i}a_4 - \mathrm{i}a_5 - a_6 = 0,\\
B_{31} &= a_1 + a_2 - a_3 + a_4 - \mathrm{i}a_5 - \mathrm{i}a_6 = 0,\\
B_{33} &= a_1 + a_2 + a_3 + a_4 - a_5 - a_6 = 0.
\end{aligned}$$

With this relation, we have $a_i = 0$ for $i = 1,2,\dots,6$, from which we also have $a_7 = a_8 = a_9 = 0$. Therefore, we have the following:

Theorem 2.2.2 *For the Choi map $\phi_{\rm ch} : M_3 \to M_3$, we have*

$$\mathbb{DEC} \cap \phi''_{\rm ch} = \mathrm{conv}\,\{\mathrm{Ad}_s : s \in E\} \subset \mathbb{CP},$$

where E is the 2-dimensional subspace of $M_{3\times3}$ consisting of all diagonal matrices with trace zero.

We also note that $\{|\xi_i\rangle|\eta_i\rangle : i = 1,2,\dots,9\}$ spans the whole space $\mathbb{C}^3 \otimes \mathbb{C}^3$, but their partial conjugates span the 7-dimensional subspace.

We close this section with a geometric interpretation of Theorem 2.2.1. To begin with general situations, we suppose that F is a subset of a closed convex cone C. For a given $x \in C$, we see that $x \in F''$ if and only if $\langle x, y\rangle = 0$ for every $y \in F'$ if and only if $\langle x, y\rangle = 0$ for every $y \in \mathcal{E}(F')$, where $\mathcal{E}(C)$ denotes the set of all extreme rays of C. Since F' is a face of C°, we have $\mathcal{E}(F') = F' \cap \mathcal{E}(C^\circ)$. Furthermore, we see that $y \in F'$ if and only if $F \subset y'$. This is also equivalent to $F'' \subset y'$ by Proposition 2.1.1. Therefore, we have

$$F'' = \bigcap \{y' : y' \supset F'',\ y \in \mathcal{E}(C^\circ)\} = \bigcap \{y' : y' \supset F,\ y \in \mathcal{E}(C^\circ)\}$$

for an arbitrary subset F of a closed convex cone C.

We restrict our attention to the case when $C = \mathbb{P}_1$. We know that every extreme ray of $\mathbb{SP}_1 = \mathbb{P}_1^\circ$ is of the form Ad_s, which is also exposed in $\mathbb{SP}_1$. Therefore, we have

$$F'' = \bigcap \{\mathrm{Ad}_s' : \mathrm{Ad}_s' \supset F\}. \tag{2.15}$$

In the next section, we will see that a face is maximal if and only if it is a dual face of an exposed face which is minimal among all exposed faces. This means that every maximal face of $\mathbb{P}_1$ is of the form Ad_s' for a rank one matrix s. Therefore, we see that the smallest exposed face F'' containing a subset F of $\mathbb{P}_1$ is the intersection of all maximal faces of $\mathbb{P}_1$ containing F.

When F is a singleton $\{\phi_{\mathrm{ch}}\}$, we see that maximal faces containing ϕ_{ch} consist of Ad_s' with $s \in S$ by Theorem 2.2.1 (i). They are also maximal faces containing ϕ_{ch}''. On the other hand, a maximal face Ad_s' with a rank one matrix s contains the subset F_E of ϕ_{ch}'' if and only if s belongs to D, with notations in Theorem 2.2.1. We note that $D \supsetneqq S$. Further, we note that there are rank one matrices in D which does not belong to S. For example, the maximal face $\mathrm{Ad}_{e_{12}}'$ contains F_E, but does not contain ϕ_{ch}'', since $e_{12} \in D\backslash S$.

References: [19]

2.3 Maximal faces

In this section, we show that a face is maximal if and only if its dual face is minimal among all exposed faces. Especially, every maximal face is a dual face. Since we know all the exposed ray of $\mathbb{P}_k^\circ = \mathbb{SP}_k$, we can characterize all maximal faces of the convex cone $\mathbb{P}_k$.

2.3.1 Boundary of convex cones

Suppose that X and Y are finite-dimensional real vector spaces, with a non-degenerate bilinear pairing $\langle\ ,\ \rangle$ on $X \times Y$. We also suppose that C is a closed convex cone in

X, and the bilinear pairing is *non-degenerate* on C, that is, the following

$$x \in C,\ \langle x, y\rangle = 0 \text{ for each } y \in C^\circ \implies x = 0 \tag{2.16}$$

holds. In the simplest case of $X = Y = \mathbb{R}^2$, the bilinear pairing $\langle x, y\rangle = x_1y_1 + x_2y_2$ is not non-degenerate on the half-plane

$$C := \{x \in X : \langle x, y_0\rangle \geqslant 0\}, \tag{2.17}$$

with $y_0 \in Y$. In this case, the dual cone is given by the single ray $\{\lambda y_0 \in Y : \lambda \geqslant 0\}$ generated by $y_0 \in Y$. In case of $X = Y = \mathbb{R}^n$, if a closed convex cone C contains two rays $\mathbb{R}^+x_0$ and $\mathbb{R}^+(-x_0)$ with the opposite directions then the standard bilinear pairing $\langle x, y\rangle = \sum_{i=1}^n x_iy_i$ is not non-degenerate on C.

If C is a closed convex cone on which the bilinear pairing is non-degenerate then for every $x \in C$ with $x \neq 0$ there exists $y \in C^\circ$ such that $\langle x, y\rangle > 0$. Every finite dimensional space may be endowed with a norm, and so we may use compactness argument to conclude that there exists $\eta \in C^\circ$ with the property

$$x \in C,\ x \neq 0 \implies \langle x, \eta\rangle > 0, \tag{2.18}$$

which is seemingly stronger than (2.16). The bilinear pairings defined in (1.39) and (1.40) are non-degenerate on any convex cones appearing in (1.45) or (1.47). In fact, either the identity matrix in $M_m \otimes M_n$ or the trace map in $L(M_m, M_n)$ play the roles of η in (2.18).

Proposition 2.3.1 *Let X and Y be finite-dimensional spaces, and C a closed convex cone in X on which $\langle\ ,\ \rangle$ is a non-degenerate bilinear pairing. For a given $y \in C^\circ$, the following are equivalent:*

(i) *y is an interior point of C°,*

(ii) *$\langle x, y\rangle > 0$ for each nonzero $x \in C$.*

(iii) *$\langle x, y\rangle > 0$ for each $x \in C$ which generates an extreme ray.*

Proof. If y is an interior point of C° then we may take $t \in [0, 1)$ and $z \in C^\circ$ such that $y = (1-t)\eta + tz$, where $\eta \in C^\circ$ is a point with the property (2.18). Then we see that

$$\langle x, y\rangle = (1-t)\langle x, \eta\rangle + t\langle x, z\rangle > 0$$

for each nonzero $x \in C$. This proves (i) $\implies$ (ii). It is clear that (ii) and (iii) are equivalent. Now, we suppose that $y \in C^\circ$ satisfies (ii), and take an arbitrary point $z \in C^\circ$. Put $C_\epsilon = \{x \in C : \|x\| = \epsilon\}$. Then since C_1 is compact, $\alpha = \sup\{\langle x, z\rangle : x \in$

$C_1\}$ is finite, and we see that $\langle x, z\rangle \leqslant 1$ for each $x \in C_{1/\alpha}$. By (ii), we also take δ with $0 < \delta < 1$ such that $\langle x, y\rangle \geqslant \delta$ for each $x \in C_{1/\alpha}$. Put

$$w = \left(1 - \frac{1}{1-\delta}\right) z + \frac{1}{1-\delta}\, y.$$

We check $\langle x, w\rangle \geqslant 0$ for each $x \in C_{1/\alpha}$, and so have $w \in C^\circ$. Because z was an arbitrary point of C°, we conclude that y is an interior point of C°, as it was required. □

The condition of non-degeneracy of the bilinear pairing on C is important in Proposition 2.3.1. In case of the convex cone C given in (2.17), we note that any point in C° does not satisfy the condition (ii) of Proposition 2.3.1. But, all the points of $C^\circ = \{\lambda y_0 : \lambda \geqslant 0\}$ is an interior point of C° except the origin.

Corollary 2.3.2 *Let X and Y be finite-dimensional spaces, and C a closed convex cone in X on which $\langle\ ,\ \rangle$ is a non-degenerate bilinear pairing. Then we have the following:*

(i) *If F is a face of C satisfying $F' = C^\circ$ then we have $F = \{0\}$,*

(ii) *If F is a face of C° satisfying $F' = \{0\}$ then we have $F = C^\circ$.*

Proof. The statement (i) is a trivial consequence of (2.16). To prove (ii), we suppose that $F \subsetneqq C^\circ$. Then we have $F \subset \partial C^\circ$, and can take a nonzero $y \in \operatorname{int} F \subset \partial C^\circ$. By the implication (ii) $\Longrightarrow$ (i) of Proposition 2.3.1, there exists a nonzero $x \in C$ such that $\langle x, y\rangle = 0$. Since $y \in \operatorname{int} F$, we have $x \in F'$. This shows that F' is nonzero. □

We apply Proposition 2.3.1 to the duality $\mathcal{S}_1^\circ = \mathbb{P}_1$, to see that ϕ is an interior point of $\mathbb{P}_1[M_m, M_n]$ if and only if $\langle \varrho, \phi\rangle > 0$ for every nonzero $\varrho = |\zeta\rangle\langle\zeta| \in \mathcal{S}_1$. Taking $|\zeta\rangle = |\xi\rangle|\eta\rangle$, we see that ϕ is an interior point of $\mathbb{P}_1$ if and only if

$$\langle |\eta\rangle\langle\eta|, \phi(|\xi\rangle\langle\xi|)\rangle > 0$$

for every nonzero $|\xi\rangle \in \mathbb{C}^m$ and $|\eta\rangle \in \mathbb{C}^n$ if and only if $\phi(|\xi\rangle\langle\xi|)$ is nonsingular for every nonzero $|\xi\rangle$ if and only if $\phi(x)$ is nonsingular for every nonzero $x \in M_n^+$ if and only if $\phi(x)$ is an interior point of M_n^+ for every nonzero $x \in M_m^+$. This shows that the converse of Proposition 1.2.5 holds.

Proposition 2.3.3 *A positive linear map $\phi : M_m \to M_n$ is an interior point of $\mathbb{P}_1$ if and only if $\phi(x)$ is an interior point of M_n^+ for every nonzero $x \in M_m^+$.*

References: [34]

2.3.2 Maximal faces and minimal exposed faces

We say that L is a *minimal exposed face* if it is an exposed face and minimal among all exposed proper faces. It is easy to see that if L is a minimal exposed face of the cone C then L' is a maximal face of C°. To see this, we suppose that F is a face of C° such that $F \supset L'$. Then we have

$$L = L'' \supset F'.$$

Since F' is an exposed face and L is minimal among exposed faces, we have either $F' = \{0\}$ or $F' = L$. If $F' = \{0\}$ then we have $F = C^\circ$ by Corollary 2.3.2 (ii). In case of $F' = L$, we have $F \subset F'' = L'$, which implies $F = L'$. Therefore, we conclude that L' is a maximal face. The following theorem tells us that the converse is also true. Especially, every maximal face is a dual face.

Theorem 2.3.4 *Let X and Y be finite-dimensional normed spaces with a non-degenerate bilinear pairing $\langle\ ,\ \rangle$ on a closed convex cone C in X. Then we have the following:*

(i) *If L is a minimal exposed face of C then L' is a maximal face of C°,*

(ii) *every maximal face of C° is the dual face of a unique minimal exposed face of C.*

Proof. It remains to prove (ii). To do this, suppose that F is a maximal face of C°. Note that we have $F \subset F'' \subset C^\circ$. If $F'' = C^\circ$ then we have $F' = F''' = (C^\circ)' = \{0\}$, and so we have $F = C^\circ$ by Corollary 2.3.2 (ii), which is not possible. By the maximality of F, we have $F = (F')'$ is the dual face of F'. In order to show that F' is minimal among exposed faces, we suppose that L is an exposed face satisfying $L \subset F'$. Then we have $F = F'' \subset L'$. By the maximality of F, we have either $L' = F$ or $L' = C^\circ$. If $L' = F$ then $L = L'' = F'$. If $L' = C^\circ$ then $L = \{0\}$ by Corollary 2.3.2 (i). Therefore, F' is minimal among exposed faces. In order to show the uniqueness, we suppose that $L_1' = L_2' = F$ for exposed faces L_1 and L_2. Then we have

$$L_1 = L_1'' = F' = L_2'' = L_2,$$

as it was required. □

We recall that Ad_s generates an exposed ray of the convex cone $\mathbb{SP}_1{}^\circ = \mathbb{P}_1$. Therefore, every $m \times n$ matrix σ gives rise to a maximal face

$$\mathrm{conv}\,\{\mathrm{Ad}_s \in \mathbb{SP}_1 : \mathrm{rank}\, s = 1, \langle s, \sigma\rangle = 0\}$$

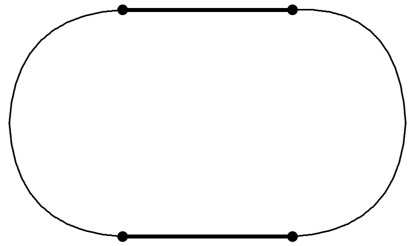

Figure 2.1: Line segments are minimal exposed faces. Four points are extreme points which are not exposed, but they are exposed in the line segments.

of the convex cone $\mathbb{SP}_1[M_m, M_n]$ by the identity (2.1). The exposed map $\mathrm{Ad}_\sigma \circ \mathrm{T}$ of $\mathbb{P}_1$ also gives rise to the corresponding maximal face. But, not every maximal face arises in this way, because there are indecomposable positive map which generates exposed ray, as we will see later.

We note that an exposed ray of a convex cone is automatically a minimal exposed face. But the converse is not true. Consider the convex cone whose section looks like an athletics track which consists of two semicircles and two parallel line segments. We note that a line segment generates a minimal exposed face. This exposed face contains an exposed ray in itself, but it is not an exposed ray in the whole convex cone. See Figure 2.1.

References: [34]

2.3.3 Boundaries of k-positive maps

We apply Theorem 2.3.4 to $\mathbb{SP}_k^\circ = \mathbb{P}_k$ to see that

$$(\mathrm{Ad}_s)' = \{\phi \in \mathbb{P}_k : \langle \phi, \mathrm{Ad}_s \rangle = 0\}$$

is a maximal face of $\mathbb{P}_k$ whenever $\operatorname{rank} s \leqslant k$, since Ad_s generates an exposed ray of $\mathbb{SP}_k$. Furthermore, every maximal face of the convex cone $\mathbb{P}_k[M_m, M_n]$ is of the form $(\mathrm{Ad}_s)'$ for an $m \times n$ matrix s with rank at most k. Equivalently, we may apply Theorem 2.3.4 to $\mathcal{S}_k^\circ = \mathbb{P}_k$ to get the following:

Corollary 2.3.5 *For a vector* $|\zeta\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ *with* $\mathrm{SR}\,|\zeta\rangle \leqslant k$*,*

$$F_k[\zeta] := \{\phi \in \mathbb{P}_k : \langle |\zeta\rangle\langle\zeta|, \phi \rangle = 0\}$$

is a maximal face of $\mathbb{P}_k[M_m, M_n]$*. Conversely, every maximal face of* $\mathbb{P}_k[M_m, M_n]$ *is of the form* $F_k[\zeta]$ *for a vector* $|\zeta\rangle$ *with* $\mathrm{SR}\,|\zeta\rangle \leqslant k$*.*

By the uniqueness part of Theorem 2.3.4 (ii), the vector $|\zeta\rangle$ in the second part of Corollary 2.3.5 is determined uniquely. This can be also seen as follows: We note

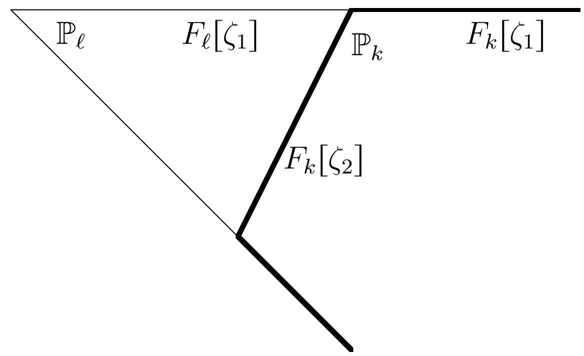

Figure 2.2: When $\mathrm{SR}\,|\zeta_2\rangle > \ell$, the maximal face $F_k[\zeta_2]$ of $\mathbb{P}_k$ is located inside of $\mathbb{P}_\ell$.

that the Choi matrices of the maps in $F_k[\zeta] \cap \mathbb{CP}$ are positive matrices ϱ satisfying $\langle |\zeta\rangle\langle\zeta|, \varrho\rangle = 0$. Therefore, we see that $F_k[\zeta] = F_k[\omega]$ implies that $|\zeta\rangle$ and $|\omega\rangle$ are parallel. In the case of $k = 1$, we see that every maximal face of $\mathbb{P}_1$ is determined by a product vector $|\zeta\rangle = |\xi\rangle \otimes |\eta\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$, and we have

$$F_1[\zeta] = \{\phi \in \mathbb{P}_1 : \langle\bar\eta|\phi(|\xi\rangle\langle\xi|)|\bar\eta\rangle = 0\}.$$

We consider a maximal face $F_k[\zeta]$ of the convex cone $\mathbb{P}_k$ with $\mathrm{SR}\,|\zeta\rangle = s \leqslant k$, and investigate how $F_k[\zeta]$ is located in the bigger convex cone $\mathbb{P}_\ell$ when $\ell < k$. In the case of $s \leqslant \ell$, we have $F_k[\zeta] = F_\ell[\zeta] \cap \mathbb{P}_k$. Because $F_k[\zeta] \subset \partial\mathbb{P}_k$, we have $F_k[\zeta] = F_\ell[\zeta] \cap \partial\mathbb{P}_k$. Especially, we have $F_k[\zeta] \subset \partial\mathbb{P}_\ell$. In general situations, it is easy to see that if C_1 is a convex subset of a convex set C, then either $C_1 \subset \partial C$ or $\mathrm{int}\, C_1 \subset \mathrm{int}\, C$ holds. We show $\mathrm{int}\, F_k[\zeta] \subset \mathrm{int}\, \mathbb{P}_\ell$ in case of $s > \ell$. See Figure 2.2.

Proposition 2.3.6 *Suppose that* $\mathrm{SR}\,|\zeta\rangle = s$ *with* $s \leqslant k$ *and* $\ell < k$. *Then the maximal face* $F_k[\zeta]$ *of* $\mathbb{P}_k$ *satisfies the following:*

(i) *if* $s \leqslant \ell$ *then we have* $F_k[\zeta] = F_\ell[\zeta] \cap \partial\mathbb{P}_k \subset \partial\mathbb{P}_\ell$,

(ii) *if* $\ell < s$ *then we have* $\mathrm{int}\, F_k[\zeta] \subset \mathrm{int}\, \mathbb{P}_\ell$.

Proof. It remains to prove (ii). Assume that there exists $\phi \in \mathrm{int}\, F_k[\zeta]$ such that $\phi \notin \mathrm{int}\, \mathbb{P}_\ell$. Then $\phi \in \partial\mathbb{P}_\ell$, and so there exists $|\omega\rangle$ with $\mathrm{SR}\,|\omega\rangle \leqslant \ell$ such that $\phi \in F_\ell[\omega]$. Since $\phi \in \mathrm{int}\, F_k[\zeta] \subset \partial\mathbb{P}_k$, we have $\phi \in F_\ell[\omega] \cap \partial\mathbb{P}_k = F_k[\omega]$ by (i). Now, $\phi \in \mathrm{int}\, F_k[\zeta]$ implies that two faces $F_k[\zeta]$ and $F_k[\omega]$ satisfy the relation $F_k[\zeta] \subset F_k[\omega]$, and we have $F_k[\zeta] = F_k[\omega]$ by maximality. Therefore, we conclude that $|\zeta\rangle$ and $|\omega\rangle$ are parallel to each other, which implies that $s \leqslant \ell$. □

The rays generated by Ad_s and $\mathrm{Ad}_s \circ \mathrm{T}$ are exposed rays of $\mathbb{DEC}$ as well as $\mathbb{P}_1$, since $\mathbb{DEC} \subset \mathbb{P}_1$. Because $\mathbb{DEC}$ is the convex hull of them, they exhaust all the exposed rays of $\mathbb{DEC}$. Since $\langle \varrho, \mathrm{Ad}_{\tilde{s}} \rangle = \langle \tilde{s}|\varrho|\tilde{s}\rangle$, we see that every maximal face of $\mathcal{PPT}$ is of the form

$$\{\varrho \in \mathcal{PPT} : \langle \zeta|\varrho|\zeta\rangle = 0\} \qquad \text{or} \qquad \{\varrho^\Gamma \in \mathcal{PPT} : \langle \zeta|\varrho|\zeta\rangle = 0\}, \tag{2.19}$$

for $|\zeta\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$. This faces of $\mathcal{PPT}$ correspond to faces $F_V \cap F^W$, with notations in (2.5) and (2.6), which are determined by pairs $(\zeta^\perp, M_{m\times n})$ and $(M_{m\times n}, \zeta^\perp)$, respectively.

References: [74], [75], [76]

2.4 Entanglement detected by positive maps

A self-adjoint matrix W in $M_m \otimes M_n$ is called an *entanglement witness* if $\langle \varrho, W\rangle \geqslant 0$ for every separable state ϱ, but $\langle \varrho_0, W\rangle < 0$ for a state ϱ_0. In this case, ϱ_0 must be an entangled state which is detected by W. An entanglement witness is nothing but the Choi matrix of a positive map which is not completely positive, or equivalently, a block-positive matrix which is not positive. In this section, we look for conditions on positive maps which detect maximal set of entanglement. We will begin with a pair (C, D) of convex cones satisfying $C \subset D$, motivated by inclusions $\mathcal{S}_1 \subset (M_m \otimes M_n)^+$ and $\mathcal{S}_1 \subset \mathcal{PPT}$.

2.4.1 Optimal entanglement witnesses

We recall that a state ϱ is entangled if and only if there exists a positive map ϕ such that $\langle \varrho, \phi\rangle < 0$ with respect to the bilinear pairing (1.39). For a given positive linear map $\phi : M_m \to M_n$, we denote by W_ϕ the set of all (unnormalized) states $\varrho \in (M_m \otimes M_n)^+$ such that $\langle \varrho, \phi\rangle < 0$. If $W_{\phi_0} \subset W_{\phi_1}$ then ϕ_0 is of little use as a detector of entanglement. In this context, it is natural to seek conditions for a pair (ϕ_0, ϕ_1) so that W_{ϕ_0} and W_{ϕ_1} are comparable.

Let X and Y be real vector spaces with a bilinear pairing. Suppose that $C \subset D$ are closed convex cones in X, with the dual cones $D^\circ \subset C^\circ$ in Y. Throughout this section, we also suppose that the bilinear pairing is non-degenerate on both D and C°. This implies that there exist $\xi_0 \in C$ and $\eta_0 \in D^\circ$ satisfying

$$x \in D,\ x \neq 0 \implies \langle x, \eta_0\rangle > 0, \qquad y \in C^\circ,\ y \neq 0 \implies \langle \xi_0, y\rangle > 0,$$

by (2.18). Then we see that η_0 is an interior point of both D° and C° by Proposition 2.3.1, and so we have $\operatorname{int} D^\circ \subset \operatorname{int} C^\circ$. In the same way, we also see that $\operatorname{int} C \subset \operatorname{int} D$.

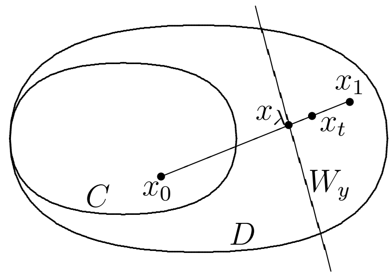

Figure 2.3: If $\langle x_\lambda, y\rangle = 0$, then x_t is a common interior point of W_y and D.

We know that $x \in X$ does not belong to C if and only if there exists $y \in C^\circ$ such that $\langle x, y\rangle < 0$. For a given $y \in C^\circ$, we define

$$W_y[D; C] := \{x \in D : \langle x, y\rangle < 0\}$$

which is a convex subset of $D\backslash C$. We use the notation W_y if there is no confusion. When $x \in W_y$, we say that y is a *witness* for $x \in D\backslash C$, or x is *detected* by y. We note that W_y is nonempty if and only if $y \notin D^\circ$. In this case, we take $x_1 \in \operatorname{int} W_y$, $x_0 \in \operatorname{int} C \subset \operatorname{int} D$, and consider the line segment $x_t = (1-t)x_0 + tx_1$. Since $\langle x_0, y\rangle > 0$ and $\langle x_1, y\rangle < 0$, there exists $\lambda \in (0,1)$ such that $\langle x_\lambda, y\rangle = 0$. Then we see that x_t is a common interior point of W_y and D for every $t \in (\lambda, 1)$, and so we see that $\operatorname{int} W_y \subset \operatorname{int} D$. See Figure 2.3.

Furthermore, the set $W_y[D; C]$ has nonzero volume in D whenever it is nonempty. To see this, it suffices to show that D is contained in the affine manifold generated by $W_y[D; C]$. If $z_1 \in D$ with $\langle z_1, y\rangle \geqslant 0$ then we take $z_0 \in W_y[D; C]$ and put $z_t = (1-t)z_0 + tz_1$. Then $\langle z_t, y\rangle = (1-t)\langle z_0, y\rangle + t\langle z_1, y\rangle < 0$ for sufficiently small $t > 0$, and so we see that $z_1 = \frac{1}{t}z_t - \frac{1-t}{t}z_0$ belongs to the affine manifold generated by $W_y[D; C]$.

We take $y_0, y_1 \in C^\circ\backslash D^\circ$ and consider the line through $y_t = (1-t)y_0 + ty_1$. If this line touches the convex set D° then the intersection with D° must be on either $(y_{-\infty}, y_0)$ or (y_0, y_1) or $(y_1, y_{+\infty})$, with the obvious meanings of notations. Suppose that the intersection is on $(y_{-\infty}, y_0)$, that is, there exists $z \in D^\circ$ such that y_0 is between z and y_1. If we take $x \in D$, then we have $\langle x, z\rangle \geqslant 0$. Therefore, $\langle x, y_0\rangle < 0$ implies $\langle x, y_1\rangle < 0$, that is $W_{y_0} \subset W_{y_1}$. In short, we can say that the further y is from D°, the bigger W_y is. See Figure 2.4. We show that the converse also holds up to scalar multiplications of y_0. We begin with the following simple lemma.

Lemma 2.4.1 *Suppose that $W_{y_0} \subset W_{y_1}$. If $x \in D$ and $\langle x, y_0\rangle = 0$, then $\langle x, y_1\rangle \leqslant 0$.*

Proof. Assume that there exists $x \in D$ such that $\langle x, y_0\rangle = 0$ and $\langle x, y_1\rangle > 0$. If we take $x' \in W_{y_0}$ then $\langle x' + \lambda x, y_0\rangle < 0$ for every real number λ. But, we have

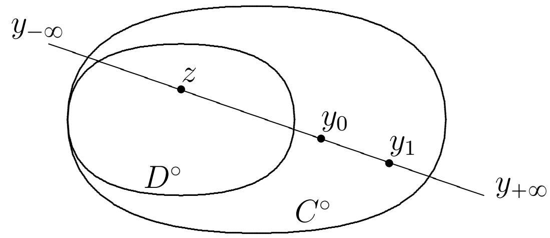

Figure 2.4: The further y is from D°, the bigger W_y is.

$\langle x' + \lambda x, y_1\rangle \geqslant 0$ for sufficiently large λ. This shows that W_{y_0} is not contained in W_{y_1}. □

The author is grateful to Yoonje Jeong for simplification of the proof for the statement (i) in the following proposition.

Proposition 2.4.2 *Let X and Y be real vector spaces with a bilinear pairing. Suppose that $C \subset D$ are closed convex cones X, and the bilinear pairing is non-degenerate on C° and D. For $y_0, y_1 \in C^\circ \backslash D^\circ$, we have the following:*

(i) *$W_{y_0} \subset W_{y_1}$ holds if and only if there exists $\lambda > 0$ and $z \in D^\circ$ such that $\lambda y_0 = y_1 + z$,*

(ii) *$W_{y_0} = W_{y_1}$ holds if and only if there exists $\lambda > 0$ such that $\lambda y_0 = y_1$,*

(iii) *$W_{y_0} \subsetneq W_{y_1}$ holds if and only if there exists $\lambda > 0$ and nonzero $z \in D^\circ$ such that $\lambda y_0 = y_1 + z$.*

Proof. For the statement (i), it remains to show the 'only if' part. Suppose that $W_{y_0} \subset W_{y_1}$. For $x \in D$ and $x' \in W_{y_0}$, we have $\langle\langle x, y_0\rangle x' - \langle x', y_0\rangle x, y_0\rangle = 0$, and so we also have

$$\langle x, y_0\rangle\langle x', y_1\rangle - \langle x', y_0\rangle\langle x, y_1\rangle = \langle\langle x, y_0\rangle x' - \langle x', y_0\rangle x, y_1\rangle \leqslant 0,$$

by Lemma 2.4.1. Therefore, we have

$$\langle x, y_0\rangle > 0,\ x' \in W_{y_0} \implies \left|\frac{\langle x', y_1\rangle}{\langle x', y_0\rangle}\right| \geqslant \frac{\langle x, y_1\rangle}{\langle x, y_0\rangle}. \tag{2.20}$$

Define

$$\lambda := \inf\left\{\left|\frac{\langle x', y_1\rangle}{\langle x', y_0\rangle}\right| : x' \in W_{y_0}\right\}.$$

Taking $\xi \in C$ such that $\langle \xi, y\rangle > 0$ for every nonzero $y \in C^\circ$, we see that $\lambda \geqslant \frac{\langle \xi, y_1\rangle}{\langle \xi, y_0\rangle}$, and so we have $\lambda > 0$. By (2.20), we have

$$\lambda\langle x, y_0\rangle - \langle x, y_1\rangle \geqslant 0, \tag{2.21}$$

whenever $x \in D$ satisfies $\langle x, y_0 \rangle > 0$. By definition of λ, we also have (2.21) for $x \in D$ with $\langle x, y_0 \rangle < 0$. If $\langle x, y_0 \rangle = 0$ then we also have (2.21) by Lemma 2.4.1. Therefore, we have $\lambda y_0 - y_1 \in D^\circ$, which completes the proof of (i).

To prove (ii), suppose that $W_{y_0} = W_{y_1}$. Then by (i), there exists $\lambda, \mu > 0$ and $z, w \in D^\circ$ such that $\lambda y_0 = y_1 + z$ and $\mu y_1 = y_0 + w$, and so $\lambda(\mu y_1 - w) = y_1 + z$. Therefore, we have $(\lambda\mu - 1)y_1 = \lambda w + z$. Because $\lambda w + z \in D^\circ$ and $y_1 \in C^\circ \backslash D^\circ$ is nonzero, we have $\lambda\mu - 1 = 0$ and $w = z = 0$. The converse is clear.

Finally, the statement (iii) is an immediate consequence of (i) and (ii). □

For y_0 and y_1 in Proposition 2.4.2, we note that $y_i' = \{x \in C : \langle x, y_i \rangle = 0\}$ for $i = 0, 1$. The condition of Proposition 2.4.2 (i) shows that $\lambda\langle x, y_0 \rangle = \langle x, y_1 \rangle + \langle x, z \rangle$ for every $x \in C$, and so we see that $W_{y_0} \subset W_{y_1}$ implies $y_0' \subset y_1'$. If $W_{y_0} = W_{y_1}$ then $y_0' = y_1'$ by Proposition 2.4.2 (ii). It should be noted that $y_0' = y_1'$ does not imply $W_{y_0} = W_{y_1}$. Indeed, we consider the pair $\mathcal{S}_1 \subset (M_m \otimes M_n)^+$ with the dual pair $\mathbb{CP} \subset \mathbb{P}_1$, and the map $\phi_{1/2} = \frac{1}{2}\phi_0 + \frac{1}{2}\phi_{\rm ch}$ with ϕ_0 in (2.14). We see that $\phi_{1/2}' = \phi_{\rm ch}'$, but $W_{\phi_{1/2}} \subsetneqq W_{\phi_{\rm ch}}$ by Proposition 2.4.2 (iii), because $\phi_0 \in \mathbb{CP}$.

Theorem 2.4.3 *Suppose that $C \subset D$ are closed convex cones in a real vector space X, and Y is a real vector space with a bilinear pairing on $X \times Y$ which is non-degenerate on C° and D. For $y_0 \in C^\circ \backslash D^\circ$, the following are equivalent:*

(i) *W_{y_0} is maximal among $\{W_y : y \in C^\circ \backslash D^\circ\}$,*

(ii) *the smallest face of C° containing y_0 has no nontrivial intersection with D°.*

Proof. We denote by F the smallest face of C° containing y_0, then y_0 is an interior point of F. Suppose that there exists a nonzero $z \in F \cap D^\circ$. Then there exist $t < 0$ such that $y_t := (1-t)y_0 + tz \in F \subset C^\circ$. This implies $(1-t)y_0 = y_t + (-t)z$ with $(-t)z \in D^\circ$, and we have $W_{y_0} \subsetneqq W_{y_t}$ by Proposition 2.4.2 (iii). This tells us that W_{y_0} is not maximal. For the converse, suppose that W_{y_0} is not maximal, and there exists $y_1 \in C^\circ$ such that $W_{y_0} \subsetneqq W_{y_1}$. Then there exist $\lambda > 0$ and nonzero $z \in D^\circ$ such that $\lambda y_0 = y_1 + z$, which implies $z \in F$. Therefore, we conclude that $F \cap D^\circ$ is nontrivial. □

For a given $y_0 \in C^\circ$, we consider the hyperplane $H_{y_0} = \{x \in X : \langle x, y_0 \rangle = 0\}$. The relations between the set $W_{y_0}[D; C]$, the hyperplane H_{y_0} and the convex cone D depend on the location of y_0. See Figure 2.5. We recall that $D^\circ \subset C^\circ$.

- If y_0 is an interior point of D° then H_{y_0} has no nontrivial intersection with D and $W_{y_0}[D; C]$ is empty.

- If y_0 is a boundary point of D° then $H_{y_0} \cap D$ is nontrivial but $W_{y_0}[D; C]$ is still empty.

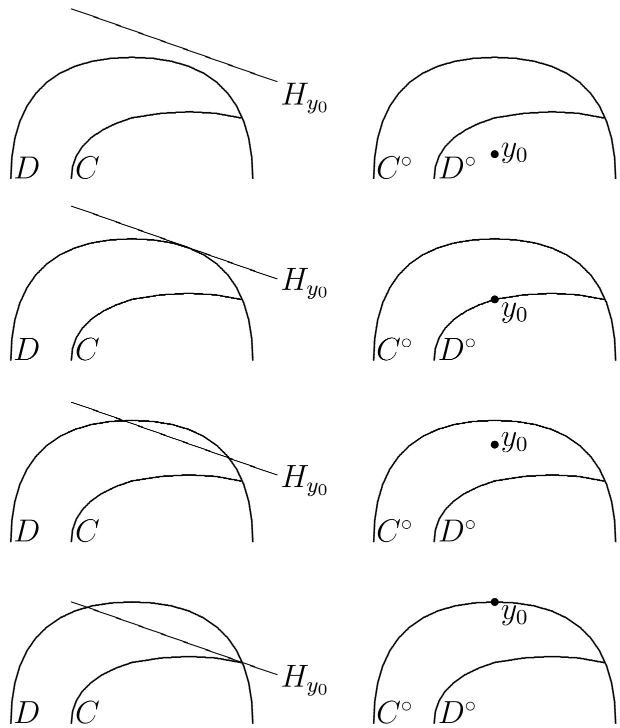

Figure 2.5: The location of the hyperplane H_{y_0} depends on the location of $y_0 \in C^\circ$

- When we take $y_0 \in \operatorname{int} C^\circ \backslash D^\circ$, the set $W_{y_0}[D; C]$ is nonempty and $\operatorname{int}(H_{y_0} \cap D) \subset \operatorname{int} D$. But $H_{y_0} \cap C$ is still trivial.
- If we take $y_0 \in \partial C^\circ \backslash D^\circ$ then $y_0' = H_{y_0} \cap C$ is nontrivial. In this case, we take the smallest face $F \subset \partial C^\circ$ of C° containing y_0. Theorem 2.4.3 tells us that the set W_{y_0} is maximal if and only if $F \cap D^\circ$ is trivial.

In order to detect entangled states, we first apply Theorem 2.4.3 to the pair of convex cones $\mathcal{S}_1 \subsetneqq (M_m \otimes M_n)^+$ with the dual pair $\mathbb{CP} \subsetneqq \mathbb{P}_1$ of linear maps, or equivalently $(M_m \otimes M_n)^+ \subsetneqq \mathcal{BP}_1$. We say that a positive map ϕ or its Choi matrix C_ϕ is optimal if the set $W_\phi[(M_m \otimes M_n)^+; \mathcal{S}_1]$ is maximal. Theorem 2.4.3 tells us that ϕ is optimal if and only if the smallest face of $\mathbb{P}_1$ containing ϕ has no nonzero completely positive map. We see that the Choi map ϕ_{ch} is optimal by Theorem 2.2.1 (iv). Every extremal positive map is optimal if it is not completely positive, and so the completely copositive maps $\mathrm{Ad}_s \circ \mathrm{T}$ are optimal whenever $\operatorname{rank} s \geqslant 2$. Such optimal positive maps are actually of little use to detect entanglement, because they can detect only non-PPT states. If we want to detect PPT entanglement then we have to use the pair $\mathcal{S}_1 \subset \mathcal{PPT}$ with the dual pair $\mathbb{DEC} \subset \mathbb{P}_1$. In this case, we see that the set $W_\phi[\mathcal{PPT}; \mathcal{S}_1]$ is maximal if and only if the smallest face of $\mathbb{P}_1$ containing ϕ has no nontrivial decomposable maps.

References: [123], [82], [102]

2.4.2 Spanning properties

Because we do not know the facial structures of the convex cone $\mathbb{P}_1$, it is not so easy to determine if a given positive map is optimal or not. We consider a slightly stronger condition than Theorem 2.4.3 (ii); the smallest exposed face ϕ'' of $\mathbb{P}_1$ containing ϕ has no nonzero completely positive maps, that is, $\phi'' \cap \mathbb{CP} = \{0\}$.

We begin with general situations. We compare the definition of $W_y[D; C]$ with $y' = \{x \in C : \langle x, y\rangle = 0\}$, which must be considered as a face of C, not a face of D since $y \in C^\circ \backslash D^\circ$.

Proposition 2.4.4 *Suppose that $C \subset D$ are closed convex cones in a real vector space X, and Y is a real vector space with a bilinear pairing on $X \times Y$. For $y_0 \in C^\circ \backslash D^\circ$, the following are equivalent:*

(i) *the smallest exposed face of C° containing y_0 has no nontrivial intersection with D°, that is, $y_0'' \cap D^\circ = \{0\}$,*

(ii) $\operatorname{int} y_0' \subset \operatorname{int} D$.

Proof. For the convex subset y_0' of D, there are two possibilities; either $\operatorname{int} y_0' \subset \operatorname{int} D$ or $y_0' \subset \partial D$. We will show that $y_0' \subset \partial D$ if and only if $y_0'' \cap D^\circ \neq \{0\}$. Suppose that $y_0' \subset \partial D$. This happens if and only if the face y_0' is contained in a maximal face y_1' of D for a point $y_1 \in D^\circ$. We see that $y_0' \subset y_1'$ if and only if $y_1'' \subset y_0''$ if and only if $y_1 \in y_0''$. This implies $y_1 \in y_0'' \cap D^\circ$. Conversely, if $y_1 \in y_0'' \cap D^\circ$ is nonzero then we have $y_0' \subset y_1' \subset \partial D$. ▫

The location of the nontrivial face $y_0' = H_{y_0} \cap C$ with $y_0 \in \partial C^\circ \backslash D^\circ$ depends on the exposed face y_0''. Proposition 2.4.4 tells us that y_0' is located inside of D if and only if y_0'' does not touch D° except zero. Whenever this is the case, the set W_{y_0} is maximal. See Figure 2.6.

Now, we look for positive maps ϕ satisfying $\phi'' \cap \mathbb{CP} = \{0\}$. Since a completely positive map is the sum of Ad_s's, we see that $\phi'' \cap \mathbb{CP} \neq \{0\}$ holds if and only if there exists $s \in M_{m\times n}$ such that $\mathrm{Ad}_s \in \phi''$, which is equivalent to the following:

$$\mathrm{SR}\,|\zeta\rangle = 1,\ \langle |\zeta\rangle\langle\zeta|, \phi\rangle = 0 \implies \langle |\zeta\rangle\langle\zeta|, \mathrm{Ad}_s\rangle = 0. \tag{2.22}$$

Motivated by this, we define the set

$$P[\phi] := \{|\zeta\rangle = |\xi\rangle|\eta\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n : \langle |\zeta\rangle\langle\zeta|, \phi\rangle = 0\},$$

for a positive map ϕ. Then, we see that ϕ'' has no nonzero completely positive map if and only if $P[\phi]$ spans the whole space, by the relation $\langle |\zeta\rangle\langle\zeta|, \mathrm{Ad}_s\rangle = |\langle\zeta|\bar{\tilde{s}}\rangle|^2$

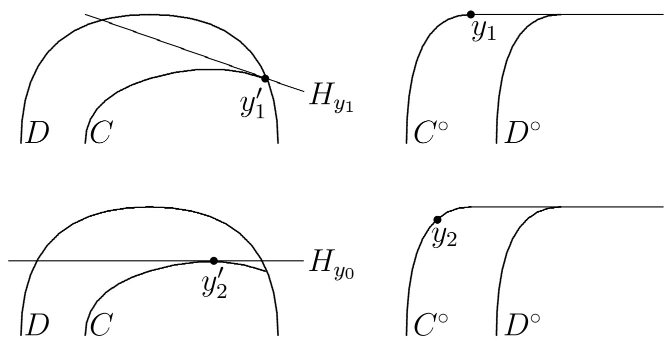

Figure 2.6: y_1'' touches D°, but y_2'' does not.

as in (2.1). A positive map ϕ satisfying this property is said to have the *spanning property*. We note that

$$|\xi\rangle|\eta\rangle \in P[\phi] \iff \langle \phi, \mathrm{Ad}_{|\bar\xi\rangle\langle\eta|}\rangle = 0 \iff \langle\bar\eta|\phi(|\xi\rangle\langle\xi|)|\bar\eta\rangle = 0 \tag{2.23}$$

by (2.11). It is clear that a positive map with the spanning property is optimal. We have seen in Theorem 2.2.1 that the Choi map ϕ_{ch} does not have the spanning property. By Proposition 2.4.4, we see that a positive linear map $\phi : M_m \to M_n$ has the spanning property if and only if $\mathrm{int}\,\phi' \subset \mathrm{int}\,(M_n \otimes M_n)^+$.

For a product vector $|\zeta\rangle = |\xi\rangle|\eta\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$, we recall the identity

$$|\zeta\rangle\langle\zeta|^\Gamma = |\bar\xi\rangle|\eta\rangle\langle\bar\xi|\langle\eta|.$$

If we replace Ad_s by $\mathrm{Ad}_s \circ \mathrm{T}$ in (2.22), then we have

$$\langle|\zeta\rangle\langle\zeta|, \mathrm{Ad}_s \circ \mathrm{T}\rangle = \langle|\zeta\rangle\langle\zeta|^\Gamma, \mathrm{Ad}_s\rangle.$$

Therefore, we see that the following are equivalent;

- ϕ'' has no nonzero completely copositive maps,
- the partial conjugates of $P[\phi]$ spans the whole space,

A positive map with such properties is called to have the *co-spanning property*. It is clear that ϕ has the spanning property if and only if $\phi \circ \mathrm{T}$ has the co-spanning property. Theorem 2.2.2 tells us that the Choi map ϕ_{ch} has the co-spanning property.

We say that a positive map ϕ is *co-optimal* when the smallest face of $\mathbb{P}_1$ containing ϕ has no nonzero completely copositive map, or equivalently $\phi \circ \mathrm{T}$ is optimal. We also say that a positive map is *bi-optimal* if it is both optimal and co-optimal. We note that ϕ is bi-optimal if and only if the smallest face of $\mathbb{P}_1$ containing ϕ has no nonzero decomposable map, and so we have the following by Theorem 2.4.3.

Proposition 2.4.5 *For a positive map $\phi : M_m \to M_n$, the following are equivalent:*

(i) *ϕ is bi-optimal,*

(ii) *the smallest face of $\mathbb{P}_1$ containing ϕ has no decomposable map,*

(iii) *ϕ detects a maximal set of PPT entanglement.*

Therefore, an bi-optimal positive map should be indecomposable automatically. On the other hand, a positive map ϕ has the *bi-spanning property* when it has both the spanning property and the co-spanning property. We apply Proposition 2.4.4 to the pair $\mathcal{S}_1 \subset \mathcal{PPT}$ to get the following:

Theorem 2.4.6 *For a positive linear map ϕ, the following are equivalent:*

(i) *ϕ has the bi-spanning property,*

(ii) *ϕ'' has no nonzero decomposable maps,*

(iii) $\operatorname{int}\phi' \subset \operatorname{int}\mathcal{PPT}$.

The property (iii) of Theorem 2.4.6 is useful to find PPT entanglement. If we take a finite family $\mathcal{I}$ of product vectors in $P[\phi]$ so that both they and their partial conjugates span the whole space, then the average ϱ_1 of $\{|\zeta_i\rangle\langle\zeta_i| : i \in \mathcal{I}\}$ lies on the boundary of $\mathcal{S}_1$, but in the interior of $\mathcal{PPT}$. If we take any ϱ_0 in the interior of $\mathcal{S}_1$ then the line segment $\varrho_\lambda = (1-\lambda)\varrho_0 + \lambda\varrho_1$ extends to PPT entanglement.

A positive map with the the bi-spanning property must be indecomposable by the condition (ii) of Theorem 2.4.6. An indecomposable positive map which generates an exposed ray has automatically the bi-spanning property. The following shows implication relations among various notions we have discussed so far.

$$
\begin{array}{ccccc}
\text{exposed indecomposable} & \Longrightarrow & \text{bi-spanning} & \Longrightarrow & \text{spanning} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{extreme indecomposable} & \Longrightarrow & \text{bi-optimal} & \Longrightarrow & \text{optimal} \\
 & & \Downarrow & & \\
 & & \text{indecomposable} & &
\end{array}
$$

References: [123], [82], [45], [77]

2.5 Positive maps of Choi type

We exhibit in this section variants of the Choi map between 3×3 matrices. Among them, we find positive maps with the bi-spanning property, and indecomposable positive maps with the spanning property without bi-optimality. Motivated by these examples, we briefly discuss the length of a separable state ϱ, which is the smallest number of pure products states whose sum gives rise to ϱ.

2.5.1 Choi type positive maps between 3×3 matrices

For a given triplet (a, b, c) of nonnegative real numbers and a 3×3 matrix $X = [x_{ij}]$, we define

$$\phi_{[a,b,c]}(X) = \begin{pmatrix} ax_{11} + bx_{22} + cx_{33} & -x_{12} & -x_{13} \\ -x_{21} & cx_{11} + ax_{22} + bx_{33} & -x_{23} \\ -x_{31} & -x_{32} & bx_{11} + cx_{22} + ax_{33} \end{pmatrix}.$$

Note that $\phi_{[1,0,1]}$ is nothing but the Choi map $\phi_{\rm ch}$ defined in (1.54), and the map $\tau_{3,k}$ in (1.52) is given by $\phi_{[k-1,k,k]}$. Especially, we have seen in Section 1.6 that $\phi_{[1,2,2]}$ is a 2-positive map which is not completely positive, and $\phi_{[0,1,1]}$ is a completely copositive map which is not 2-positive. The Choi matrix of the map $\phi_{[a,b,c]}$ is given by

$$\varrho_{[a,b,c]} := \begin{pmatrix} a & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & -1 \\ \cdot & c & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot \\ -1 & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & a \end{pmatrix}. \tag{2.24}$$

Note that $\phi_{[a,b,c]}$ is completely positive if and only if $\varrho_{[a,b,c]}$ is positive if and only if the following

$$a \geqslant 2$$

holds. For example, the map

$$\phi_{[2,0,0]} = \mathrm{Ad}_{e_{11}-e_{22}} + \mathrm{Ad}_{e_{22}-e_{33}} + \mathrm{Ad}_{e_{33}-e_{11}}$$

has been discussed in (2.14), where we use $\{e_{ij}\}$ for the standard matrix units. On the other hand, $\phi_{[a,b,c]}$ is completely copositive if and only if

$$\varrho^{\Gamma}_{[a,b,c]} = \begin{pmatrix} a & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & c & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & -1 & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot & -1 & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & c & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & b & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a \end{pmatrix}$$

is positive if and only if

$$bc \geqslant 1.$$

The completely copositive map $\phi_{[0,1,1]}$ may be written by

$$\phi_{[0,1,1]} = \mathrm{T} \circ (\mathrm{Ad}_{e_{12}-e_{21}} + \mathrm{Ad}_{e_{23}-e_{32}} + \mathrm{Ad}_{e_{31}-e_{13}}).$$

We note that the map $\phi_{[a,b,c]}$ may be expressed by

$$\phi_{[a,b,c]} = \psi_{[a,b,c]} - \mathrm{id}_3$$

with the map $\psi_{[a,b,c]}$, which sends $x = [x_{ij}]$ to the diagonal matrix with the diagonal entries

$$\begin{aligned} &(a+1)x_{11} + bx_{22} + cx_{33}, \\ &cx_{11} + (a+1)x_{22} + bx_{33}, \\ &bx_{11} + cx_{22} + (a+1)x_{33}. \end{aligned}$$

We apply Proposition 1.6.5, to see that $\phi_{[a,b,c]}$ is positive if and only if

$$\psi_{[a,b,c]}(|\xi_0\rangle\langle\xi_0|) \geqslant |\xi_0\rangle\langle\xi_0|$$

for every unit vector $|\xi_0\rangle$ if and only if

$$\langle\xi_0|\psi_{[a,b,c]}(|\xi_0\rangle\langle\xi_0|)^{-1}|\xi_0\rangle \leqslant 1$$

for every unit vector $|\xi_0\rangle$ if and only if the inequality

$$\frac{\alpha}{(a+1)\alpha + b\beta + c\gamma} + \frac{\beta}{(a+1)\beta + b\gamma + c\alpha} + \frac{\gamma}{(a+1)\gamma + b\alpha + c\beta} \leqslant 1 \qquad (2.25)$$

holds for all positive real numbers α, β and γ.

We take $\alpha = \beta = \gamma = 1$ to get $a + b + c \geqslant 2$. We also take $\beta = \alpha^{-1}$ and $\gamma \to 0$, then we have

$$ca\alpha^4 + (a^2 - 1 + bc)\alpha^2 + ab \geqslant 0$$

for each positive α. If $a \geqslant 1$ then this is true. If $0 \leqslant a \leqslant 1$ then this implies

$$a^2 - 1 + bc \geqslant 0 \quad \text{or} \quad (a^2 - 1 + bc)^2 - 4bca^2 \leqslant 0,$$

which implies $bc \geqslant (1-a)^2$. Therefore, we get necessary conditions

$$a + b + c \geqslant 2, \qquad 0 \leqslant a \leqslant 1 \to bc \geqslant (1-a)^2, \tag{2.26}$$

where $p \to q$ means that p implies q. Proving the following lemma, we may conclude that the map $\phi_{[a,b,c]}$ is positive if and only if the conditions in (2.26) are satisfied.

Lemma 2.5.1 *If nonnegative numbers a, b and c satisfy the conditions in* (2.26) *then inequality* (2.25) *holds for all positive α, β, γ.*

Proof. Suppose that a, b and c satisfy (2.26), and α, β and γ are positive numbers. Put

$$x = \frac{b\beta + c\gamma}{\alpha}, \qquad y = \frac{b\gamma + c\alpha}{\beta}, \qquad z = \frac{b\alpha + c\beta}{\gamma},$$

and consider the system of linear equations

$$\begin{aligned} x\alpha - b\beta - c\gamma &= 0, \\ -c\alpha + y\beta - b\gamma &= 0, \\ -b\alpha - c\beta + z\gamma &= 0, \end{aligned}$$

with unknowns α, β and γ. This system of equations has already a nontrivial solution, and so we have

$$xyz - bc(x + y + z) - (b^3 + c^3) = 0. \tag{2.27}$$

On the other hand, the left-side of (2.25) becomes

$$\frac{1}{a+1+x} + \frac{1}{a+1+y} + \frac{1}{a+1+z},$$

and so, it suffices to show the inequality

$$F(x,y,z) = xyz + a(xy + yz + zx) + (a^2 - 1)(x + y + z) + (a+1)^2(a-2) \geqslant 0$$

holds under the condition (2.27). To see this, we slice the surface (2.27) by the plane

$$x + y + z = d. \tag{2.28}$$

We first note that the surface (2.27) and the plane (2.28) has nonempty intersection if and only if $d \geqslant 3(b+c)$. When $d = 3(b+c)$, the intersection is the one point $(b+c, b+c, b+c)$. When $d > 3(b+c)$, the intersection is a compact curve.

One can show that if (x, y, z) is a critical point of F under the constraints (2.27) and (2.28) then

$$x = y, \quad y = z \quad \text{or} \quad z = x. \tag{2.29}$$

Therefore, it suffices to find minimum of F on the curve given by (2.27) and (2.29). In case of $x = y$, the curve is given by $(x^2 - bc)z = 2bcx + b^3 + c^3$ from which we have $x > \sqrt{bc}$, and F becomes a function of one variable x. By a direct computation, we have

$$\begin{aligned} F'(x) = &\frac{1}{(x^2 - bc)^2} \left[x^2 + (b+c)x + (b^2 - bc + c^2)\right] \\ &\times [x - (b+c)][ax^2 + (a^2 - 1 + bc)x + abc]. \end{aligned}$$

The first two factors are nonnegative, and the last factor is also nonnegative by the second conditions in (2.26). Now, we conclude that F has the minimum at $x = b+c$, which implies $y = z = b + c$ by (2.27) and $x = y$. Therefore, we have

$$\begin{aligned} F &\geqslant 3a(b+c)^3 + 3(a^2 - 1 + bc)(b+c) + ((a+1)^2(a-1) + b^3 + c^3) \\ &= (a+b+c-2)(a+b+c+1)^2 \geqslant 0, \end{aligned}$$

by the first condition of (2.26). The cases of $y = z$ and $z = x$ can be done in the same way. □

In order to deal with decomposability of the map $\phi_{[a,b,c]}$, we consider the PPT state ϱ_p defined in (1.36) We take bilinear pairing with $\varrho_{[a,b,c]}$ to get

$$0 \leqslant \langle \varrho_p, \phi_{[a,b,c]} \rangle = 3[cp^2 + (a-2)p + b]$$

for every $p > 0$. Therefore, we get the following necessary condition

$$0 \leqslant a \leqslant 2 \to bc \geqslant \left(\frac{2-a}{2}\right)^2. \tag{2.30}$$

Compare with the condition (2.26). Now, we show that the condition (2.30) implies that the map $\phi_{[a,b,c]}$ is decomposable. If $a \geqslant 2$ or $bc \geqslant 1$ then there is nothing to prove since $\phi_{[a,b,c]}$ is completely positive or completely copositive in each case. In case of $0 \leqslant a < 2$ and $bc < 1$, we have

$$\phi_{[a,b,c]} = (1 - \sqrt{bc})\, \phi_{[\alpha,0,0]} + \sqrt{bc}\, \phi_{[0,\sqrt{b/c},\sqrt{c/b}]},$$

with $\alpha = \frac{a}{1-\sqrt{bc}}$. We note that $\alpha \geqslant 2$ by (2.30), and so $\phi_{[\alpha,0,0]}$ is completely positive. Since $\phi_{[0,\sqrt{b/c},\sqrt{c/b}]}$ is completely copositive, we have seen that $\phi_{[a,b,c]}$ is decomposable. We summarize as follows:

Theorem 2.5.2 *Let a, b and c be nonnegative real numbers. Then the linear map $\phi_{[a,b,c]}$ is*

(i) *positive if and only if* $a + b + c \geqslant 2$ *and* $0 \leqslant a \leqslant 1 \to bc \geqslant (1-a)^2$*,*

(ii) *completely positive if and only if* $a \geqslant 2$*,*

(iii) *completely copositive if and only if* $bc \geqslant 1$*,*

(iv) *decomposable if and only if* $0 \leqslant a \leqslant 2 \to bc \geqslant \left(\frac{2-a}{2}\right)^2$.

For further examples of Choi type positive linear maps in various directions, see Section 7 of the survey paper [30] and references there. See also [71, 105] for recent development.

References: [18], [30], [71], [105]

2.5.2 Spanning properties of Choi type positive maps

In order to figure out the region given by (2.26), we look at the curve given by

$$0 \leqslant a < 1, \qquad a + b + c = 2, \qquad bc = (1-a)^2.$$

We parameterize by $t = \frac{b}{1-a}$. Then $0 < t < \infty$, and we have

$$at + b = t, \qquad (1-t)a + c = 2 - t, \qquad a + ct = 1, \tag{2.31}$$

from which we have

$$\gamma(t) = (a(t), b(t), c(t)) = \left(\frac{(1-t)^2}{1-t+t^2}, \frac{t^2}{1-t+t^2}, \frac{1}{1-t+t^2} \right).$$

One may easily check that γ makes a part of the circle centered at $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$ with the radius $\sqrt{\frac{2}{3}}$. See Figure 2.7. When $t = 0$, we have $\gamma(0) = (1, 0, 1)$ gives rise to the Choi map ϕ_{ch}, and we also get the completely copositive map $\phi_{[0,1,1]}$ at $t = 1$. We write

$$\phi_t := \phi_{[a(t),b(t),c(t)]}, \qquad 0 < t < \infty.$$

Now, we use (2.23) to look for product vectors in $P[\phi_t]$. We write $|\xi\rangle = (x, y, z)^{\mathrm{T}} \in \mathbb{C}^3$. When $|x| = |y| = |z| = 1$, we see that

$$\phi_t(|\xi\rangle\langle\xi|) = \begin{pmatrix} 2 & -x\bar{y} & -x\bar{z} \\ -y\bar{x} & 2 & -y\bar{z} \\ -z\bar{x} & -z\bar{y} & 2 \end{pmatrix}$$

has the kernel vector $|\bar{\eta}\rangle = (x, y, z)^{\mathrm{T}} \in \mathbb{C}^3$. Therefore, we have

$$|\xi\rangle \otimes |\eta\rangle = (\alpha, \beta, \gamma)^{\mathrm{T}} \otimes (\bar{\alpha}, \bar{\beta}, \bar{\gamma})^{\mathrm{T}} \in P[\phi_t], \tag{2.32}$$

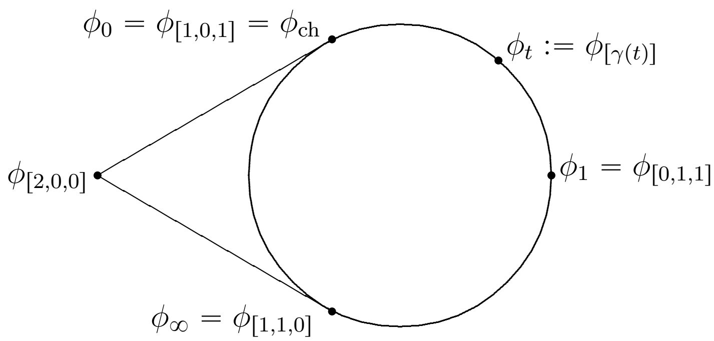

Figure 2.7: Two surfaces $a+b+c=2$ and $bc=(1-a)^2$ make a circle.

whenever $|\alpha|=|\beta|=|\gamma|=1$. When one of x,y,z, say $z=0$, the determinant $\det\phi_t(|\xi\rangle\langle\xi|)$ is given by

$$\begin{aligned}&(b|x|^2+c|y|^2)(ac|x|^4+|x|^2|y|^2a^2+|x|^2|y|^2bc+ab|y|^4-|y|^2|x|^2)\\&=\frac{(t-1)^2}{(t^2-t+1)^3}(t^2|x|^2+|y|^2)(-|y|^2t+|x|^2)^2,\end{aligned}$$

which becomes zero when $|x|=\sqrt{t}$ and $|y|=1$. In this case,

$$\phi_t(|\xi\rangle\langle\xi|)=\begin{pmatrix}at+b & -x\bar{y} & 0\\ -y\bar{x} & ct+a & 0\\ 0 & 0 & bt+c\end{pmatrix}$$

has the kernel vector $|\bar{\eta}\rangle=(x,ty,0)^{\rm T}\in\mathbb{C}^3$. Therefore, we have

$$|\xi\rangle\otimes|\eta\rangle=(\sqrt{t}\alpha,\beta,0)^{\rm T}\otimes(\sqrt{t}\bar{\alpha},t\bar{\beta},0)^{\rm T}\in P[\phi_t]. \tag{2.33}$$

By the same way, we also have

$$\begin{aligned}|\xi\rangle\otimes|\eta\rangle&=(0,\sqrt{t}\alpha,\beta)^{\rm T}\otimes(0,\sqrt{t}\bar{\alpha},t\bar{\beta})^{\rm T}\in P[\phi_t],\\|\xi\rangle\otimes|\eta\rangle&=(\beta,0,\sqrt{t}\alpha,)^{\rm T}\otimes(t\bar{\beta},0,\sqrt{t}\bar{\alpha})^{\rm T}\in P[\phi_t],\end{aligned} \tag{2.34}$$

with $|\alpha|=|\beta|=1$.

Among them, we take ten product vectors with real coefficients as follows: We first take

$$\begin{aligned}|\xi_1\rangle\otimes|\eta_1\rangle&=(1,1,1)^{\rm T}\otimes(1,1,1)^{\rm T},\\|\xi_2\rangle\otimes|\eta_2\rangle&=(1,1,-1)^{\rm T}\otimes(1,1,-1)^{\rm T},\\|\xi_3\rangle\otimes|\eta_3\rangle&=(1,-1,1)^{\rm T}\otimes(1,-1,1)^{\rm T},\\|\xi_4\rangle\otimes|\eta_4\rangle&=(-1,1,1)^{\rm T}\otimes(-1,1,1)^{\rm T},\end{aligned} \tag{2.35}$$

among product vectors in (2.32). They span 4-dimensional space whose orthogonal complement is spanned by

$$|12\rangle - |21\rangle, \quad |23\rangle - |32\rangle, \quad |31\rangle - |13\rangle, \quad |11\rangle - |22\rangle, \quad |22\rangle - |33\rangle.$$

Next, we also take

$$\begin{aligned} |\xi_5\rangle \otimes |\eta_5\rangle &= (\sqrt{t}, 1, 0)^{\mathrm{T}} \otimes (\sqrt{t}, t, 0)^{\mathrm{T}}, \\ |\xi_6\rangle \otimes |\eta_6\rangle &= (\sqrt{t}, -1, 0)^{\mathrm{T}} \otimes (\sqrt{t}, -t, 0)^{\mathrm{T}}, \end{aligned} \tag{2.36}$$

among (2.33), and

$$\begin{aligned} |\xi_7\rangle \otimes |\eta_7\rangle &= (0, \sqrt{t}, 1)^{\mathrm{T}} \otimes (0, \sqrt{t}, t)^{\mathrm{T}}, \\ |\xi_8\rangle \otimes |\eta_8\rangle &= (0, \sqrt{t}, -1)^{\mathrm{T}} \otimes (0, \sqrt{t}, -t)^{\mathrm{T}}, \\ |\xi_9\rangle \otimes |\eta_9\rangle &= (1, 0, \sqrt{t})^{\mathrm{T}} \otimes (t, 0, \sqrt{t})^{\mathrm{T}}, \\ |\xi_{10}\rangle \otimes |\eta_{10}\rangle &= (-1, 0, \sqrt{t})^{\mathrm{T}} \otimes (-t, 0, \sqrt{t})^{\mathrm{T}}, \end{aligned} \tag{2.37}$$

among (2.34). These six vectors span a 6-dimensional space whose orthogonal complement is spanned by

$$|12\rangle - t|21\rangle, \quad |23\rangle - t|32\rangle, \quad |31\rangle - t|13\rangle. \tag{2.38}$$

It is easily seen that the above ten product vectors span the whole space $\mathbb{C}^3 \otimes \mathbb{C}^3$ unless $t \neq 1$. Therefore, we have the following:

Theorem 2.5.3 *The positive map $\phi_{[a,b,c]}$ has the bi-spanning property, whenever (a, b, c) satisfies*

$$0 < a < 1, \qquad a + b + c = 2, \qquad bc = (1-a)^2.$$

In fact, it is known [47] that the maps in Theorem 2.5.3 generates an exposed ray of the convex cone $\mathbb{P}_1$.

Now, we turn our attention to the surface $bc = (1-a)^2$ with $0 \leqslant a \leqslant 1$ in the condition (2.26) of positivity of the map $\phi_{[a,b,c]}$. To do this, we fix $t > 0$ with $t \neq 1$, and consider the line segment

$$L_t := \left\{ L_t(s) = \left(1 - s, st, \frac{s}{t}\right) : \frac{t}{1 - t + t^2} \leqslant s \leqslant 1 \right\},$$

which is a part of the line segment from $(1, 0, 0)$ to $\delta(t) := (0, t, \frac{1}{t})$. Precisely, it is the line segment between two points between $\gamma(t)$ and $\delta(t)$. It is easily seen that this line segment is contained in the surface $bc = (1-a)^2$ in (2.26). See Figure 2.8. We note that $L_t(1) = \delta(t) = (0, t, \frac{1}{t})$ gives rise the the completely copositive map $\phi_{[0,t,\frac{1}{t}]}$, and so it is clear that maps on the line segment L_t do not satisfy the co-optimal property. Nevertheless, we will show that they satisfy the spanning property.

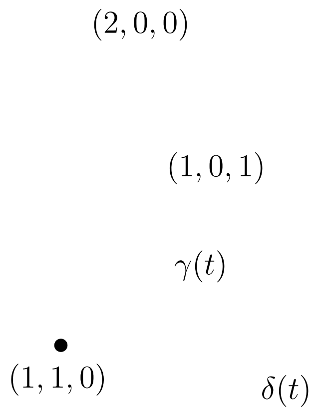

Figure 2.8: The surface $bc = (1-a)^2$ consists of line segments.

Theorem 2.5.4 *The positive map $\phi := \phi_{[a,b,c]}$ has the spanning property, whenever (a, b, c) satisfies*

$$0 \leqslant a < 1, \qquad a + b + c > 2, \qquad bc = (1-a)^2.$$

Proof. We define vectors in $\mathbb{C}^3$ as follows;

$$\begin{aligned}
|\xi^1_{\theta,\sigma}\rangle &= (0, e^{\mathrm{i}\theta} b^{1/4}, e^{\mathrm{i}\sigma} c^{1/4})^{\mathrm{T}}, \qquad & |\eta^1_{\theta,\sigma}\rangle &= (0, e^{-\mathrm{i}\theta}(bc)^{1/4}, e^{-\mathrm{i}\sigma} b^{1/2})^{\mathrm{T}},\\
|\xi^2_{\theta,\sigma}\rangle &= (e^{\mathrm{i}\sigma} c^{1/4}, 0, e^{\mathrm{i}\theta} b^{1/4})^{\mathrm{T}}, \qquad & |\eta^2_{\theta,\sigma}\rangle &= (e^{-\mathrm{i}\sigma} b^{1/2}, 0, e^{-\mathrm{i}\theta}(bc)^{1/4})^{\mathrm{T}},\\
|\xi^3_{\theta,\sigma}\rangle &= (e^{\mathrm{i}\theta} b^{1/4}, e^{\mathrm{i}\sigma} c^{1/4}, 0)^{\mathrm{T}}, \qquad & |\eta^3_{\theta,\sigma}\rangle &= (e^{-\mathrm{i}\theta}(bc)^{1/4}, e^{-\mathrm{i}\sigma} b^{1/2}, 0)^{\mathrm{T}}.
\end{aligned}$$

It is easily checked that

$$\langle \xi^k_{\theta,\sigma}|\langle \eta^k_{\theta,\sigma}|C_\phi|\xi^k_{\theta,\sigma}\rangle|\eta^k_{\theta,\sigma}\rangle = -2(1-a)bc^{1/2} + 2b^{3/2}c = 0$$

for $k = 1, 2, 3$. Therefore, the vectors $|\xi^k_{\theta,\sigma}\rangle|\eta^k_{\theta,\sigma}\rangle$ belong to $P[\phi_{[a,b,c]}]$ for $k = 1, 2, 3$. We take $\sigma_1 = 0$, $\sigma_2 = \pi/2$ and $\sigma_3 = \pi$, and consider the 9×9 matrix whose columns consist of nine vectors $|\xi^k_{0,\sigma_\ell}\rangle|\eta^k_{0,\sigma_\ell}\rangle$ for k, $\ell = 1, 2, 3$. Then the absolute value of the determinant is given by $128\, b^{\frac{9}{2}} c^{\frac{9}{4}}$ which is nonzero. □

Therefore, every triplet (a, b, c) in Theorem 2.5.4 gives rise to an indecomposable positive map $\phi := \phi_{[a,b,c]}$ with the following properties;

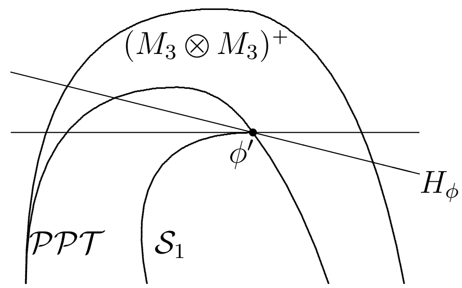

Figure 2.9: Entanglement $W_\phi[(M_m \otimes M_n)^+, \mathcal{S}_1]$ detected by ϕ is maximal, but PPT entanglement $W_\phi[\mathcal{PPT}, \mathcal{S}_1]$ detected by ϕ is not maximal.

- ϕ has the spanning properties, so it is optimal and detects a maximal set $W_\phi((M_3 \otimes M_3)^+, \mathcal{S}_1(M_3 \otimes M_3))$ of entanglement. The interior of the face ϕ' of $\mathcal{S}_1$ is sitting in the interior of $(M_3 \otimes M_3)^+$ by Proposition 2.4.4.
- ϕ does not have co-optimal properties, especially the smallest face of $\mathbb{P}_1$ containing ϕ has a nonzero decomposable map, so it does not detect a maximal set $W_\phi(\mathcal{PPT}[M_3 \otimes M_3], \mathcal{S}_1(M_3 \otimes M_3))$ of PPT entanglement by Theorem 2.4.3. The face ϕ' of $\mathcal{S}_1$ is sitting on the boundary of $\mathcal{PPT}$ by Proposition 2.4.4. See Figure 2.9.

References: [31], [44], [45], [46], [47], [48]

2.5.3 Lengths of separable states

We return to the ten product vectors $|\zeta_i\rangle := |\xi_i\rangle|\eta_i\rangle$ listed in (2.35), (2.36) and (2.37). We note that

$$\varrho_t := \sum_{i=1}^{10} |\zeta_i\rangle\langle\zeta_i| = \begin{pmatrix} \alpha & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \gamma & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \beta & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \beta & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \gamma & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \gamma & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \beta & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \alpha \end{pmatrix},$$

with

$$\alpha = \frac{2t^2+2}{t^2+2}, \qquad \beta = \frac{t+2}{t^2+2}, \qquad \gamma = \frac{t^3+2}{t^2+2},$$

up to scalar multiplications. So far, we have seen that ϱ_t is a separable state on the boundary of $\mathcal{S}_1$. Furthermore, ϱ_t is in the interior of $\mathcal{PPT}$ unless $t = 1$. In fact, one may see both ϱ_t and ϱ_t^Γ have full ranks when $t \neq 1$.

We have found product vectors $|\xi\rangle|\eta\rangle \in P[\phi_t]$ in (2.32), (2.33) and (2.34). It is known [46] that they are all product vectors in $P[\phi_t]$, and there is no more product vectors in $P[\phi_t]$. Now, we consider the map

$$\psi_t := \phi_t + \phi_t \circ \mathrm{T}, \qquad t > 0,\ t \neq 1.$$

Then $|\xi\rangle|\eta\rangle \in P[\psi_t]$ if and only if $|\xi\rangle|\eta\rangle \in P[\phi_t]$ and $|\bar{\xi}\rangle|\eta\rangle \in P[\phi_t]$ if and only if $|\xi\rangle|\eta\rangle$ is one of ten product vectors $|\zeta_1\rangle, \dots, |\zeta_{10}\rangle$ listed in (2.35), (2.36) and (2.37), with $|\zeta_i\rangle = |\xi_i\rangle \otimes |\eta_i\rangle$. Therefore, the convex cone generated by ten product states $|\zeta_i\rangle\langle\zeta_i|$ with $i = 1, \dots, 10$ is a face of $\mathcal{S}_1$, which is the dual face

$$\psi_t' = \phi_t' \cap (\phi_t \circ \mathrm{T})' \tag{2.39}$$

of the positive map ψ_t. Because these ten product vectors span the whole space $\mathbb{C}^3 \otimes \mathbb{C}^3$, we see that the map ψ_t has the bi-spanning property for $t \neq 1$. Note that ψ_t does not generate an extreme ray of $\mathbb{P}_1$.

Even though ten vectors $|\zeta_1\rangle, \dots, |\zeta_{10}\rangle$ are linearly dependent in $\mathbb{C}^3 \otimes \mathbb{C}^3$, the corresponding ten states are linearly independent in $M_3 \otimes M_3$. To see this, we suppose that $\sum_{i=1}^{10} \alpha_i |\zeta_i\rangle\langle\zeta_i| = 0$. Take $|\omega\rangle \in \mathbb{C}^3 \otimes \mathbb{C}^3$ among vectors in (2.38), then we have

$$0 = \sum_{i=1}^{10} \alpha_i |\zeta_i\rangle\langle\zeta_i|\omega\rangle = \sum_{i=1}^{4} \alpha_i \langle\zeta_i|\omega\rangle|\zeta_i\rangle.$$

Because $\{|\zeta_i\rangle : i = 1, 2, 3, 4\}$ is linearly independent, we have $\alpha_i\langle\zeta_i|\omega\rangle = 0$, and $\alpha_i = 0$ for $i = 1, 2, 3, 4$. Since $\{|\zeta_i\rangle : i = 5, 6, \dots, 10\}$ is also linearly independent, we have $\alpha_i = 0$ for $i = 5, 6, \dots, 10$.

Therefore, the separable state ϱ_t with $t \neq 1$ is decomposed into the sum of pure product states in a unique way. Especially, we need ten pure product states in order to express ϱ_t as the sum of them. The *length* of a separable state ϱ is defined as the smallest number of pure product states whose convex sum is ϱ. We see that the length of the separable state ϱ_t is 10, which is strictly greater than $9 = 3 \cdot 3$. It is known [101, 15] that the lengths of $2 \otimes 2$ and $2 \otimes 3$ separable states are less than or equal to 4 and 6, respectively.

Linear independence of $\{|\zeta_i\rangle\langle\zeta_i| : i = 1, 2, \dots, 10\}$ has a geometric interpretation. We restrict ourselves on the normalized states. Then the face ψ_t' in (2.39) is the 9-dimensional simplex with the ten extreme points which is normalized states $|\zeta_i\rangle\langle\zeta_i|$ with $i = 1, 2, \dots, 10$. All the states in this face has also a unique decomposition into the convex sum of pure product states.

References: [101], [1], [106], [2], [15], [46], [16], [47], [48]

2.6 Exposed positive maps by Woronowicz

In this section, we exhibit positive linear maps from M_2 to M_4 which generate exposed rays of the convex cone $\mathbb{P}_1$ of all positive maps. For this purpose, we also provide a sufficient condition for exposedness which is relatively easy to check.

2.6.1 A dimension condition for exposed positive maps

For a linear map $\phi : M_m \to M_n$, we define the linear map $\hat{\phi} : M_m \otimes \mathbb{C}^n \to \mathbb{C}^n$ by

$$\hat{\phi}(a \otimes \xi) = \phi(a)|\xi\rangle, \qquad a \in M_m,\ \xi \in \mathbb{C}^n,$$

and the subspace N_ϕ in $M_m \otimes \mathbb{C}^n$ by

$$N_\phi = \operatorname{span}\{a \otimes |\eta\rangle \in M_m^+ \otimes \mathbb{C}^n : \phi(a)|\eta\rangle = 0\}.$$

Then we have $N_\phi \subset \ker \hat{\phi}$, in general. By the relation (2.11), we have

$$|\xi\rangle\langle\xi| \otimes |\eta\rangle \in N_\phi \iff |\xi\rangle \otimes |\bar{\eta}\rangle \in P[\phi],$$

for $|\xi\rangle \in \mathbb{C}^m$ and $|\eta\rangle \in \mathbb{C}^n$.

Because $\mathcal{S}_1$ is generated by $M_m^+ \otimes M_n^+$, we see that $\psi \in \phi''$ is equivalent to

$$a \in M_m^+,\ |\eta\rangle \in \mathbb{C}^n,\ \langle a \otimes |\eta\rangle\langle\eta|, \phi\rangle = 0 \implies \langle a \otimes |\eta\rangle\langle\eta|, \psi\rangle = 0, \tag{2.40}$$

with the duality between $\mathcal{S}_1$ and $\mathbb{P}_1$. By the identity

$$\langle\eta|\phi(a)|\eta\rangle = \langle|\bar{\eta}\rangle\langle\bar{\eta}|, \phi(a)\rangle = \langle a \otimes |\bar{\eta}\rangle\langle\bar{\eta}|, \phi\rangle,$$

we also see that $a \otimes |\bar{\eta}\rangle\langle\bar{\eta}| \in \phi'$ if and only if $a \otimes |\eta\rangle \in N_\phi$ for $a \in M_m^+$ and $|\eta\rangle \in \mathbb{C}^n$. Therefore, we see that

$$\psi \in \phi'' \iff N_\phi \subset N_\psi.$$

If ϕ satisfies the condition

$$\ker \hat{\phi} = N_\phi, \tag{2.41}$$

then we have $\ker \hat{\phi} = N_\phi \subset N_\psi \subset \ker \hat{\psi}$. Therefore, we conclude that there exists a linear map $X : \mathbb{C}^n \to \mathbb{C}^n$ such that $\hat{\psi} = X \circ \hat{\phi}$, or equivalently, there exists $X \in M_n$ such that $\psi(a)|\eta\rangle = X\phi(a)|\eta\rangle$ for each $a \in M_n$ and $|\eta\rangle \in \mathbb{C}^m$. This implies $\psi(a) = X\phi(a)$. Therefore, if a positive linear map $\phi : M_m \to M_n$ satisfies the condition (2.41) together with the following condition

(C) If $X \in M_n$ and $a \mapsto X\phi(a)$ is positive then X is a scalar matrix,

then we conclude that ϕ is exposed. Because $(X\phi(a))^* = X\phi(a^*)$ if and only if $X\phi(a) = \phi(a)X^*$, we see that the following two conditions

($\mathrm{C_h}$) If $X \in M_n$ and $X\phi(a) = \phi(a)X^*$ for every $a \in M_m$ then X is a scalar matrix,

($\mathrm{C'_h}$) If $X \in M_n$ and $a \mapsto X\phi(a)$ is Hermiticity preserving then X is a scalar matrix

are equivalent. It is clear that ($\mathrm{C_h}$) or equivalent condition ($\mathrm{C'_h}$) implies (C).

For a subset S of M_n, we define the *commutant* S' by

$$S' := \{a \in M_n : ax = xa \text{ for every } x \in S\}.$$

Proposition 1.2.1 tells us that the commutant of M_n itself is trivial, that is, consists of scalar matrices. It is clear that S' is a subalgebra of M_n, and it is a $*$-subalgebra if S is *$*$-invariant*, that is, $x \in S$ implies $x^* \in S$. It is also clear that $S \subset S''$ and $S' = S'''$. A positive linear map $\phi : M_m \to M_n$ is called *irreducible* if the range $\phi(M_m)$ of ϕ has the trivial commutant. Note that the range of a positive map $\phi : M_m \to M_n$ is a $*$-invariant subspace, and so its commutant is a $*$-subalgebra of M_n. Suppose that the range of an irreducible positive map ϕ contains the identity. If we take $a \in M_m$ with $\phi(a) = I_m$ then $X\phi(a) = \phi(a)X^*$ implies that $X = X^*$, and so the irreducibility implies the condition ($\mathrm{C_h}$). Therefore, we have the following:

Theorem 2.6.1 (Woronowicz [131]) *Suppose that $\phi : M_m \to M_n$ is an irreducible positive linear map satisfying the condition* (2.41). *If the range of ϕ contains the identity then ϕ generates an exposed ray of the convex cone $\mathbb{P}_1$ of all positive maps.*

If the range of ϕ contains the idnetity then we see that the map $\hat{\phi}$ is surjective. Therefore, we have $\dim\ker\hat{\phi} = n(m^2-1)$, and so the condition $\ker\hat{\phi} = N_\phi$ in (2.41) holds in Theorem 2.6.1 if and only if the following dimension condition

$$\dim N_\phi = n(m^2-1) \tag{2.42}$$

is satisfied. We call this the *Woronowicz dimension condition*.

It is easy to see that the condition ($\mathrm{C_h}$) of a positive map ϕ implies irreducibility. To see this, we suppose that ϕ satisfies ($\mathrm{C_h}$) and $X\phi(a) = \phi(a)X$ for every $a \in M_m$. Because the range of the positive map ϕ is $*$-invariant, we also have $X^*\phi(a) = \phi(a)X^*$ for every $a \in M_m$. If we write $Y = \frac{1}{2}(X + X^*)$ and $Z = \frac{1}{2\mathrm{i}}(X - X^*)$, then we have $Y\phi(a) = \phi(a)Y = \phi(a)Y^*$, which implies that Y is a scalar matrix by condition ($\mathrm{C_h}$). By the same argument, we also see that Z is a scalar matrix, and so we conclude that ϕ is irreducible. It should be noted that irreducibility does not imply the condition (C) in general. To see this, we define $\phi : M_2 \to M_2$ by

$$\phi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & a \\ a & a+d \end{pmatrix},$$

which is easily seen to be irreducible. But, we have

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & a \\ a & a+d \end{pmatrix} = \begin{pmatrix} a & a \\ a & a \end{pmatrix} = \begin{pmatrix} a & a \\ a & a+d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

and so we see that ϕ violates the condition ($\mathrm{C_h}$) as well as (C). It is straightforward to check that the Woronowicz map $\phi_{\rm wo}$ given by (1.56) satisfies the condition ($\mathrm{C_h}$), and so it is irreducible.

In the remainder of this section, we show that the identity map $\mathrm{id}_n : M_n \to M_n$ satisfies the dimension condition (2.42). This will give rise to an another proof of Theorem 2.1.8. We fix $i = 2, 3, \dots, n$ for a moment, and put

$$\begin{aligned} |\xi\rangle &= (1, \alpha_2, \dots, \alpha_i, \dots, \alpha_n)^{\rm T} \in \mathbb{C}^n, \\ |\eta_i\rangle &= (\bar\alpha_i, 0, \dots, 0, -1, 0, \dots, 0)^{\rm T} \in \mathbb{C}^n, \end{aligned}$$

with -1 at the i-th entry. Then we have $|\xi\rangle\langle\xi| \otimes |\eta_i\rangle \in N_{\mathrm{id}_n}$. We identity $|\xi\rangle\langle\xi| \otimes |\eta\rangle$ in $M_n \otimes \mathbb{C}^n$ and $|\xi\rangle \otimes |\bar\xi\rangle \otimes |\eta\rangle$ in $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ to count dimensions. For each $i = 2, \dots, n$, we define the space V_i by

$$V_i = \mathrm{span}\,\{|\xi\rangle \otimes |\bar\xi\rangle \otimes |\eta_i\rangle : \alpha_2, \alpha_3, \dots, \alpha_n \in \mathbb{C}\} \subset \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n.$$

Lemma 2.6.2 *The set $\{\alpha^k \bar\alpha^\ell : k, \ell = 0, 1, 2, \dots\}$ of monomials is linearly independent.*

Proof. Suppose that $P(\alpha)$ is a linear combination of finitely many monomials among $\alpha^k \bar\alpha^\ell$. Then $P(t\alpha)$ is a polynomial with respect to the variable t whose coefficients are homogeneous polynomials in the variables α and $\bar\alpha$. Therefore, it is enough to show that homogeneous polynomials with a fixed degree n are linearly independent. We multiply α^n to all of homogeneous polynomials of degree n, and use the identity theorem on the unit circle, to get the conclusion. □

Because monomials with variables 1, α_i and $\bar\alpha_i$, $i = 2, 3, \dots, n$, are linearly independent, we see that the dimension of V_i is the number of monomials which appear in the entries of $|\xi\rangle \otimes |\bar\xi\rangle \otimes |\eta_i\rangle$. Noting that $|\bar\xi\rangle$ and $|\eta_i\rangle$ have n and 2 variables and the variable $\bar\alpha_i$ appears two times, we see that $|\bar\xi\rangle \otimes |\eta_i\rangle$ has $2n-1$ monomials in the entries, and

$$\dim V_i = n(2n-1), \qquad i = 2, \dots, n.$$

The orthogonal complement $V_i^\perp$ is spanned by orthogonal vectors

$$\begin{aligned} &|j11\rangle + |jii\rangle, \qquad j = 1, 2, \dots, n, \\ &|xyz\rangle, \qquad x, y = 1, 2, \dots, n, \quad z = 2, \dots, i-1, i+1, \dots, n, \end{aligned} \tag{2.43}$$

whose cardinality is $n+n^2(n-2)=n^3-n(2n-1)$.

In order to count the dimension of $N_{\mathrm{id}_n}^{\perp}=V_2^{\perp}\cap V_3^{\perp}\cap\cdots\cap V_n^{\perp}$, we write down the generators of $V_i^{\perp}$ as follows

$$
\begin{aligned}
V_2^{\perp}: &\quad |111\rangle+|122\rangle,\ |211\rangle+|222\rangle,\dots,\ |n11\rangle+|n22\rangle, \quad |xy3\rangle,|xy4\rangle,\cdots,|xyn\rangle,\\
V_3^{\perp}: &\quad |111\rangle+|133\rangle,\ |211\rangle+|233\rangle,\dots,\ |n11\rangle+|n33\rangle, \quad |xy2\rangle,|xy4\rangle,\cdots,|xyn\rangle,\\
\cdots&\\
V_n^{\perp}: &\quad |111\rangle+|1nn\rangle,\ |211\rangle+|2nn\rangle,\dots,\ |n11\rangle+|nnn\rangle, \quad |xy2\rangle,|xy3\rangle,\cdots,|xy(n-1)\rangle.
\end{aligned}
$$

Then we see that $V_2^{\perp}\cap V_2^{\perp}\cap\cdots\cap V_n^{\perp}$ is spanned by following n orthogonal vectors

$$
\begin{aligned}
|\zeta_1\rangle :=&|111\rangle+|122\rangle+|133\rangle+\cdots+|1nn\rangle,\\
|\zeta_2\rangle :=&|211\rangle+|222\rangle+|233\rangle+\cdots+|2nn\rangle,\\
&\cdots\\
|\zeta_n\rangle :=&|n11\rangle+|n22\rangle+|n33\rangle+\cdots+|nnn\rangle.
\end{aligned}
$$

Therefore, we have $\dim N_{\mathrm{id}_n}=n^3-n$ which coincides with the dimension of $\ker\widehat{\mathrm{id}_n}$. By Theorem 2.6.1, we conclude that the identity map id_n is exposed.

References: [130], [131], [49], [81]

2.6.2 Exposed positive maps between 2×2 and 4×4 matrices

The Woronowicz map ϕ_{wo} in (1.56) can be parameterized as follows: We first take positive numbers a,b,c and d with $ab>1$, and we define positive numbers e,f,g,h and k by the relations

$$
(ab-1)\begin{pmatrix} e\\ f\end{pmatrix}=a(c+d)\begin{pmatrix} c\\ d\end{pmatrix},\quad g^2=acd,\quad h=be-c^2,\quad k=bf-d^2. \tag{2.44}
$$

We define $\phi: M_2\to M_4$ by

$$
\phi\begin{pmatrix} x & y\\ z & w\end{pmatrix}=\begin{pmatrix} hx-cd(y+z)+kw & -gx+gz & 0 & 0\\ -gx+gy & ax & z & 0\\ 0 & y & bw & -cz-dw\\ 0 & 0 & -cy-dw & ex+fw\end{pmatrix}, \tag{2.45}
$$

If $(a,b,c,d)=(2,2,2,1)$, then we get the Woronowicz map ϕ_{wo}.

For each complex number α, we consider the following positive rank one matrix

$$
|\xi_\alpha\rangle\langle\xi_\alpha|=\begin{pmatrix} 1 & \bar\alpha\\ \alpha & |\alpha|^2\end{pmatrix}
$$

onto the vector $|\xi_\alpha\rangle=(1,\alpha)^{\mathrm T}\in\mathbb{C}^2$, and consider the determinant $\Delta_i(\alpha)$ of right-below $i\times i$ submatrix of $\phi(|\xi_\alpha\rangle\langle\xi_\alpha|)$ for each $i=1,2,3,4$. By a direct calculation, we have

$$\begin{aligned}
\Delta_1(\alpha) &= e + f|\alpha|^2,\\
\Delta_2(\alpha) &= |\alpha|^2\left[h - cd(\alpha+\bar{\alpha}) + k|\alpha|^2\right],\\
\Delta_3(\alpha) &= acd|\alpha|^2|1-\alpha|^2,\\
\Delta_4(\alpha) &= 0.
\end{aligned}$$

By the relations between e, f, g, h and k in (2.44), we see that

$$\Delta_1(\alpha) > 0, \qquad \Delta_2(\alpha) > 0, \qquad \Delta_3(\alpha) > 0, \qquad \Delta_4(\alpha) = 0$$

for any complex numbers $\alpha \neq 0, 1$, and so, we see that $\phi(|\xi_\alpha\rangle\langle\xi_\alpha|)$ is positive for $\alpha \neq 0, 1$. We also have

$$\phi(|\xi_0\rangle\langle\xi_0|) = \begin{pmatrix} h & -g & 0 & 0 \\ -g & a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e \end{pmatrix}$$

and

$$\phi(|\xi_1\rangle\langle\xi_1|) = \begin{pmatrix} h - 2cd + k & 0 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 1 & b & -c-d \\ 0 & 0 & -c-d & e+f \end{pmatrix}.$$

We write $|\xi_\infty\rangle := (0,1)^{\mathrm{t}} \in \mathbb{C}^2$, then we have

$$\phi(|\xi_\infty\rangle\langle\xi_\infty|) = \begin{pmatrix} k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b & -d \\ 0 & 0 & -d & f \end{pmatrix}.$$

Since $\phi(|\xi_\alpha\rangle\langle\xi_\alpha|)$ is positive for every $\alpha \in \mathbb{C} \cup \{\infty\}$, we conclude that ϕ is a positive map from M_2 into M_4.

It is straightforward to see that ϕ satisfies the condition ($\mathrm{C_h}$). Take a matrix $\sigma = [\sigma_{ij}] \in M_4$, and suppose that

$$[\sigma_{ij}]\,\phi\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \phi\begin{pmatrix} x & y \\ z & w \end{pmatrix}[\bar{\sigma}_{ji}]$$

for every $x, y, z, w \in \mathbb{C}$. Compare the entries of both sides, one may check that σ is a scalar matrix.

We also see that $\phi(|\xi_\alpha\rangle\langle\xi_\alpha|) \in M_4$ has the one dimensional kernel space which is generated by

$$|\eta_\alpha\rangle := \begin{pmatrix} g\alpha(1-\alpha) \\ \alpha\left[h - cd(\alpha+\bar{\alpha}) + k|\alpha|^2\right] \\ -e - f|\alpha|^2 \\ -\bar{\alpha}(c + d\alpha) \end{pmatrix} \in \mathbb{C}^4 \tag{2.46}$$

for $\alpha \in \mathbb{C}$, and the kernel of $\phi(|\xi_\infty\rangle\langle\xi_\infty|)$ is spanned by

$$|\eta_\infty\rangle = (0,1,0,0)^{\mathrm{T}}.$$

Now, we proceed to determine the dimension of the space N_ϕ, or equivalently, the dimension of the space

$$\mathrm{span}\,\{|\zeta_\alpha\rangle := |\xi_\alpha\rangle \otimes |\bar{\xi}_\alpha\rangle \otimes |\eta_\alpha\rangle : \alpha \in \mathbb{C} \cup \{\infty\}\} \subset \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^4.$$

We first note that each entries of $|\zeta_\alpha\rangle$ are linear combinations of the following 12 monomials

$$1,\ \alpha,\ \bar{\alpha},\ \alpha^2,\ \alpha\bar{\alpha},\ \bar{\alpha}^2,\ \alpha^3,\ \alpha^2\bar{\alpha},\ \alpha\bar{\alpha}^2,\ \alpha^3\bar{\alpha},\ \alpha^2\bar{\alpha}^2,\ \alpha^3\bar{\alpha}^2. \tag{2.47}$$

In order to show that ϕ satisfies the condition (2.42), we consider the vector

$$(f_1(\alpha), f_2(\alpha), \dots, f_n(\alpha)) \in \mathbb{C}^n, \tag{2.48}$$

for complex functions $f_1, \dots, f_n$. In order to find the dimension of the span of the vectors in (2.48) through $\alpha \in \mathbb{C}$, it suffices to consider the rank of the 'coefficient matrix'. The coefficient matrix of $|\xi_\alpha\rangle|\bar{\xi}_\alpha\rangle$ is given by

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdots \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdots \end{pmatrix}$$

with respect to the monomials $1, \alpha, \bar{\alpha}, \alpha^2, \alpha\bar{\alpha}, \bar{\alpha}^2, \dots$, which is rank four. On the other hand, the coefficient matrix for $|\eta_\alpha\rangle$ is given by

$$\begin{pmatrix} \cdot & g & \cdot & -g & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & h & \cdot & -cd & -cd & \cdot & \cdot & k & \cdot & \cdots \\ -e & \cdot & \cdot & \cdot & -f & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & -c & \cdot & -d & \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix}$$

with respect to monomials $1, \alpha, \bar{\alpha}, \alpha^2, \alpha\bar{\alpha}, \bar{\alpha}^2, \alpha^3, \alpha^2\bar{\alpha}, \dots$. This is of rank 4, which means that the set $\{|\eta_\alpha\rangle : \alpha \in \mathbb{C}\}$ spans the whole space $\mathbb{C}^4$. Now, one may produce the coefficient matrix of $|\zeta_\alpha\rangle := |\xi_\alpha\rangle \otimes |\bar{\xi}_\alpha\rangle \otimes |\eta_\alpha\rangle$ with respect to the monomials in (2.47) which is 16×12 matrix, and check that this matrix has rank 12. This shows that the map ϕ satisfies the Woronowicz dimension condition (2.42), and we have the following:

Theorem 2.6.3 *The map* $\phi : M_2 \to M_4$ *defined by* (2.44) *and* (2.45) *is an indecomposable exposed positive map for every positive* a, b, c *and* d *with* $ab > 1$.

Proof. It remains to show that the map ϕ is not decomposable. If it were decomposable, then $\phi = \mathrm{Ad}_s$ or $\phi = \mathrm{Ad}_s \circ \mathrm{T}$. Then either C_ϕ or C_ϕ^Γ is of rank one, which is not the case. ▫

References: [49]

2.7 Exposed positive maps by Robertson

In this section, we exhibit the Robertson map between 4×4 matrices which is an indecomposable positive map generating an exposed ray. The construction use the notion of Jordan homomorphisms.

2.7.1 Robertson's positive maps between 4×4 matrices

In this section, we construct an example of indecomposable exposed positive map between 4×4 matrices. We begin with the map $\tau_{2,1}$

$$\tau_{2,1} = \mathrm{Tr}_2 - \mathrm{id}_2 : \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$$

between 2×2 matrices, as it was defined in (1.52). This is an anti-automorphism of order two, that is, it satisfies $\tau_{2,1}(xy) = \tau_{2,1}(y)\tau_{2,1}(x)$ for $x, y \in M_2$ and $\tau_{2,1} \circ \tau_{2,1} = \mathrm{id}$, as well as it is a completely copositive map.

We recall that quaternion numbers may be expressed as

$$\mathbb{H} = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in M_2(\mathbb{C}) : \alpha, \beta \in \mathbb{C} \right\}. \tag{2.49}$$

In other words, the quaternion $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H}$ corresponds to

$$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \longleftrightarrow \begin{pmatrix} a + \mathrm{i}b & c + \mathrm{i}d \\ -c + \mathrm{i}d & a - \mathrm{i}b \end{pmatrix} \in M_2(\mathbb{C}).$$

It is worthwhile to note the following correspondences

$$\mathbf{i} \leftrightarrow \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} = \mathrm{i}\sigma_z, \qquad \mathbf{j} \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathrm{i}\sigma_y, \qquad \mathbf{k} \leftrightarrow \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \mathrm{i}\sigma_x,$$

in terms of Pauli matrices in (1.16).

We define the linear map $\pi : M_4(\mathbb{C}) \to M_4(\mathbb{C})$ by

$$\pi = \frac{1}{2}(\mathrm{id}_4 + \mathrm{T}_2 \otimes \tau_{2,1}),$$

that is, we define

$$\pi \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x + \tau_{2,1}(x) & y + \tau_{2,1}(z) \\ z + \tau_{2,1}(y) & w + \tau_{2,1}(w) \end{pmatrix}, \qquad \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in M_4(\mathbb{C}).$$

Then π is a positive map since $\tau_{2,1}$ is completely copositive. With the expression (2.49), we have

$$\mathbb{H} = \{x \in M_2(\mathbb{C}) : \tau_{2,1}(x) = x^*\},$$

and one may check that the range of $M_4^{\rm h}(\mathbb{C})$ under π is

$$\begin{aligned} M_2^{\rm h}(\mathbb{H}) &= \left\{ \begin{pmatrix} x & y \\ y^* & w \end{pmatrix} : x, y \in \mathbb{H} \cap M_2^{\rm h}(\mathbb{C}),\ w \in \mathbb{H} \right\} \\ &= \left\{ \begin{pmatrix} a & 0 & \alpha & \beta \\ 0 & a & -\bar\beta & \bar\alpha \\ \bar\alpha & -\beta & b & 0 \\ \bar\beta & \alpha & 0 & b \end{pmatrix} : a, b \in \mathbb{R},\ \alpha, \beta \in \mathbb{C} \right\} \subset M_4^{\rm h}(\mathbb{C}), \end{aligned} \tag{2.50}$$

and $\pi|_{M_2^{\rm h}(\mathbb{H})}$ is the identity. In short, π is a positive projection from $M_4^{\rm h}(\mathbb{C})$ onto $M_2^{\rm h}(\mathbb{H})$, which is a six dimensional subspace of the real space $M_4^{\rm h}(\mathbb{C})$.

We note that $M_2^{\rm h}(\mathbb{H})$ is a *JC-subalgebra* of $M_4(\mathbb{C})$, that is, it is a real subspace of Hermitian matrices in $M_4^{\rm h}(\mathbb{C})$ which is closed under the binary operation

$$a \circ b = \frac{1}{2}(ab + ba).$$

By the relation

$$a \circ b = \frac{1}{2}[(a+b)^2 - a^2 - b^2],$$

we see that a real subspace of Hermitian matrices is a JC-algebra if and only if it is closed under the operation of square. Furthermore, a linear map ϕ between JC-algebras is a *Jordan homomorphism*, that is, $\phi(a \circ b) = \phi(a) \circ \phi(b)$ if and only if it preserves the operation of square. Because a Hermitian matrix is positive if and only if it is the square of a Hermitian matrix, we see that every Jordan homomorphism defined on the full matrix algebra is positive.

We define the Jordan automorphism $\theta : M_2^{\rm h}(\mathbb{H}) \to M_2^{\rm h}(\mathbb{H})$ by

$$\theta \begin{pmatrix} x & y \\ y^* & w \end{pmatrix} = \begin{pmatrix} w & y \\ y^* & x \end{pmatrix}$$

of order two. This is also positive, since the 4×4 matrix in (2.50) is positive if and only if $ab \geqslant |\alpha|^2 + |\beta|^2$. Finally, we define the map

$$\phi_{\rm rb} := \theta \circ \pi$$

on $M_4^{\rm h}(\mathbb{C})$, which is an extension of θ defined on $M_2^{\rm h}(\mathbb{H})$ to the whole Hermitian parts $M_4^{\rm h}(\mathbb{C})$. See Figure 2.10. We extend the map $\phi_{\rm rb}$ to the whole matrix algebra $M_4(\mathbb{C})$ by (1.11). Then we have

$$\begin{aligned} \phi_{\rm rb} \begin{pmatrix} x & y \\ z & w \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} w + \tau_{2,1}(w) & y + \tau_{2,1}(z) \\ z + \tau_{2,1}(y) & x + \tau_{2,1}(x) \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} {\rm Tr}_2(w) & y + \tau_{2,1}(z) \\ z + \tau_{2,1}(y) & {\rm Tr}_2(x) \end{pmatrix}, \end{aligned} \tag{2.51}$$

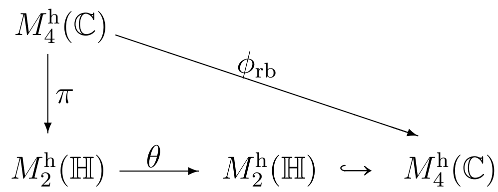

Figure 2.10: The Robertson map ϕ_{rb} is an extension of Jordan automorphism θ.

for x, y, z and w in $M_2(\mathbb{C})$. This map ϕ_{rb} is called the *Robertson map* which is a positive map between $M_4(\mathbb{C})$. It should be noted that the map

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \mapsto \begin{pmatrix} w & y \\ z & x \end{pmatrix}$$

is not a positive map on the whole matrix algebra $M_4(\mathbb{C})$, although it is positive on the JC-subalgebra $M_2^{\mathrm{h}}(\mathbb{H})$ of $M_4(\mathbb{C})$.

The Choi matrix of $2\phi_{\mathrm{rb}}$ is a 16×16 matrix, whose first four rows coming from $\phi_{\mathrm{rb}}(|1\rangle\langle j|)$ with $j = 1, 2, 3, 4$ are given by

$$\left[\begin{array}{cccc cccc cccc cccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right].$$

The next four rows are given by

$$\left[\begin{array}{cccc cccc cccc cccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right].$$

The other rows are given by

$$\left[\begin{array}{cccc cccc cccc cccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right]$$

and

$$\left[\begin{array}{cccc cccc cccc cccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right].$$

In order to show that ϕ_{rb} is not decomposable, we define

$$\sigma_1 = \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i}^* & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & \mathbf{j} \\ \mathbf{j}^* & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & \mathbf{k} \\ \mathbf{k}^* & 0 \end{pmatrix}, \quad \sigma_4 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix},$$

where $\mathbf{1}$ is the identity for quaternion. We also put

$$\sigma_5 = \sigma_1\sigma_2\sigma_3\sigma_4 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}.$$

Then we have

$$\phi_{\rm rb}(\sigma_i) = \sigma_i, \quad i = 1,2,3,4, \qquad \phi_{\rm rb}(\sigma_5) = -\sigma_5.$$

Now, we define $\Sigma \in M_4(M_4)$ by

$$\Sigma = \begin{pmatrix} 0 & 0 & 0 & \sigma_1 \\ \sigma_4 & 0 & 0 & 0 \\ 0 & \sigma_3 & 0 & 0 \\ 0 & 0 & \sigma_2 & 0 \end{pmatrix}.$$

Then we have

$$\Sigma^4 = \begin{pmatrix} \sigma_5 & 0 & 0 & 0 \\ 0 & -\sigma_5 & 0 & 0 \\ 0 & 0 & \sigma_5 & 0 \\ 0 & 0 & 0 & -\sigma_5 \end{pmatrix}, \qquad \Sigma^8 = I_{16},$$

and eigenvalues of Σ consist of eighth roots of unity. Define

$$\begin{aligned} \varrho_\pm &= \Sigma + \Sigma^* \pm \Sigma^4 + (1+\sqrt{2})I_{16} \\ &= \begin{pmatrix} \pm\sigma_5 & \sigma_4^* & 0 & \sigma_1 \\ \sigma_4 & \mp\sigma_5 & \sigma_3^* & 0 \\ 0 & \sigma_3 & \pm\sigma_5 & \sigma_2^* \\ \sigma_1^* & 0 & \sigma_2 & \mp\sigma_5 \end{pmatrix} + (1+\sqrt{2})I_{16}, \end{aligned}$$

then we see that

$$(\mathrm{id}_4 \otimes \phi_{\rm rb})(\varrho_+) = \varrho_-$$

The eigenvalues of $\varrho_\pm$ are given by

$$\{\omega + \bar{\omega} \pm \omega^4 + 1 + \sqrt{2} : \omega^8 = 1, \omega \in \mathbb{C}\}.$$

Therefore, we see that ϱ_+ is of PPT, because the partial transpose of ϱ_+ is nothing but its conjugate. We also note that $(\mathrm{id}_4 \otimes \phi_{\rm rb})(\varrho_+) = \varrho_-$ is not positive. By Corollary 1.8.6 (i), we conclude that $\phi_{\rm rb}$ is not decomposable.

Theorem 2.7.1 *The map $\phi_{\rm rb}$ defined in* (2.51) *is an indecomposable positive map.*

References: [95], [96], [97], [98], [118]

2.7.2 Exposedness of the Robertson map

In this section, we show that the Robertson map $\phi_{\rm rb}$ generates an exposed ray of he convex cone $\mathbb{P}_1[M_4, M_4]$. First of all, it is straightforward to check that the map $\phi_{\rm rb}$ is irreducible.

We write $V = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \in M_4$. Then we have

$$\mathrm{Ad}_V \circ \phi_{\rm rb} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathrm{Tr}_2(w) & -y - \tau_{2,1}(z) \\ -z - \tau_{2,1}(y) & \mathrm{Tr}_2(x) \end{pmatrix}.$$

We note that

$$\begin{aligned} &\begin{pmatrix} \mathrm{Tr}_2(w) & -y - \tau_{2,1}(z) \\ -z - \tau_{2,1}(y) & \mathrm{Tr}_2(x) \end{pmatrix} \\ &\quad = \begin{pmatrix} \mathrm{Tr}_2(x) + \mathrm{Tr}_2(w) & 0 \\ 0 & \mathrm{Tr}_2(x) + \mathrm{Tr}_2(w) \end{pmatrix} - \begin{pmatrix} x & y \\ z & w \end{pmatrix} - \begin{pmatrix} \tau_{2,1}(x) & \tau_{2,1}(z) \\ \tau_{2,1}(y) & \tau_{2,1}(w) \end{pmatrix} \end{aligned}$$

We also note that $\tau_{2,1}(x) = u^* x^{\rm T} u$ for $x \in M_2$, with $u = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in M_2$, and so, it follows that

$$\mathrm{Ad}_V \circ \phi_{\rm rb}(X) = \frac{1}{2} \left(\mathrm{Tr}_4(X) - X - U^* X^{\rm T} U \right)$$

with

$$U = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} = \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & -1 & \cdot \end{pmatrix} \in M_4. \tag{2.52}$$

We note that both $\phi_{\rm rb}$ and $\mathrm{Ad}_V \circ \phi_{\rm rb}$ are unital, and the map $\mathrm{Ad}_V \circ \phi_{\rm rb}$ is also irreducible. We proceed to show that $\mathrm{Ad}_V \circ \phi_{\rm rb}$ is also exposed, which implies that $\phi_{\rm rb}$ is exposed.

Theorem 2.7.2 *For the matrix U in* (2.52)*, the map*

$$\phi_U := \mathrm{Tr}_4 - \mathrm{id}_4 - \mathrm{Ad}_U \circ \mathrm{T} \tag{2.53}$$

is exposed.

Proof. We show that ϕ_U satisfies the dimension condition (2.42). We note that

$$\phi(|\xi\rangle\langle\xi|) = I_4 - |\xi\rangle\langle\xi| - U^* |\bar{\xi}\rangle\langle\bar{\xi}| U,$$

for a unit vector $|\xi\rangle \in \mathbb{C}^4$. Because $|\xi\rangle$ and $U^*|\bar{\xi}\rangle$ are mutually orthogonal, we see that $\phi(|\xi\rangle\langle\xi|)|\eta\rangle = 0$ if and only if $|\eta\rangle$ belongs to the span of $|\xi\rangle$ and $U^*|\bar{\xi}\rangle$. Therefore, we have to show that

$$N_\phi = \mathrm{span}\, \{ |\xi\rangle \otimes |\bar{\xi}\rangle \otimes |\xi\rangle,\ |\xi\rangle \otimes |\bar{\xi}\rangle \otimes U^* |\bar{\xi}\rangle \in \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4 : |\xi\rangle \in \mathbb{C}^4 \}$$

is of 60 dimensional subspace of $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$. To see this, we write $|\xi\rangle = (1, \alpha, \beta, \gamma)^{\mathrm{T}} \in \mathbb{C}^4$. Then the following 16 monomials

$$\begin{array}{cccc} 1, & \alpha, & \beta, & \gamma, \\ \bar{\alpha}, & \alpha\bar{\alpha}, & \beta\bar{\alpha}, & \gamma\bar{\alpha}, \\ \bar{\beta}, & \alpha\bar{\beta}, & \beta\bar{\beta}, & \gamma\bar{\beta}, \\ \bar{\gamma}, & \alpha\bar{\gamma}, & \beta\bar{\gamma}, & \gamma\bar{\gamma} \end{array} \tag{2.54}$$

appear to write down the entries of $|\xi\rangle \otimes |\bar{\xi}\rangle$. To list up monomials appearing in the entries of $|\xi\rangle \otimes |\bar{\xi}\rangle \otimes |\xi\rangle$, we multiply the monomials in the first row of (2.54) with $(1, \alpha, \beta, \gamma)^{\mathrm{T}}$ to get ten monomials

$$1, \quad \alpha, \quad \beta, \quad \gamma, \quad \alpha^2, \quad \beta^2, \quad \gamma^2, \quad \alpha\beta, \quad \beta\gamma, \quad \gamma\alpha.$$

We also get ten monomials from each row in (2.54), and so we have 40 monomials in the entries of $|\xi\rangle \otimes |\bar{\xi}\rangle \otimes |\xi\rangle$. Therefore, we see that the span V of $|\xi\rangle \otimes |\bar{\xi}\rangle \otimes |\xi\rangle$ is of 40 dimension. We have

$$|\xi\rangle|\bar{\xi}\rangle|\xi\rangle = (1, \alpha, \beta, \gamma)^{\mathrm{T}} \otimes (1, \bar{\alpha}, \bar{\beta}, \bar{\gamma})^{\mathrm{T}} \otimes (1, \alpha, \beta, \gamma)^{\mathrm{T}},$$

and so we also see that vectors in the family

$$\{|ijk\rangle - |kji\rangle : i, j, k = 1, 2, 3, 4, \ i \neq k\}$$

belong to $V^{\perp}$. These 24 vectors, up to scalar multiplications, are easily seen to be mutually orthogonal, and so we see that it is a basis of $V^{\perp}$. The orthogonal complement $W^{\perp}$ of the space W spanned by

$$|\xi\rangle \otimes |\bar{\xi}\rangle \otimes U^*|\bar{\xi}\rangle = (1, \alpha, \beta, \gamma)^{\mathrm{T}} \otimes (1, \bar{\alpha}, \bar{\beta}, \bar{\gamma})^{\mathrm{T}} \otimes (-\bar{\alpha}, 1, -\bar{\gamma}, \bar{\beta})^{\mathrm{T}}$$

has the following 24 vectors

$$\begin{aligned} &|i11\rangle + |i22\rangle, \quad && |i33\rangle + |i44\rangle, \\ &|i13\rangle + |i42\rangle, \quad && |i14\rangle - |i32\rangle, \\ &|i23\rangle - |i41\rangle, \quad && |i24\rangle + |i31\rangle, \end{aligned}$$

with $i = 1, 2, 3, 4$. They make a basis of $W^{\perp}$.

One may check directly that $V^{\perp} \cap W^{\perp}$ is a four dimensional subspace of $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ which is spanned by the following four vectors

$$\begin{aligned} &|114\rangle - |411\rangle + |224\rangle - |422\rangle + |231\rangle - |132\rangle, \\ &|423\rangle - |324\rangle + |133\rangle - |331\rangle + |144\rangle - |441\rangle, \\ &|233\rangle - |332\rangle + |244\rangle - |442\rangle + |314\rangle - |413\rangle, \\ &|142\rangle - |241\rangle + |113\rangle - |311\rangle + |223\rangle - |322\rangle. \end{aligned}$$

Hence, we conclude that $\dim(V + W) = 60$, as it was required. □

Therefore, we see that the Robertson map $\phi_{\rm rb}$ generates an exposed ray of the convex cone $\mathbb{P}_1[M_4, M_4]$ of all positive maps between 4×4 matrices. It is known [28] that the map ϕ_U defined in (2.53) is exposed whenever U is a anti-symmetry. We note that

$$\phi_U = \tau_{4,1} - \mathrm{Ad}_U \circ \mathrm{T}$$

with the map $\tau_{4,1}$ in (1.51). The map ϕ_U is usually called the Breuer–Hall map. For further related examples of positive maps, see [103] and Section 8 of [30].

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