실해석(대학원) 시험

2001 년 6 월 16 일

1. State 'Monotone Convergence Theorem', 'Fatou Lemma', and 'Lebesgue Dominated Convergence Theorem'. Prove that if $\langle f_n \rangle$ is a sequence in $L^1(\mu)$ with $\sum_{n=1}^{\infty} \|f_n\|_1 < \infty$ then $\sum_{n=1}^{\infty} f_n$ converges in $L^1(\mu)$.

2. Let $f(x,y) = \dfrac{xy}{(x^2+y^2)^2}$ for $x, y \in [0,1]$.
   (i) Show that $\int_0^1 \int_0^1 f(x,y)dxdy = \int_0^1 \int_0^1 f(x,y)dydx$.
   (ii) Show that f is not integrable on $[0,1] \times [0,1]$.

3. Show that there exists no inner product on $L^1[0,1]$ with $\langle f, f \rangle = \|f\|_1^2$.

4. Assume that a function g on $[0,1]$ has the property:
$$h \in L^q[0,1] \implies gh \in L^1[0,1],$$
where $1 \le p \le \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Show that $g \in L^p[0,1]$.

5. Let μ be a measure on $\mathbb{R}$ defined by $\mu(E) = \int_E f\,dm$ for every Lebesgue measurable subset E of $\mathbb{R}$, where f is a nonnegative function in $L^1(\mathbb{R})$, and m is the Lebesgue measure. Show that $\int_{\mathbb{R}} g\,d\mu = \int_{\mathbb{R}} gf\,dm$ for every $g \in L^1(\mu)$.

6. Investigate all the possible implications between the following statements for continuous functions $f, f_n \in C[0,1]$:
   (a) $f_n(x) \to f(x)$ as $n \to \infty$ for each $x \in [0,1]$.
   (b) $\|f_n - f\|_{\sup} \to 0$ as $n \to \infty$.
   (c) $\int_0^1 f_n d\mu \to \int_0^1 f d\mu$ as $n \to \infty$ for each $\mu \in M[0,1]$.
   (d) $\int_0^1 f_n(x)g(x)dx \to \int_0^1 f(x)g(x)dx$ as $n \to \infty$ for each $g \in L^1[0,1]$.

7.
   (i) Show that the Banach algebra $L^1(\mathbb{T})$ under the convolution has no identity element.
   (ii) Give the definition of the Fejér kernel $\langle K_n \rangle$ in $L^1(\mathbb{T})$.
   (iii) Show that the Dirac measure $\delta_0 \in M(\mathbb{T})$ is the identity element under the convolution of $M(\mathbb{T})$.
   (iv) Explain how can we say that the Fejér kernel converges to δ_0.

8.
   (i) Show the identity $\widehat{fg}(n) = \widehat{f} * \widehat{g}(n)$ holds for every $g, h \in L^2(\mathbb{T})$ and $n \in \mathbb{Z}$.
   (ii) Show that $\{\widehat{f} : f \in L^1(\mathbb{T})\} = \{a * b : a, b \in \ell^2(\mathbb{Z})\}$.

9. Draw the graph of the function $f_k = \chi_{(-k,k)} * \chi_{(-1,1)}$, and find the Fourier transform $\widehat{f_k}$ of f_k, where $k = 1, 2, \ldots$. Show that the range of the Fourier transform is a proper subset of $C_0(\mathbb{R})$.

10. Write down anything.