실해석(대학원) 시험

1999 년 6 월 19 일

1. Show that the Banach algebra $L^1(\mathbb{R})$ under the convolution has no identity element. Discuss the notion of 'approximate identity'.

2. Draw the graph of the function $f_k = \chi_{(-k,k)} * \chi_{(-1,1)}$, and find the Fourier transform $\widehat{f_k}$ of f_k, where $k = 1, 2, \ldots$. Show that the range of the Fourier transform is a proper subset of $C_0(\mathbb{R})$.

3. Write down the Poisson Summation Formula, and find $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2}$, where $a > 0$.

4.
   (i) For $g, h \in L^2(\mathbb{R})$, show the identity $\widehat{g}\,\widehat{h} = \widehat{g * h}$.
   (ii) Show that
$$\{\widehat{f} : f \in L^1(\mathbb{R})\} = \{g * h : g, h \in L^2(\mathbb{R})\}.$$

5. For $\phi \in L^\infty[0,1]$, define the operator M_ϕ on $L^2[0,1]$ by
$$M_\phi(\xi) = \phi\xi, \qquad \xi \in L^2[0,1].$$
   (i) Show that M_ϕ is a bounded operator.
   (ii) Find conditions on ϕ for which M_ϕ has an eigenvalue.
   (iii) Let T be a bounded operator on $L^2[0,1]$ such that $TM_\phi = M_\phi T$ for each $\phi \in L^\infty[0,1]$. Show that $T = M_\psi$ for a $\psi \in L^\infty[0,1]$.

6. For $f \in L^1(\mathbb{T})$, define the bounded operator C_f on $L^2(\mathbb{T})$ by
$$C_f(\xi) = f * \xi, \qquad \xi \in L^2(\mathbb{T}).$$
   (i) Is the oprator C_f compact?
   (ii) Find the condition on f for which the operator C_f is self-adjoint.
   (iii) Find the norm of C_f in terms of the Fourier coefficients of f.

7. Show that the Volterra V on $L^2[0,1]$ defined by
$$(V\xi)(t) = \int_0^t \xi(s)ds, \qquad t \in [0,1]$$
is a bounded linear operator with $\|V\| \le \dfrac{1}{\sqrt{2}}$. Show also that V is a compact operator.