FACIAL STRUCTURES FOR UNITAL POSITIVE LINEAR MAPS IN THE TWO DIMENSIONAL MATRIX ALGEBRA

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Abstract. We completely determine the lattice structure for the faces of the convex set of all unital positive linear maps between the $2 \times 2$ matrix algebras.

1. Introduction

After Stinespring’s paper [10], the completely positive linear maps have been considered as the right order-preserving morphisms between self-adjoint operator algebras, and studied extensively. On the other hand, the structures of the convex set of all positive linear maps between operator algebras are extremely complicated even in the cases of low dimensional matrix algebras. Recently, the importance of positive linear maps between matrix algebras has been recognized by mathematical physicists, in relation with the so called ‘entanglement’ in quantum information theory. See [3] for a survey in this area.

In the paper [1], we have completely determined the facial structures of the convex cone $\mathbb{P}[M_2, M_2]$ of all positive linear maps in the $C^*$-algebra $M_2$ of all $2 \times 2$ matrices over the complex field. In this paper, we will determine the facial structures of the convex set $\mathbb{P}_I[M_2, M_2]$ of all unital positive linear maps between $M_2$.

Because every positive linear map in $\mathbb{P}[M_2, M_2]$ is decomposable [12], that is, every positive linear map between $M_2$ is the sum of completely positive linear maps and completely copositive linear maps, we know that every face in $\mathbb{P}[M_2, M_2]$ corresponds to a pair of subspaces in $M_2$ by [7]. We have also characterized in [1] the pairs of subspaces which correspond to faces of $\mathbb{P}[M_2, M_2]$. In order to classify the faces in $\mathbb{P}_I[M_2, M_2]$, we characterize the faces of $\mathbb{P}[M_2, M_2]$ which contain unital maps in Section 3, and investigate in Section 4 the relations between faces to understand the whole lattice structures of the convex set $\mathbb{P}_I[M_2, M_2]$.

The maximal faces of the convex set $\mathbb{P}_I[M_m, M_n]$ of all unital positive linear maps from $M_m$ into $M_n$ has been characterized in [5]. On the other hand, Størmer [11] found all extreme points of the convex set $\mathbb{P}_I[M_2, M_2]$ whose affine dimension is 12. We already know [5] that maximal faces are affine isomorphic each other, and they

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are parameterized by the four dimensional manifolds $\mathbb{CP}^1 \times \mathbb{CP}^1$. In this note, we will pay attention on a one specific maximal face $F$ whose affine dimension is 7. The main part of the boundary of $F$ consists of maximal faces which are two affine dimensional convex sets without line segments in their boundaries. The convex body $F$ has two exceptional maximal faces. One of them is a single point, and another one is affine isomorphic to the unit ball of the two dimensional complex $\ell^1$-space which is the four affine dimensional convex body whose maximal faces consist of line segments. We refer [9] for the description of completely positive trace-preserving maps in $\mathbb{P}[M_2, M_2]$. For another information on the convex set $\mathbb{P}_I[M_m, M_n]$, see the references in [1].

From now on, we denote by $\mathbb{P}$ (respectively $\mathbb{P}_I$) the convex set $\mathbb{P}[M_2, M_2]$ (respectively $\mathbb{P}_I[M_2, M_2]$). In this paper, every vector in $\mathbb{C}^2$ will be considered as a $2 \times 1$ matrix, and so every rank one $2 \times 2$ matrix is of the form $xy^*$ with $x, y \in \mathbb{C}^2$. We also denote by $\overline{x}$ the vector whose entries are the conjugates of the corresponding entries of $x$. The same notations will be used for matrices. Throughout this note, $(\cdot | \cdot)$ denotes the usual inner product in $\mathbb{C}^n$ or in $M_2$. The notation $x \parallel y$ means that two vectors $x$ and $y$ are parallel each other. We also denote by $\{e_1, e_2\}$ the usual othonormal basis of $\mathbb{C}^2$, and $\{e_{ij} : i, j = 1, 2\}$ the usual matrix units of $M_2$.

2. Preliminaries

It is well-known known [2], [4] that every completely positive (respectively completely copositive) linear map from $M_m$ into $M_n$ is of the form

$$\phi_V : X \mapsto \sum_i V_i^* X V_i,$$

(respectively $\phi^V : X \mapsto \sum_i V_i^* X^t V_i$)

for a subset $V = \{V_1, \ldots, V_\nu\} \subset M_{m \times n}$, where $X^t$ denote the transpose of $X$. We denote by $\mathbb{P}_2$ (respectively $\mathbb{P}^2$) the convex cone of all completely positive (respectively completely copositive) linear maps between $M_2$. Then we have $\mathbb{P} = \text{conv} (\mathbb{P}_2, \mathbb{P}^2)$, where $\text{conv} \{C_1, C_2\}$ denotes the convex hull of $C_1$ and $C_2$, and every positive linear map in $\mathbb{P}$ is of the form

$$\phi = \phi_V + \phi^W,$$

with $V = \{V_1, V_2, \ldots, V_\nu\}$ and $W = \{W_1, W_2, \ldots, W_t\}$ in $M_2$. It is also known [6] that every face of $\mathbb{P}_2$ (respectively $\mathbb{P}^2$) is of the form

$$\Phi_D = \{\phi_V : \text{span } V \subset D\} \quad \text{(respectively } \Phi^E = \{\phi^V : \text{span } V \subset E\}),$$

for a subspace $D$ (respectively $E$) of $M_2$, and the interior is given by

$$\text{int } \Phi_D = \{\phi_V : \text{span } V = D\} \quad \text{(respectively } \text{int } \Phi^E = \{\phi^V : \text{span } V = E\}).$$

Finally, it was shown in [7] that every face of $\mathbb{P}$ is of the form

$$\sigma(D, E) = \text{conv} \{\Phi_D, \Phi^E\}.$$
for a pair \((D, E)\) of subspaces of \(M_2\) and this expression is unique under the conditions
\[
\sigma(D, E) \cap \mathbb{P}_2 = \Phi_D, \quad \sigma(D, E) \cap \mathbb{P}^2 = \Phi_E.
\]

We have classified in [1] the pairs \((D, E)\) of subspaces in \(M_2\) which give rise to faces of \(P\) as follows:

- I \((3, 3)\) \(D = (xy)^\perp, \ E = (\overline{xy})^\perp\)
- II \((2, 2)\) \(D = \{xy^*, zw^*\}^\perp, \ E = \{\overline{xy^*}, \overline{zw^*}\}^\perp (x \parallel z \text{ or } y \parallel w)\)
- III \((2, 2)\) \(D, E \) are not spanned by rank one matrices
- IV \((2, 1)\) \(D \) is not spanned by rank one matrices, \(E \) is spanned by a rank one matrix
- V \((1, 2)\) \(D \) is spanned by a rank one matrix, \(E \) is not spanned by rank one matrices
- VI \((1, 1)\) \(D, E \) are spanned by rank two matrices
- VII \((1, 1)\) \(D = Cxy^*, \ E = C\overline{xy}^*\)
- VIII \((1, 0)\) \(D \) is spanned by a rank two matrix, \(E = \{0\}\)
- IX \((0, 1)\) \(D = \{0\}, \ E \) is spanned by a rank two matrix,

where the second column denotes \((\dim D, \dim E)\). We refer to [1] for more detailed description of the faces. We just mention that \(\sigma(D_0, E_0) \subset \sigma(D, E)\) if and only if \(D_0 \subset D\) and \(E_0 \subset E\).

We conclude this section with the following simple proposition, which says that the faces of \(P_I\) are determined by the faces of \(P\) whose interiors have unital maps.

**Proposition 2.1.** Let \(C\) be a convex set and \(L\) a hyperplane in an affine space. Then \(G\) is a face of \(C \cap L\) if and only if \(G = F \cap L\) for a face \(F\) of \(C\) whose interior has elements of \(L\).

**Proof.** If \(F\) is a face of \(C\) then it is clear that \(F \cap L\) is a face of \(C \cap L\). Conversely, assume that \(G\) is a face of \(C \cap L\). Let \(F\) be the smallest face of \(C\) containing \(G\). Then \(\text{int} \ G \subset \text{int} \ F\) by [5] Lemma 2.2 (see also [8], Theorem 18.2.), and so \(\text{int} \ F\) contains an element of \(L\). To show \(G = F \cap L\), let \(x \in \text{int} \ G\). Then, for any \(y \in F \cap L\) there is \(t > 1\) such that \(tx + (1 - t)y \in F\) since \(x \in \text{int} \ F\). We also have \(tx + (1 - t)y \in L\) since \(L\) is a hyperplane. This shows that \(tx + (1 - t)y \in F \cap L\) and \(x \in \text{int} (F \cap L)\). Since two faces \(G\) and \(F \cap L\) of \(C \cap L\) share an interior point \(x\), we conclude that \(G = F \cap L\). \(\square\)

### 3. Characterization of faces

Now, we proceed to determine which faces of \(P\) contain unital maps as interior points. The faces of type I are determined by two unit vectors \(x, y\) (more precisely by the one dimensional subspaces of \(M_2\) spanned by \(xy^*\) and \(\overline{xy}^*\), respectively). Denote
this face by

\[ F[x, y] = \sigma(\Phi_{xy}^{\perp}, \Phi_{yx}^{\perp}) \]

\[ = \{ \phi_V + \phi_W : \text{span } V \perp xy^*, \text{span } W \perp xy^* \} \]

\[ = \{ \phi_V + \phi_W : V_iy \perp x, W_jy \perp x (i = 1, 2, \ldots, s, j = 1, 2, \ldots, t) \} \]

where \( \phi_V + \phi_W \) is the map as in (1). Since

\[ V_y \perp x \iff x^*Vy \iff (\phi_V(xx^*)y \mid y) = 0 \]

and \( W_y \perp x \iff (\phi_W(xx^*)y \mid y) = 0 \) similarly, we also have

\[ F[x, y] = \{ \phi \in \mathbb{P} : (\phi(xx^*)y \mid y) = 0 \} \]

as was introduced in [5]. If we take unit vectors \( \xi \) and \( \eta \) so that \( x \perp \xi \) and \( y \perp \eta \), then we see that

\[ \phi = \frac{1}{2}\phi_{xy} + 1 \phi_{y^*} + \phi_{y^*}, \quad \psi = \frac{1}{2}\phi_{yx} + 1 \phi_{x^*} + \phi_{x^*} \]

are unital maps which are interior points of \( \Phi_{xy}^{\perp} \) and \( \Phi_{yx}^{\perp} \), respectively. See [6] for general cases. Therefore, any nontrivial convex combination of \( \phi \) and \( \psi \) is a unital map in the interior of \( F[x, y] \).

Note that a face of type II is of the form \( F[x_0, y_0] \cap F[x_1, y_1] = \sigma(D, E) \) with

\[ D = \{ x_0y_0^*, x_1y_1^* \}^\perp, \quad E = \{ x_0y_0^*, x_1y_1^* \}^\perp \]

Proposition 3.1. Let \( x_i, y_i \) \((i = 0, 1)\) be unit vectors. Then the following are equivalent:

(i) \( F[x_0, y_0] \cap F[x_1, y_1] \) has a unital map in the interior,

(ii) \(|(y_0 \mid y_1)| < |(x_0 \mid x_1)| \) or \( (y_0 \mid y_1) = (x_0 \mid x_1) = 0 \), or \(|(y_0 \mid y_1)| = |(x_0 \mid x_1)| = 1 \).

Proof. Let \( \phi = \phi_V + \phi_W \) in (1) belong to the face \( F[x_0, y_0] \cap F[x_1, y_1] \). If we take unit vectors \( \xi_i \) orthogonal to \( x_i \) \((i = 0, 1)\), then we may write

\[ V_jy_i = a_{ji}\xi_i, \quad W_ky_i = b_{ki}\xi_i \]

for \( i = 0, 1, j = 1, 2, \ldots, s \) and \( k = 1, 2, \ldots, t \). Write

\[ a_i = \sum_{j=1}^{s} a_{ji}e_j \in \mathbb{C}^s, \quad b_i = \sum_{k=1}^{t} b_{ki}e_k \in \mathbb{C}^t \]

for \( i = 0, 1 \). If we define \( U : \mathbb{C}^2 \to \mathbb{C}^{2(s+t)} \) by

\[ U\eta = (V_1\eta, V_2\eta, \ldots, V_s\eta, W_1\eta, W_2\eta, \ldots, W_t\eta), \]

then we see that \( \phi \) is unital if and only if \( U \) is an isometry if and only if

\[ (y_i \mid y_\ell) = (Uy_i \mid Uy_\ell) = (a_i \mid a_\ell)\xi_i + (b_i \mid b_\ell)\xi_\ell, \quad i, \ell = 0, 1. \]

If we take \( i = \ell \) then \( 1 = \|a_i\|^2 + \|b_i\|^2 \) for \( i = 0, 1 \), and so we see that

\[ |(y_0 \mid y_1)| \leq \|(a_0 \mid a_1) + (b_0 \mid b_1)\| \|\xi_0 \mid \xi_1\| \]

\[ \leq \|(a_0\|a_1\| + \|b_0\|b_1\|)\|\xi_0 \mid \xi_1\| \leq \|\xi_0 \mid \xi_1\|. \]
This proves the following

\[(6) \quad F[x_0, y_0] \cap F[x_1, y_1] \text{ has a unital map } \implies |(y_0 \mid y_1)| \leq |(x_0 \mid x_1)|,\]

because \(|(x_0 \mid x_1)| = |(\xi_0 \mid \xi_1)|\).

Now, assume the conditions in (ii). If \(|(y_0 \mid y_1)| = |(x_0 \mid x_1)| = 1\) or equivalently \(y_0 \parallel y_1\) and \(x_0 \parallel x_1\), then we have already seen that \(F[x_0, y_0] \cap F[x_1, y_1] = F[x_0, y_0] = F[x_1, y_1]\) has a unital map in the interior. Assume that \(\{y_0, y_1\}\) is linearly independent. In this case, we see that \(\phi\) is unital if and only if the relation (4) holds. Put \(b_0 = b_1 = 0\) and take unit vectors \(a_0, a_1 \in \mathbb{C}^2\) so that the equality (4) holds. Write the entries of \(a_0\) and \(a_1\) by (3) and then we may take \(V_j : \mathbb{C}^2 \to \mathbb{C}^2\) satisfying (2) for \(j = 1, 2\). If \(|(y_0 \mid y_1)| < |(x_0 \mid x_1)|\) or \((y_0 \mid y_1) = (x_0 \mid x_1) = 0\) then we may take above \(a_0\) and \(a_1\) so that \(a_0 \| a_1\). Then we see that \(\{V_1, V_2\}\) is linearly independent and so \(\text{span} \{V_1, V_2\} = D\). Indeed, \(\alpha V_1 + \beta V_2 = 0\) implies that

\[
0 = (\alpha V_1 + \beta V_2)(y_0) = (\alpha a_{10} + \beta a_{20})\xi_0, \\
0 = (\alpha V_1 + \beta V_2)(y_1) = (\alpha a_{11} + \beta a_{21})\xi_1,
\]

and so, we have \(\alpha = \beta = 0\). Therefore, \(\phi = \phi_{V_1} + \phi_{V_2}\) is a unital map which belongs to the interior of \(\Phi_D\). By the same argument, we may take a unital map \(\psi\) in the interior of \(\Phi^E\), and any nontrivial convex combination of \(\phi\) and \(\psi\) is a unital map in the interior of the face \(F[x_0, y_0] \cap F[x_1, y_1]\). This proves the direction (ii) \(\implies\) (i).

Now, we show that if \(0 < |(y_0 \mid y_1)| = |(x_0 \mid x_1)| < 1\) then every unital map in \(F[x_0, y_0] \cap F[y_0, y_1]\) belongs to a proper face. This will completes the proof of (i) \(\implies\) (ii) in view of (6). In this case, the inequality (5) becomes an equality, and we have \(a_0 \| a_1\) and \(b_0 \| b_1\). Therefore, any unital map in the face is of the form \(\phi = \phi_V + \phi^W\), where \(V\) and \(W\) are given by

\[
V y_0 = a_0 \xi_0, \quad V y_1 = a_1 \xi_1, \quad W y_0 = b_0 \xi_0, \quad W y_1 = b_1 \xi_1,
\]

and \(a_0, a_1, b_0, b_1\) are scalars with the relations

\[
|a_0|^2 + |b_0|^2 = 1, \quad |a_1|^2 + |b_1|^2 = 1, \quad (y_0 \mid y_1) = a_0 \overline{a_1} (\xi_0 \mid \xi_1) + b_0 \overline{b_1} (\xi_0 \mid \xi_1).
\]

Write \((y_0 \mid y_1) = re^{i\theta}\) and \((\xi_0 \mid \xi_1) = re^{i\phi}\). Then we have

\[
1 = |a_0 a_1 e^{i(\theta + \phi)} + b_0 b_1 e^{i(\phi + \theta)}| = |a_0|^2 + |b_0|^2 = |a_1|^2 + |b_1|^2,
\]

and so we have

\[
a_0 = a_1 e^{i(\theta + \phi)}, \quad b_0 = b_1 e^{i(\phi + \theta)}.
\]

Therefore, we have

\[
V (y_0 + e^{i\theta} y_1) = a_1 e^{i\theta} (e^{-i\phi} \xi_0 + \xi_1) \| b_1 e^{i\theta} (e^{-i\phi} \xi_0 + \xi_1) = W (y_0 + e^{i\theta} y_1).
\]

This shows that \(\phi\) lies in a proper face of \(F[x_0, y_0] \cap F[x_1, y_1]\) as was required, by [1], Proposition 3.6. □
A face $\sigma(D, E)$ of type III is determined by two dimensional subspaces $D$ and $E$ which are not spanned by rank one matrices. We have shown that $D$ and $E$ are of the forms

$$D = \text{span} \{xy^*, xw^* + zy^*\}, \quad E = \text{span} \{\overline{x}y^*, \overline{x}w^* + \mu\overline{z}y^*\},$$

respectively, where $x \perp z$, $y \perp w$, $\|x\| = \|z\| = 1$ and $|\mu| = 1$. See [1], Corollary 2.2 and Proposition 3.5. We also note that a face of type IV (respectively of type V) is of the form $\Phi_D$ (respectively $\Phi_E$).

**Proposition 3.2.** Let $x, y, z, w, \mu$ and the space $D, E$ be given as above. Then the following are equivalent:

(i) The face $\Phi_D$ has a unital map in the interior,
(ii) The face $\Phi_E$ has a unital map in the interior,
(iii) The face $\sigma(D, E)$ has a unital map in the interior,
(iv) $\|y\| < \|w\|$.

**Proof.** Let

$$P = a_1(xy^* + zy^*) + a_2xy^*$$
$$Q = b_1(xy^* + zy^*) + b_2xy^*$$
$$R = c_1(\overline{x}y^* + \mu\overline{z}y^*) + c_2\overline{x}y^*$$
$$S = d_1(\overline{x}y^* + \mu\overline{z}y^*) + d_2\overline{x}y^*.$$

If we define $\xi_1, \xi_2 \in \mathbb{C}^4$ by

$$\xi_1 = (a_1, b_1, c_1, d_1), \quad \xi_2 = (a_2, b_2, c_2, d_2),$$

and put $y_0 = \frac{y}{\|y\|}$ and $w_0 = \frac{w}{\|w\|}$, then we have

$$P^*P + Q^*Q + R^*R + S^*S = \|\xi_1\|^2\|w\|^2 w_0w_0^* + (\|\xi_1\|^2 + \|\xi_2\|^2)\|y\|^2y_0y_0^* + (\xi_1|\xi_2)wy^* + (\xi_1|\overline{\xi_2})wy^*.$$

Note that $\phi = \phi_P + \phi_Q + \phi_R + \phi_S$ is unital if and only if $P^*P + Q^*Q + R^*R + S^*S = I$ if and only if the following relations hold:

$$\langle \xi_1, \xi_2 \rangle = 0, \quad \|\xi_1\|^2\|w\|^2 = 1, \quad (\|\xi_1\|^2 + \|\xi_2\|^2)\|y\|^2 = 1.$$  \hspace{1cm} (7)

Therefore, we see that $\sigma(D, E)$ has a unital map if and only if there exist vectors $\xi_1, \xi_2 \in \mathbb{C}^4$ with the relations in (7) if and only if $\|y\| \leq \|w\|$.

If $\|y\| < \|w\|$ then we may take nonzero vectors $\xi_1, \xi_2$ satisfying (7) so that $\text{span} \{P, Q\} = D$ and $\text{span} \{R, S\} = E$. Then $\phi_P + \phi_Q$ and $\phi_R + \phi_S$ lie in the interior of $\Phi_D$ and $\Phi_E$, respectively, and so, we see that $\phi$ lies in the interior of $\sigma(D, E)$. If $\|w\| = \|y\|$ then we have $\xi_2 = 0$. In this case, every unital map in $\sigma(D, E)$ is a convex combination of $\phi_V$ and $\phi_W$, where

$$V = xw^* + zy^*, \quad W = \overline{x}w^* + \mu\overline{z}y^*.$$
Note that

\[ V y = \mu W y, \quad V w = W w, \quad V(y + e^{i\theta}w) = \mu W(y + e^{i\theta}w), \]

where \( \theta \) is given by \( \mu = e^{2i\theta} \), and so we see that the cone generated by \( \phi_V \) and \( \phi_W \) is a proper face of \( \sigma(D, E) \) which is of type VI by [1], Proposition 3.6. Therefore, we conclude that there is no unital map in the interior of \( \sigma(D, E) \), and this completes the proof of (iii) \( \iff \) (iv). The equivalences (i) \( \iff \) (iv) and (ii) \( \iff \) (iv) are easier with the same methods. □

Next, we consider the face of type VI. This face is a two dimensional convex cone generated by \( \phi_V \) and \( \phi_W \), where \( V \) and \( W \) are nonsingular matrices satisfying conditions in [1], Proposition 3.6. Note that every interior point of this face is of the form \( \phi_V + \phi_W \). Take a unit vector \( y_0 \) such that \( V y_0 \parallel W y_0 \). Then \( V y_0 = ax_0 \) and \( W y_0 = bx_0 \) with scalars \( a, b \) and a unit vector \( x_0 \). Take unitaries \( S \) and \( T \) such that \( Se_1 = y_0 \) and \( Te_1 = x_0 \). Then we have

\[ T^* V S e_1 = ae_1, \quad \overline{T^* W S e_1} = be_1. \]

We may write

\[ V_0 = \begin{pmatrix} a & x \\ 0 & y \end{pmatrix}, \quad W_0 = \begin{pmatrix} b & z \\ 0 & w \end{pmatrix}, \]

so that the relations

\[ V = TV_0 S^*, \quad W = \overline{T} W_0 S^* \]

hold, where we may assume that \( a \) and \( b \) are positive numbers. Note that an interior point \( \phi_V + \phi_W \) is unital if and only if \( V^* V + W^* W = I \) if and only if \( V_0^* V_0 + W_0^* W_0 = I \) if and only if the following relations hold:

\[ a^2 + b^2 = 1 \]

\[ |x|^2 + |y|^2 + |z|^2 + |w|^2 = 1 \]

\[ ax + bz = 0 \]

(8)

Now, we note that \( V y_i \parallel W y_i \) if and only if \( TV_0 S^* y_i \parallel \overline{TW_0 S^*} y_i \) if and only if \( V_0(S^* y_i) \parallel W_0(S^* y_i) \). Therefore, two matrices \( V_0, W_0 \) also satisfy the conditions in [1], Proposition 3.6. Since

\[ \overline{W_0}^{-1} V_0 = \frac{1}{b \overline{w}} \begin{pmatrix} a \overline{w} & x \overline{w} - y \overline{z} \\ 0 & by \end{pmatrix}, \]

we have also the relations

\[ a|w| = b|y| \]

\[ a \overline{w}(\overline{x} w - \overline{y} z) + (x \overline{w} - y \overline{z}) b \overline{y} = 0, \]

by the equation (5) in [1]. Write

\[ \gamma = |x|^2 + |z|^2, \quad \delta = |y|^2 + |w|^2. \]

(10)
From the relations (8), (9) and (10), we have

\[
 a = \frac{1}{\sqrt{\delta}} |y|, \quad b = \frac{1}{\sqrt{\delta}} |w|, \quad |x| = \frac{\sqrt{\gamma}}{\sqrt{\delta}} |w|, \quad |z| = \frac{\sqrt{\gamma}}{\sqrt{\delta}} |y|.
\]

If we write \( y = |y| e^{i\theta}, \ w = |w| e^{i\phi}, \) then we have

\[
 V_0 = \begin{pmatrix} \frac{1}{\sqrt{\delta}} |y| & \frac{\sqrt{\gamma}}{\sqrt{\delta}} |w| e^{i\psi} \\ 0 & \frac{1}{\sqrt{\delta}} |w| \end{pmatrix}, \quad W_0 = \begin{pmatrix} \frac{1}{\sqrt{\delta}} |w| & -\frac{\sqrt{\gamma}}{\sqrt{\delta}} |y| e^{i\psi} \\ 0 & |w| e^{i\phi} \end{pmatrix},
\]

for a \( \psi \in \mathbb{R} \), by the third equation of (8). Finally, we see by a direct calculation that the second equation of (9) holds if and only if

\[
 \gamma = 0 \quad \text{or} \quad e^{2i\psi} = -e^{i(\theta + \phi)}.
\]

This shows the following:

**Proposition 3.3.** Assume that \( V, W \) are non-singular matrices and convex cone generated by \( \phi_V \) and \( \phi^V \) is a face of \( \mathbb{P}[M_2, M_2] \). Then this face has a unital map in the interior if and only if \( V \) and \( W \) are of the forms

\[
 V^\pm = T \begin{pmatrix} \frac{1}{\sqrt{\delta}} |y| & \frac{\sqrt{\gamma}}{\sqrt{\delta}} |w| e^{i\psi \pm} \\ 0 & \frac{1}{\sqrt{\delta}} |w| \end{pmatrix} S^*, \quad W^\pm = T \begin{pmatrix} \frac{1}{\sqrt{\delta}} |w| & -\frac{\sqrt{\gamma}}{\sqrt{\delta}} |y| e^{i\psi \pm} \\ 0 & |w| e^{i\phi} \end{pmatrix} S^*,
\]

respectively, where

\[
 0 < |y|^2 + |w|^2 = \delta \leq 1, \quad \gamma + \delta = 1, \quad (e^{i\psi \pm})^2 = -e^{i(\theta + \phi)},
\]

and \( S \) and \( T \) are unitaries.

We use the convention

\[
 \phi + \frac{\pi}{2} \leq \psi^+ < \phi + \frac{3\pi}{2}, \quad \phi - \frac{\pi}{2} \leq \psi^- < \phi + \frac{\pi}{2}.
\]

Then we have

\[
 \psi^\pm = \begin{cases} \theta + \phi & + \frac{\pi}{2}, & \text{if } 0 \leq \frac{\theta - \phi}{2} < \pi, \\ \frac{\theta + \phi}{2} & + \frac{\pi}{2}, & \text{if } -\pi \leq \frac{\theta - \phi}{2} < 0. \end{cases}
\]

It is clear that a face of type VII does not have a unital map at all. The remaining cases of types VIII and IX are extreme rays generated by \( \phi_U \) or \( \phi^U \), and these faces have unital maps in the interiors if and only if \( U \) is a unitary.

### 4. Relations between faces

We recall [5] that every maximal face of \( \mathbb{P}_I \) is of the form \( F[x, y] \cap \mathbb{P}_I \) for a \( (x, y) \in \mathbb{C}P^1 \times \mathbb{C}P^1 \), where \( \mathbb{C}P^1 \) is the one dimensional complex projective plane, and any two maximal faces are affine isomorphic each other. Therefore, we consider the
single maximal face $F = F[e_2,e_1] \cap P_I$, which is associated with the pair $(D,E)$ of subspaces

\[ D = E = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a,b,d \in \mathbb{C} \right\}. \]

Note that $D$ and $E$ are orthogonal to the matrix $e_2e_1^* = e_{21}$. Throughout this section, we identify a linear map $\phi : M_2 \to M_2$ with the the $4 \times 4$ matrix;

\[
\phi = \begin{pmatrix} \phi(e_{11}) & \phi(e_{12}) \\ \phi(e_{21}) & \phi(e_{22}) \end{pmatrix}.
\]

We begin with the faces of types VI. Note that a face of type VI in $F[e_2,e_1]$ is two dimensional convex cone generated by $\phi V_0$ and $\phi W_0$, where $V_0$ and $W_0$ are defined in (11) with the condition (12). Hence, we see that this face gives rise to an extreme point if and only if $V_0$ is not an unitary if and only if $W_0$ is not an unitary if and only if $\gamma \neq 0$. In this case, we see by a direct calculation that the corresponding extreme point $\phi V_0 + \phi W_0$ is given by

\[
\phi V_0 + \phi W_0 = \begin{pmatrix} 1 & 0 & 0 & \alpha \\ 0 & \gamma & \beta & \varepsilon^\pm \\ 0 & \beta & 0 & 0 \\ \alpha & \varepsilon^\pm & 0 & \delta \end{pmatrix} =: E^\pm_{\alpha,\beta},
\]

with

\[
\gamma = 1 - \delta, \quad \alpha = \frac{|y|}{\sqrt{\delta}} y, \quad \beta = \frac{|w|}{\sqrt{\delta}} w, \quad \varepsilon^\pm = 2 \frac{\sqrt{\gamma}}{\sqrt{\delta}} |yw| e^{i(\theta - \varphi \pm)}. \tag{14}
\]

Note that the number $\delta$ is given by

\[
|\alpha| + |\beta| = \frac{1}{\sqrt{\delta}} (|y|^2 + |w|^2) = \sqrt{\delta}, \quad 0 < \delta < 1
\]

by (10), and the relation (14) defines a one-to-one correspondence between $(y,w)$ with $0 < |y|^2 + |w|^2 < 1$ and $(\alpha,\beta)$ with $0 < |\alpha| + |\beta| < 1$. The reverse relations are given by

\[
y = \delta^{1/4} |\alpha|^{-1/2} \alpha, \quad w = \delta^{1/4} |\beta|^{-1/2} \beta.
\]

We also note that

\[
|\varepsilon^\pm|^2 = 4 \frac{\gamma}{\delta} |yw|^2 = 2 \frac{\gamma}{\delta} \left( (|y|^2 + |w|^2)^2 - (|y|^4 + |w|^4) \right) = 2 \gamma (\delta - |\alpha|^2 - |\beta|^2),
\]

and

\[
\arg \varepsilon^\pm = \begin{cases} 
\frac{\theta - \varphi + \pi}{2}, & \text{if } 0 \leq \frac{\theta - \varphi}{2} < \pi, \\
\frac{\theta - \varphi}{2} \pm \frac{\pi}{2}, & \text{if } -\pi \leq \frac{\theta - \varphi}{2} < 0
\end{cases}
\]

by (13).
When $\alpha = 0$ or $\beta = 0$, we define
\[
E_{\alpha,\beta} = \begin{pmatrix}
1 & 0 & 0 & \alpha \\
0 & \gamma & \beta & 0 \\
0 & \beta & 0 & 0 \\
\alpha & 0 & 0 & \delta
\end{pmatrix}
\]
with
\[
\sqrt{\delta} = |\alpha| + |\beta|, \quad \gamma = 1 - \delta,
\]
and continue to consider the faces of type IV with $\gamma = 0$. Recall that $\gamma = 0$ if and only if $V_0$ and $W_0$ in (11) are unitaries. In this case, it is clear that these faces give rise to the line segments;
\[
L_{\alpha,\beta} = \text{conv}\{E_{\alpha,0}, E_{0,\beta}\}
\]
with $|\alpha| = |\beta| = 1$.

It is also clear that the map $\phi_U$ and $\phi_U'$ in faces of type VIII and IX also give rise to extreme points $E_{\alpha,0}$ and $E_{0,\beta}$ with $|\alpha| = |\beta| = 1$, respectively.

We proceed to consider the faces of types III, IV and V. By Proposition 3.2, we see that every face of this type is associated with a pair $(D_{\lambda,\mu}, E_{\lambda,\mu})$ of subspaces
\[
D_{\lambda,\mu} = \text{span}\left\{\begin{pmatrix}0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix}\lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right\}, \quad E_{\lambda,\mu} = \text{span}\left\{\begin{pmatrix}0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix}\lambda^{-1} & 0 \\ 0 & \mu \end{pmatrix}\right\},
\]
where $\lambda, \mu \in \mathbb{C}$ with $0 < |\lambda| < 1$ and $|\mu| = 1$. Note that
\[(D_{\lambda,\mu}, E_{\lambda,\mu}) = (D_{\lambda',\mu'}, E_{\lambda',\mu'}) \iff (\lambda, \mu) = (\lambda', \mu').\]

We write this face by $F_{\lambda,\mu}$:
\[
F_{\lambda,\mu} = \sigma(D_{\lambda,\mu}, E_{\lambda,\mu}) \cap P_I = \text{conv}\{\Phi_{D_{\lambda,\mu}}, \Phi_{E_{\lambda,\mu}}\} \cap P_I.
\]
If $\phi = \sum_i \phi_i A_i \in \Phi_{D_{\lambda,\mu}}$ is unital with $A_i = \begin{pmatrix}a_i & -b_i \\ 0 & a_i \end{pmatrix}$ then we have
\[
\sum_i |a_i|^2 = |\lambda|^2, \quad \sum_i |b_i|^2 = 1 - |\lambda|^2, \quad \sum_i a_i \bar{b}_i = 0,
\]
from the relation $\sum_i A_i^* A_i = I$. It follows that $\phi = E_{\lambda,0}$ is a unique unital map in the face $\Phi_{D_{\lambda,\mu}}$, and this gives rise to a unique extreme point. By the same argument, we also have a unique extreme point $E_{0,\lambda,\mu}$ in the face $\Phi_{E_{\lambda,\mu}}$. We conclude that faces of types IV and V give rise to extreme points
\[
E_{\alpha,0} \quad \text{and} \quad E_{0,\beta} \quad \text{with} \quad 0 < |\alpha|, |\beta| < 1,
\]
respectively.

We have shown that the face $F_{\lambda,\mu}$ has extreme points $E_{\lambda,0}$ and $E_{0,\lambda,\mu}$. Because the corresponding spaces $D_{\lambda,\mu}$ and $E_{\lambda,\mu}$ do not have an unitary, it is now clear that every proper face $F_{\lambda,\mu}$ which is neither $E_{\lambda,0}$ nor $E_{0,\lambda,\mu}$ with $0 < |\alpha| < 1$ comes out
from the faces of types VI with \( \gamma \neq 0 \). Note that \( E_{\alpha,\beta}^\pm \in F_{\lambda,\mu} \) if and only if \( V_0 \in D_{\lambda,\mu} \) and \( W_0 \in E_{\lambda,\mu} \), where \( V_0, W_0 \) are given in (11). This is the case if and only if

\[
\frac{1}{\sqrt{\delta}} = \lambda^{-1} e^{i\theta} = \overline{\mu}^{-1} e^{i\phi}.
\]

Since \( \alpha = |\alpha| e^{i\theta}, \beta = |\beta| e^{i\phi} \), we see that \( E_{\alpha,\beta}^\pm \) is an extreme point of \( F_{\lambda,\mu} \) if and only if the following relations

\[
|\alpha| + |\beta| = |\lambda|, \quad \arg \alpha = \arg \lambda, \quad \arg \beta = \arg \mu + \arg \lambda, \quad \alpha, \beta \neq 0
\]

hold. We conclude that every proper face of \( F_{\lambda,\mu} \) is an extreme point. For each fixed \((\alpha, \beta)\) with the relations (16) there are exactly two extreme points \( E_{\alpha,\beta}^+ \) and \( E_{\alpha,\beta}^- \) in the face \( F_{\lambda,\mu} \), and we have

\[
\frac{1}{2} E_{\alpha,\beta}^+ + \frac{1}{2} E_{\alpha,\beta}^- = \frac{|\alpha|}{|\lambda|} E_{\lambda,0} + \frac{|\beta|}{|\lambda|} E_{0,\mu}, \quad E_{\lambda,0} = \lim_{\beta \to 0} E_{\alpha,\beta}^+, \quad E_{0,\mu} = \lim_{\alpha \to 0} E_{\alpha,\beta}^-.
\]

Finally, we consider the faces of types II. By Proposition 3.1, we see that a face of this type corresponds to the pair \((D_{a,b}, E_{a,b})\) of subspaces

\[
D_{a,b} = \text{span} \left\{ \begin{pmatrix} a & -r \\ 0 & s \end{pmatrix}, \begin{pmatrix} 0 & -s \\ 0 & b \end{pmatrix} \right\}, \quad E_{a,b} = \text{span} \left\{ \begin{pmatrix} a & -r \\ 0 & s \end{pmatrix}, \begin{pmatrix} 0 & -s \\ 0 & b \end{pmatrix} \right\},
\]

where \( \xi = \begin{pmatrix} b \\ s \end{pmatrix} \) and \( \eta = \begin{pmatrix} r \\ a \end{pmatrix} \), with the restrictions

\[
r^2 + |a|^2 = s^2 + |b|^2 = 1, \quad 0 \leq |b| < |a| \leq 1 \quad \text{or} \quad |a| = |b| = 0 \quad \text{or} \quad |a| = |b| = 1.
\]

We denote this face by \( G_{a,b} \):

\[
G_{a,b} = \sigma(D_{a,b}, E_{a,b}) \cap \mathbb{P}_I.
\]

Recall that if \( |a| = |b| = 0 \) then \( F[e_2, e_1] = F[\xi, \eta] \) and \( G_{a,b} = F[e_2, e_1] \cap \mathbb{P}_I = F \). Therefore, we may ignore this case. Note that any proper face of \( G_{a,b} \) comes out from faces of types VI, VIII and IX. We see that \( G_{a,b} \) has an extreme point arising from a face of type VIII (respectively of type IX) if and only if \( D_{a,b} \) (respectively \( E_{a,b} \)) has an unitary if and only if there are \( p, q \in \mathbb{C} \) such that

\[
|ap| = 1, \quad |bq| = 1, \quad rp + sq = 0
\]

if and only if \( r = s \) if and only if \( |a| = |b| = 1 \).

In the case of \( |a| = |b| = 1 \), we see that \( D_{a,b} = E_{a,b} = \text{span} \{e_{11}, e_{22}\} \) is independent of the choice of \( a \) and \( b \). We denote this face by \( \mathbb{H} \):

\[
\mathbb{H} = \sigma(D, D) \cap \mathbb{P}_I,
\]

with \( D = \text{span} \{e_{11}, e_{22}\} \). Since \( e_2 e_1^* \) and \( e_1 e_2^* \) are orthogonal to \( D \), we see that

\[
\mathbb{H} = F[e_2, e_1] \cap F[e_1, e_2] \cap \mathbb{P}_I.
\]
Since \( \phi(e_{22}) = e_{22} \) for every \( \phi \in \mathbb{H} \), we also see that no extreme points of the forms \( E_{a,b}^\pm \) belong to \( \mathbb{H} \), and every maximal face of \( \mathbb{H} \) is of the form \( \mathbb{L}_{a,b} \). Note that \( \mathbb{L}_{a,b} \subset \mathbb{H} \) if and only if
\[
\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \in D, \quad \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} \in D.
\]
Since this is always the case, we see that every \( \mathbb{L}_{a,b} \) is a maximal face of \( \mathbb{H} \). We conclude that
\[
\mathbb{H} = \left\{ \begin{pmatrix} 1 & 0 & 0 & \alpha \\ 0 & 0 & \beta & 0 \\ 0 & \beta & 0 & 0 \\ \alpha & 0 & 0 & 1 \end{pmatrix} : |\alpha| + |\beta| \leq 1 \right\},
\]
and \( \mathbb{H} \) is affine isomorphic to the unit ball of the two dimensional \( \ell^1 \)-space over the complex field.

Now, we consider the case of \( |b| = 0 \) (and \( a \neq 0 \) of course). In this case, \( E_{a,b}^\pm \in \mathbb{G}_{a,0} \) implies that \( y = w = 0 \) in (11), which is absurd. This shows that \( \mathbb{G}_{a,0} \) has no proper face and is an extreme point in itself. It is easy to see that this extreme point is independent of the choice of \( a \) with \( 0 < |a| \leq 1 \), and \( \mathbb{G}_{a,0} = E_{a,0} \).

Next, we assume that \( 0 < |b| < |a| \leq 1 \), and proceed to determine which extreme point \( E_{a,b}^\pm \) belongs to the face \( \mathbb{G}_{a,b} \). First of all, it is easy to see that
\[
(D_{a,b}, E_{a,b}) = (D_{a',b'}, E_{a',b'}) \iff \begin{cases} (a, b) = (a', b'), & \text{if } 0 < |b| < |a| < 1, \\ b = b' & \text{if } 0 < |b| < |a| = 1. \end{cases}
\]
Note that \( E_{a,b}^\pm \in \mathbb{G}_{a,b} \) if and only if \( V_0 \in D_{a,b} \) and \( W_0 \in E_{a,b} \) with \( V_0 \) and \( W_0 \) in (11) if and only if the equations
\[
\begin{align*}
\frac{1}{\sqrt{\alpha^2}} |a|^{1/2} e^{-i\theta/2} + \frac{s}{b} |a|^{1/2} e^{i\theta/2} &= -\frac{\sqrt{\gamma}}{\sqrt{\delta}} |\beta|^{1/2} e^{i(\psi + \phi - \theta/2)} = \mp i \frac{\sqrt{\gamma}}{\sqrt{\delta}} |\beta|^{1/2} e^{i\phi/2} \\
\frac{1}{\sqrt{\delta}} |\beta|^{1/2} e^{-i\phi/2} + \frac{s}{\beta} |\beta|^{1/2} e^{i\phi/2} &= \frac{\sqrt{\gamma}}{\sqrt{\delta}} |\alpha|^{1/2} e^{i(\psi - \phi/2)} = \pm i \frac{\sqrt{\gamma}}{\sqrt{\delta}} |\alpha|^{1/2} e^{i\phi/2}
\end{align*}
\]
hold by (13) when \( 0 \leq \frac{\theta - \phi}{2} < \pi \). If \( -\pi \leq \frac{\theta - \phi}{2} < 0 \) then \( E_{a,b}^\pm \in \mathbb{G}_{a,b} \) if and only if \( \mp \) is replaced by \( \pm \) and vice versa in (17). We put \( |\alpha|^{1/2} e^{i\phi/2} \) in the second equation into the first equation to get
\[
\frac{r^2}{|a|^2} - \delta \frac{s^2}{|b|^2} + \gamma = 0, \quad \text{or equivalently} \quad \frac{|b|}{|a|} = \sqrt{\delta} = |\alpha| + |\beta|,
\]
since \( \gamma + \delta = 1 \). We also note that the first and second equations in (17) are equivalent under the relations in (18).

We consider the case \( 0 < |b| < |a| = 1 \) or equivalently \( r = 0 \). In this case, we have \( \sqrt{\delta} = |\alpha| + |\beta| = |b| \) and \( \gamma = 1 - |b|^2 = s^2 \), and so it follows from the first
equation of (17) that
\[ \frac{s^2}{|b|^2} |\alpha| = \frac{\gamma}{\delta} |\beta| = \frac{s^2}{|b|^2} |\beta|, \quad \frac{\theta - \phi}{2} = \begin{cases} \arg b \pm \frac{\pi}{2}, & \text{if } 0 \leq \frac{\theta - \phi}{2} < \pi, \\ \arg b \pm \frac{\pi}{2}, & \text{if } -\pi \leq \frac{\theta - \phi}{2} < 0. \end{cases} \]

If \(0 \leq \frac{\theta - \phi}{2} < \pi\) then we have \(E_{\alpha,\beta}^+ \in G_{1,b}\) if and only if \(0 \leq \arg b - \frac{\pi}{2} < \pi\) if and only if \(\frac{\pi}{2} \leq \arg b < \frac{3}{2} \pi\), and the same conclusion holds when \(-\pi \leq \frac{\theta - \phi}{2} < 0\). Therefore, we see that
\[ E_{\alpha,\beta}^+ \in G_{1,b} \iff \pi \leq \arg b < \frac{3}{2} \pi, \quad E_{\alpha,\beta}^- \in G_{1,b} \iff -\pi \leq \arg b < \frac{\pi}{2}, \]
in any cases, and we have
\[ |\alpha| = |\beta| = \frac{1}{2} |b|, \quad \arg \alpha - \arg \beta = 2 \arg b + \pi. \]

Now, assume that \(0 < |b| < |a| < 1\). Taking the square of the absolute value of the first equation in (17), we have
\[ r^2 + s^2 + 2 \text{Re} \frac{|ab|}{\overline{ab}} rs e^{i\theta} = (|a|^2 - |b|^2) |\beta|/|\alpha|. \]
Therefore, there exist \(\theta\) satisfying (20) if and only if \(r^2 + s^2 + 2rs \geq (|a|^2 - |b|^2) |\beta|/|\alpha| \)
if and only if the relation \(\frac{|\beta|}{|\alpha|} \leq \frac{(r+s)^2}{s^2 - r^2} = \frac{s+r}{s-r} \) holds. By the same argument for
the second equation of (17), we also have \(\frac{|\alpha|}{|\beta|} \leq \frac{s+r}{s-r} \). Hence, we conclude that if
\(E_{\alpha,\beta}^\pm \) belongs to \(G_{a,b} \) then
\[ |\alpha| + |\beta| = \frac{|b|}{|a|}, \quad \frac{s-r}{s+r} \leq \frac{|\beta|}{|\alpha|} \leq \frac{s+r}{s-r}. \]
Let \((|\alpha|, |\beta|)\) be fixed with the relations in (18) and (21), and write
\[ A(a, b, \alpha) = \frac{1}{\sqrt{\delta}} \frac{r}{|a|^{1/2}} |\alpha|^{1/2} e^{-i\theta/2} + \frac{s}{b |\alpha|^{1/2}} e^{i\phi/2}. \]
Then there exists a \(\theta\) (recall again that we wrote \(\theta = \arg \alpha\) and \(\phi = \arg \beta\)) such that
\[ |A(a, b, \alpha)| = \frac{\sqrt{r}}{\sqrt{\delta}} |\beta|^{1/2}. \] Therefore, there exist a unique \(\phi\) such that
\[ A(a, b, \alpha) = \begin{cases} -i \frac{\sqrt{r}}{\sqrt{\delta}} |\beta|^{1/2} e^{i\phi/2}, & \text{if } 0 \leq \frac{\theta - \phi}{2} < \pi, \\ +i \frac{\sqrt{r}}{\sqrt{\delta}} |\beta|^{1/2} e^{i\phi/2}, & \text{if } -\pi \leq \frac{\theta - \phi}{2} < 0. \end{cases} \]
or there exists a unique $\phi$ such that

$$A(a, b, \alpha) = \begin{cases} +i\sqrt{\gamma}/\sqrt{\delta}|\beta|/2e^{i\phi/2}, & \text{if } 0 \leq \theta - \phi < \pi, \\ -i\sqrt{\gamma}/\sqrt{\delta}|\beta|/2e^{i\phi/2}, & \text{if } -\pi \leq \theta - \phi < 0. \end{cases}$$

and so we see that there is $\beta$ with $E^+_{\alpha, \beta} \in G_{a,b}$ or there is $\beta$ with $E^-_{\alpha, \beta} \in G_{a,b}$. Furthermore, both cases do not occur simultaneously. By the continuity, we see that the extreme points of $G_{a,b}$ consist of $E^+_{\alpha, \beta}$, or consists of $E^-_{\alpha, \beta}$. If the both strict inequalities hold in (21) then two pairs $(\alpha, \beta)$ satisfy $E^\pm_{\alpha, \beta} \in G_{a,b}$. On the other hand, if the one equality holds in (21) then there is a unique pair $(\alpha, \beta)$ satisfying $E^\pm_{\alpha, \beta} \in G_{a,b}$.

Now, we summarize the above discussion as follows:

**Theorem 4.1.** Every proper face of $F = F[e_2, e_1] \cap P_1[M_2, M_2]$ is one of the following:

- $F_{\lambda, \mu}$ ($0 < |\lambda| < 1, |\mu| = 1$)
- $G_{a,b}$ ($0 < |b| < |a| < 1$ or $0 < |b| < 1, a = 1$)
- $H$
- $L_{\alpha, \beta}$ ($|\alpha| = |\beta| = 1$)
- $E^\pm_{\alpha, \beta}$ ($0 < |\alpha| + |\beta| < 1$)
- $E_{\alpha, \beta}$ ($0 \leq |\alpha| + |\beta| \leq 1, \alpha = 0$ or $\beta = 0$)

We also have the following:

(i) Maximal faces of $F$ consist of $F_{\lambda, \mu}, G_{a,b}, H$ and $E_{0,0}$.
(ii) Maximal faces of $F_{\lambda, \mu}$ consist of extreme points $E_{\lambda,0}, E_{0,\lambda}$ and $E^\pm_{\alpha, \beta}$ with (16).
(iii) Maximal faces of $G_{1,b}$ consist of extreme points $E^+_{\alpha, \beta}$ or $E^-_{\alpha, \beta}$ with (19).
(iv) Maximal faces of $G_{a,b}$ ($|a| < 1$) consist of extreme points $E^+_{\alpha, \beta}$ or $E^-_{\alpha, \beta}$ with (21) and (18).
(v) Maximal faces of $H$ consist of $L_{\alpha, \beta}$.
(vi) Maximal faces of $L_{\alpha, \beta}$ consist of two extreme points $E_{\alpha,0}$ and $E_{0,\beta}$.
(vii) $E^\pm_{\alpha, \beta}$ and $E_{\alpha, \beta}$ are extreme points of $F$.

We remark here that extreme points $E^\pm_{\alpha, \beta}$ and $E_{\alpha, \beta}$ exhaust all extreme points discussed in [11], Theorem 8.2. Since $\phi(e_{22}) = \delta e_{22}$ for any $\phi \in F$, we may slice $F$ by $\delta \in [0, 1]$. For a fixed $\delta \in [0, 1]$, put

$$S_\delta = \{ \phi \in F : \phi(e_{22}) = \delta e_{22} \}.$$

It is clear that

$$S_0 = E_{0,0}, \quad S_1 = H \supset L_{\alpha, \beta}.$$
We also have

\[ \mathbb{F}_{\lambda,\mu} \subset S_\delta \iff |\lambda| = \sqrt{\delta}, \quad \mathbb{G}_{a,b} \subset S_\delta \iff \frac{|b|}{|a|} = \sqrt{\delta}, \]

\[ \mathbb{E}^{\pm}_{\alpha,\beta}, \mathbb{E}_{\alpha,\beta} \subset S_\delta \iff |\alpha| + |\beta| = \sqrt{\delta} \]

For a fixed \( \delta \in (0,1) \), we see that maximal faces of \( S_\delta \) consist of unexposed faces \( \mathbb{F}_{\lambda,\mu} \)'s and exposed faces \( \mathbb{G}_{a,b} \)'s, which are parametrized by

\[ I = \{(\lambda, \mu) \in \mathbb{C}^2 : |\lambda| = \sqrt{\delta}, \ |\mu| = 1\} \]

\[ J = \{(a, b) \in \mathbb{C}^2 : 0 < |a| < 1, |b| = \sqrt{\delta}|a| \quad \text{or} \quad a = 1, |b| = \sqrt{\delta}\}, \]

respectively. Note that \( I \) is the torus and \( J \) is the solid torus without boundary. It is also apparent that \( \mathbb{F}_{\lambda,\mu} \) and \( \mathbb{G}_{a,b} \) are two dimensional convex bodies and \( S_\delta \) is a six dimensional convex body for \( \delta \in (0,1) \). Finally, we see that \( \mathbb{F} \) is a seven dimensional convex body.

**References**


