FACIAL STRUCTURES FOR UNITAL POSITIVE LINEAR MAPS IN THE TWO DIMENSIONAL MATRIX ALGEBRA

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Abstract. We completely determine the lattice structure for the faces of the convex set of all unital positive linear maps between the 2×2 matrix algebras.

1. Introduction

After Stinespring's paper [10], the completely positive linear maps have been considered as the right order-preserving morphims between self-adjoint operator algebras, and studied extensively. On the other hand, the structures of the convex set of all positive linear maps between operator algebras are extremely complicated even in the cases of low dimensional matrix algebras. Recently, the importance of positive linear maps between matrix algebras has been recognized by mathematical physicists, in relation with the so called 'entanglement' in quantum information theory. See [3] for a survey in this area.

In the paper [1], we have completely determined the facial structures of the convex cone $\mathbb{P}[M_2, M_2]$ of all positive linear maps in the C^*-algebra M_2 of all 2×2 matrices over the complex field. In this paper, we will determine the facial structures of the convex set $\mathbb{P}_I[M_2, M_2]$ of all unital positive linear maps between M_2.

Because every positive linear map in $\mathbb{P}[M_2, M_2]$ is decomposable [12], that is, every positive linear map between M_2 is the sum of completely positive linear maps and completely copositive linear maps, we know that every face in $\mathbb{P}[M_2, M_2]$ corresponds to a pair of subspaces in M_2 by [7]. We have also characterized in [1] the pairs of subspaces which correspond to faces of $\mathbb{P}[M_2, M_2]$. In order to classify the faces in $\mathbb{P}_I[M_2, M_2]$, we characterize the faces of $\mathbb{P}[M_2, M_2]$ which contain unital maps in Section 3, and investigate in Section 4 the relations between faces to understand the whole lattice structures of the convex set $\mathbb{P}_I[M_2, M_2]$.

The maximal faces of the convex set $\mathbb{P}_I[M_m, M_n]$ of all unital positive linear maps from M_m into M_n has been characterized in [5]. On the other hand, Størmer [11] found all extreme points of the convex set $\mathbb{P}_I[M_2, M_2]$ whose affine dimension is 12. We already know [5] that maximal faces are affine isomorphic each other, and they

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are parameterized by the four dimensional manifolds $\mathbb{CP}^1 \times \mathbb{CP}^1$. In this note, we will pay attention on a one specific maximal face $\mathbb{F}$ whose affine dimension is 7. The main part of the boundary of $\mathbb{F}$ consists of maximal faces which are two affine dimensional convex sets without line segments in their boundaries. The convex body $\mathbb{F}$ has two exceptional maximal faces. One of them is a single point, and another one is affine isomorphic to the unit ball of the two dimensional complex ℓ^1-space which is the four affine dimensional convex body whose maximal faces consist of line segments. We refer [9] for the description of completely positive trace-preserving maps in $\mathbb{P}[M_2, M_2]$. For another information on the convex set $\mathbb{P}_I[M_m, M_n]$, see the references in [1].

From now on, we denote by $\mathbb{P}$ (respectively $\mathbb{P}_I$) the convex set $\mathbb{P}[M_2, M_2]$ (respectively $\mathbb{P}_I[M_2, M_2]$). In this paper, every vector in $\mathbb{C}^2$ will be considered as a 2×1 matrix, and so every rank one 2×2 matrix is of the form xy^* with $x, y \in \mathbb{C}^2$. We also denote by $\overline{x}$ the vector whose entries are the conjugates of the corresponding entries of x. The same notations will be used for matrices. Throughout this note, $(\cdot \mid \cdot)$ denotes the usual inner product in $\mathbb{C}^\nu$ or in M_2. The notation $x \parallel y$ means that two vectors x and y are parallel each other. We also denote by $\{e_1, e_2\}$ the usual othonormal basis of $\mathbb{C}^2$, and $\{e_{ij} : i, j = 1, 2\}$ the usual matrix units of M_2.

2. Preliminaries

It is well-known known [2], [4] that every completely positive (respectively completely copositive) linear map from M_m into M_n is of the form

$$\phi_{\mathcal{V}} : X \mapsto \sum_i V_i^* X V_i, \qquad (\text{respectively } \phi^{\mathcal{V}} : X \mapsto \sum_i V_i^* X^{\mathrm{t}} V_i)$$

for a subset $\mathcal{V} = \{V_1, \ldots, V_\nu\} \subset M_{m \times n}$, where X^{t} denote the transpose of X. We denote by $\mathbb{P}_2$ (respectively $\mathbb{P}^2$) the convex cone of all completely positive (respectively completely copositive) linear maps between M_2. Then we have $\mathbb{P} = \mathrm{conv}\,(\mathbb{P}_2, \mathbb{P}^2)$, where $\mathrm{conv}\,\{C_1, C_2\}$ denotes the convex hull of C_1 and C_2, and every positive linear map in $\mathbb{P}$ is of the form

$$\phi = \phi_{\mathcal{V}} + \phi^{\mathcal{W}}, \tag{1}$$

with $\mathcal{V} = \{V_1, V_2, \ldots, V_s\}$ and $\mathcal{W} = \{W_1, W_2, \ldots, W_t\}$ in M_2. It is also known [6] that every face of $\mathbb{P}_2$ (respectively $\mathbb{P}^2$) is of the form

$$\Phi_D = \{\phi_{\mathcal{V}} : \mathrm{span}\,\mathcal{V} \subset D\} \qquad (\text{respectively } \Phi^E = \{\phi^{\mathcal{V}} : \mathrm{span}\,\mathcal{V} \subset E\}),$$

for a subspace D (respectively E) of M_2, and the interior is given by

$$\mathrm{int}\,\Phi_D = \{\phi_{\mathcal{V}} : \mathrm{span}\,\mathcal{V} = D\} \qquad (\text{respectively } \mathrm{int}\,\Phi^E = \{\phi^{\mathcal{V}} : \mathrm{span}\,\mathcal{V} = E\}).$$

Finally, it was shown in [7] that every face of $\mathbb{P}$ is of the form

$$\sigma(D, E) = \mathrm{conv}\,\{\Phi_D, \Phi^E\}$$

for a pair (D, E) of subspaces of M_2 and this expression is unique under the conditions

$$\sigma(D, E) \cap \mathbb{P}_2 = \Phi_D, \qquad \sigma(D, E) \cap \mathbb{P}^2 = \Phi^E.$$

We have classified in [1] the pairs (D, E) of subspaces in M_2 which give rise to faces of $\mathbb{P}$ as follows:

I	$(3,3)$	$D = (xy^*)^\perp,\ E = (\overline{x}y^*)^\perp$
II	$(2,2)$	$D = \{xy^*, zw^*\}^\perp,\ E = \{\overline{x}y^*, \overline{z}w^*\}^\perp$ $(x \not\parallel z$ or $y \not\parallel w)$
III	$(2,2)$	D, E are not spanned by rank one matrices
IV	$(2,1)$	D is not spanned by rank one matrices, E is spanned by a rank one matrix
V	$(1,2)$	D is spanned by a rank one matrix, E is is not spanned by rank one matrices
VI	$(1,1)$	D, E are spanned by rank two matrices
VII	$(1,1)$	$D = \mathbb{C}xy^*, E = \mathbb{C}\overline{x}y^*$
VIII	$(1,0)$	D is spanned by a rank two matrix, $E = \{0\}$
IX	$(0,1)$	$D = \{0\}, E$ is spanned by a rank two matrix,

where the second column denotes $(\dim D, \dim E)$. We refer to [1] for more detailed description of the faces. We just mention that $\sigma(D_0, E_0) \subset \sigma(D, E)$ if and only if $D_0 \subset D$ and $E_0 \subset E$.

We conclude this section with the following simple proposition, which says that the faces of $\mathbb{P}_I$ are determined by the faces of $\mathbb{P}$ whose interiors have unital maps.

Proposition 2.1. *Let C be a convex set and L a hyperplane in an affine space. Then G is a face of $C \cap L$ if and only if $G = F \cap L$ for a face F of C whose interior has elements of L.*

Proof. If F is a face of C then it is clear that $F \cap L$ is a face of $C \cap L$. Conversely, assume that G is a face of $C \cap L$. Let F be the smallest face of C containing G. Then $\operatorname{int} G \subset \operatorname{int} F$ by [5] Lemma 2.2 (see also [8], Theorem 18.2.), and so $\operatorname{int} F$ contains an element of L. To show $G = F \cap L$, let $x \in \operatorname{int} G$. Then, for any $y \in F \cap L$ there is $t > 1$ such that $tx + (1-t)y \in F$ since $x \in \operatorname{int} F$. We also have $tx + (1-t)y \in L$ since L is a hyperplane. This shows that $tx + (1-t)y \in F \cap L$ and $x \in \operatorname{int}(F \cap L)$. Since two faces G and $F \cap L$ of $C \cap L$ share an interior point x, we conclude that $G = F \cap L$. □

3. Characterization of faces

Now, we proceed to determine which faces of $\mathbb{P}$ contain unital maps as interior points. The faces of type I are determined by two unit vectors x, y (more precisely by the one dimensional subspaces of M_2 spanned by xy^* and $\overline{x}y^*$, respectively). Denote

this face by

$$\begin{aligned} F[x,y] &= \sigma(\Phi_{(xy^*)^\perp}, \Phi^{(\overline{x}y^*)^\perp}) \\ &= \{\phi_{\mathcal{V}} + \phi^{\mathcal{W}} : \operatorname{span}\mathcal{V} \perp xy^*,\ \operatorname{span}\mathcal{W} \perp \overline{x}y^*\} \\ &= \{\phi_{\mathcal{V}} + \phi^{\mathcal{W}} : V_i y \perp x,\ W_j y \perp \overline{x}\ (i = 1, 2, \dots, s, j = 1, 2, \dots, t)\}, \end{aligned}$$

where $\phi_{\mathcal{V}} + \phi^{\mathcal{W}}$ is the map as in (1). Since

$$Vy \perp x \iff x^*Vy \iff (\phi_V(xx^*)y \,|\, y) = 0$$

and $Wy \perp \overline{x} \iff (\phi^W(xx^*)y \,|\, y) = 0$ similarly, we also have

$$F[x,y] = \{\phi \in \mathbb{P} : (\phi(xx^*)y \,|\, y) = 0\}$$

as was introduced in [5]. If we take unit vectors ξ and η so that $x \perp \xi$ and $y \perp \eta$, then we see that

$$\phi = \frac{1}{2}\phi_{x\eta^*} + \frac{1}{2}\phi_{\xi\eta^*} + \phi_{\xi y^*}, \qquad \psi = \frac{1}{2}\phi^{\overline{x}\eta^*} + \frac{1}{2}\phi^{\overline{\xi}\eta^*} + \phi^{\overline{\xi}y^*}$$

are unital maps which are interior points of $\Phi_{(xy^*)^\perp}$ and $\Phi^{(\overline{x}y^*)^\perp}$, respectively. See [6] for general cases. Therefore, any nontrivial convex combination of ϕ and ψ is a unital map in the interior of $F[x,y]$.

Note that a face of type II is of the form $F[x_0,y_0] \cap F[x_1,y_1] = \sigma(D,E)$ with

$$D = \{x_0y_0^*, x_1y_1^*\}^\perp, \qquad E = \{\overline{x_0}y_0^*, \overline{x_1}y_1^*\}^\perp.$$

Proposition 3.1. *Let x_i, y_i $(i = 0, 1)$ be unit vectors. Then the following are equivalent:*

(i) *$F[x_0,y_0] \cap F[x_1,y_1]$ has a unital map in the interior,*
(ii) *$|(y_0 \,|\, y_1)| < |(x_0 \,|\, x_1)|$ or $(y_0 \,|\, y_1) = (x_0 \,|\, x_1) = 0$, or $|(y_0 \,|\, y_1)| = |(x_0 \,|\, x_1)| = 1$.*

Proof. Let $\phi = \phi_{\mathcal{V}} + \phi^{\mathcal{W}}$ in (1) belong to the face $F[x_0,y_0] \cap F[x_1,y_1]$. If we take unit vectors ξ_i orthogonal to x_i $(i = 0, 1)$, then we may write

$$V_j y_i = a_{ji}\xi_i, \qquad W_k y_i = b_{ki}\overline{\xi_i} \tag{2}$$

for $i = 0, 1$, $j = 1, 2, \dots, s$ and $k = 1, 2, \dots, t$. Write

$$a_i = \sum_{j=1}^{s} a_{ji}e_j \in \mathbb{C}^s, \qquad b_i = \sum_{k=1}^{t} b_{ki}e_k \in \mathbb{C}^t \tag{3}$$

for $i = 0, 1$. If we define $U : \mathbb{C}^2 \to \mathbb{C}^{2(s+t)}$ by

$$U\eta = (V_1\eta, V_2\eta, \dots, V_s\eta, W_1\eta, W_2\eta, \dots, W_t\eta),$$

then we see that ϕ is unital if and only if U is an isometry if and only if

$$(y_i \,|\, y_\ell) = (Uy_i \,|\, Uy_\ell) = (a_i \,|\, a_\ell)(\xi_i \,|\, \xi_\ell) + (b_i \,|\, b_\ell)(\overline{\xi_i} \,|\, \overline{\xi_\ell}), \qquad i, \ell = 0, 1. \tag{4}$$

If we take $i = \ell$ then $1 = \|a_i\|^2 + \|b_i\|^2$ for $i = 0, 1$, and so we see that

$$\begin{aligned} |(y_0 \,|\, y_1)| &\le (|(a_0 \,|\, a_1)| + |(b_0 \,|\, b_1)|)|(\xi_0 \,|\, \xi_1)| \\ &\le (\|a_0\|\|a_1\| + \|b_0\|\|b_1\|)|(\xi_0 \,|\, \xi_1)| \le |(\xi_0 \,|\, \xi_1)|. \end{aligned} \tag{5}$$

This proves the following

$$F[x_0, y_0] \cap F[x_1, y_1] \text{ has a unital map } \Longrightarrow \ |(y_0 \,|\, y_1)| \leq |(x_0 \,|\, x_1)|, \tag{6}$$

because $|(x_0 \,|\, x_1)| = |(\xi_0 \,|\, \xi_1)|$.

Now, assume the conditions in (ii). If $|(y_0 \,|\, y_1)| = |(x_0 \,|\, x_1)| = 1$ or equivalently $y_0 \parallel y_1$ and $x_0 \parallel x_1$, then we have already seen that $F[x_0, y_0] \cap F[x_1, y_1] = F[x_0, y_0] = F[x_1, y_1]$ has a unital map in the interior. Assume that $\{y_0, y_1\}$ is linearly independent. In this case, we see that ϕ is unital if and only if the relation (4) holds. Put $b_0 = b_1 = 0$ and take unit vectors $a_0, a_1 \in \mathbb{C}^2$ so that the equality (4) holds. Write the entries of a_0 and a_1 by (3) and then we may take $V_j : \mathbb{C}^2 \to \mathbb{C}^2$ satisfying (2) for $j = 1, 2$. If $|(y_0 \,|\, y_1)| < |(x_0 \,|\, x_1)|$ or $(y_0 \,|\, y_1) = (x_0 \,|\, x_1) = 0$ then we may take above a_0 and a_1 so that $a_0 \not\parallel a_1$. Then we see that $\{V_1, V_2\}$ is linearly independent and so $\operatorname{span}\{V_1, V_2\} = D$. Indeed, $\alpha V_1 + \beta V_2 = 0$ implies that

$$\begin{aligned} 0 &= (\alpha V_1 + \beta V_2)(y_0) = (\alpha a_{10} + \beta a_{20})\xi_0, \\ 0 &= (\alpha V_1 + \beta V_2)(y_1) = (\alpha a_{11} + \beta a_{21})\xi_1, \end{aligned}$$

and so, we have $\alpha = \beta = 0$. Therefore, $\phi = \phi_{V_1} + \phi_{V_2}$ is a unital map which is belongs to the interior of Φ_D. By the same argument, we may take a unital map ψ in the inetrior of Φ^E, and any nontrivial convex combination of ϕ and ψ is a unital map in the interior of the face $F[x_0, y_0] \cap F[x_1, y_1]$. This proves the direction (ii) $\Longrightarrow$ (i).

Now, we show that if $0 < |(y_0 \,|\, y_1)| = |(x_0 \,|\, x_1)| < 1$ then every unital map in $F[x_0, y_0] \cap F[y_0, y_1]$ belongs to a proper face. This will completes the proof of (i) $\Longrightarrow$ (ii) in view of (6). In this case, the inequality (5) becomes an equality, and we have $a_0 \parallel a_1$ and $b_0 \parallel b_1$. Therefore, any unital map in the face is of the form $\phi = \phi_V + \phi^W$, where V and W are given by

$$Vy_0 = a_0\xi_0, \quad Vy_1 = a_1\xi_1, \qquad Wy_0 = b_0\overline{\xi_0}, \quad Wy_1 = b_1\overline{\xi_1},$$

and a_0, a_1, b_0, b_1 are scalars with the relations

$$|a_0|^2 + |b_0|^2 = 1, \qquad |a_1|^2 + |b_1|^2 = 1, \qquad (y_0 \,|\, y_1) = a_0\overline{a_1}(\xi_0 \,|\, \xi_1) + b_0\overline{b_1(\xi_0 \,|\, \xi_1)}.$$

Write $(y_0 \,|\, y_1) = re^{i\theta}$ and $(\xi_0 \,|\, \xi_1) = re^{i\phi}$. Then we have

$$1 = a_0\overline{a_1 e^{i(-\phi+\theta)}} + b_0\overline{b_1 e^{i(\phi+\theta)}} = |a_0|^2 + |b_0|^2 = |a_1|^2 + |b_1|^2,$$

and so we have

$$a_0 = a_1 e^{i(-\phi+\theta)}, \qquad b_0 = b_1 e^{i(\phi+\theta)}.$$

Therefore, we have

$$V(y_0 + e^{i\theta}y_1) = a_1 e^{i\theta}(e^{-i\phi}\xi_0 + \xi_1) \parallel \overline{b_1 e^{i\theta}}(e^{-i\phi}\xi_0 + \xi_1) = \overline{W(y_0 + e^{i\theta}y_1)}.$$

This shows that ϕ lies in a proper face of $F[x_0, y_0] \cap F[x_1, y_1]$ as was required, by [1], Proposition 3.6. $\square$

A face $\sigma(D,E)$ of type III is determined by two dimensional subspaces D and E which are not spanned by rank one matrices. We have shown that D and E are of the forms

$$D = \text{span}\,\{xy^*, xw^* + zy^*\}, \qquad E = \text{span}\,\{\overline{x}y^*, \overline{x}w^* + \mu\overline{z}y^*\},$$

respectively, where $x \perp z$, $y \perp w$, $\|x\| = \|z\| = 1$ and $|\mu| = 1$. See [1], Corollary 2.2 and Proposition 3.5. We also note that a face of type IV (respectively of type V) is of the form Φ_D (respectively Φ^E).

Proposition 3.2. *Let x, y, z, w, μ and the space D, E be given as above. Then the following are equivalent:*

(i) *The face Φ_D has a unital map in the interior,*
(ii) *The face Φ^E has a unital map in the interior,*
(iii) *The face $\sigma(D,E)$ has a unital map in the interior,*
(iv) $\|y\| < \|w\|$.

Proof. Let

$$\begin{aligned} P &= a_1(xw^* + zy^*) + a_2xy^* \\ Q &= b_1(xw^* + zy^*) + b_2xy^* \\ R &= c_1(\overline{x}w^* + \mu\overline{z}y^*) + c_2\overline{x}y^* \\ S &= d_1(\overline{x}w^* + \mu\overline{z}y^*) + d_2\overline{x}y^*. \end{aligned}$$

If we define $\xi_1, \xi_2 \in \mathbb{C}^4$ by

$$\xi_1 = (a_1, b_1, c_1, d_1), \qquad \xi_2 = (a_2, b_2, c_2, d_2),$$

and put $y_0 = \dfrac{y}{\|y\|}$ and $w_0 = \dfrac{w}{\|w\|}$, then we have

$$\begin{aligned} P^*P + Q^*Q + R^*R + S^*S = \|\xi_1\|^2\|w\|^2 w_0w_0^* + (\|\xi_1\|^2 + \|\xi_2\|^2)\|y\|^2 y_0y_0^* \\ + (\xi_1 \,|\, \xi_2)yw^* + \overline{(\xi_1 \,|\, \xi_2)}wy^*. \end{aligned}$$

Note that $\phi = \phi_P + \phi_Q + \phi^R + \phi^S$ is unital if and only if $P^*P + Q^*Q + R^*R + S^*S = I$ if and only if the following relations hold:

$$(7) \qquad (\xi_1 \,|\, \xi_2) = 0, \qquad \|\xi_1\|^2\|w\|^2 = 1, \qquad (\|\xi_1\|^2 + \|\xi_2\|^2)\|y\|^2 = 1.$$

Therefore, we see that $\sigma(D,E)$ has a unital map if and only if there exist vectors $\xi_1, \xi_2 \in \mathbb{C}^4$ with the relations in (7) if and only if $\|y\| \le \|w\|$.

If $\|y\| < \|w\|$ then we may take nonzero vectors ξ_1, ξ_2 satisfying (7) so that $\text{span}\,\{P, Q\} = D$ and $\text{span}\,\{R, S\} = E$. Then $\phi_P + \phi_Q$ and $\phi^R + \phi^S$ lie in the interior of Φ_D and Φ^E, respectively, and so, we see that ϕ lies in the inetrior of $\sigma(D,E)$. If $\|w\| = \|y\|$ then we have $\xi_2 = 0$. In this case, every unital map in $\sigma(D,E)$ is a convex combination of ϕ_V and ϕ^W, where

$$V = xw^* + zy^*, \qquad W = \overline{x}w^* + \mu\overline{z}y^*.$$

Note that

$$Vy = \mu\overline{Wy}, \qquad Vw = \overline{Ww}, \qquad V(y + e^{i\theta}w) = \mu\overline{W(y + e^{i\theta}w)},$$

where θ is given by $\mu = e^{2i\theta}$, and so we see that the cone generated by ϕ_V and ϕ^W is a proper face of $\sigma(D, E)$ which is of type VI by [1], Proposition 3.6. Therefore, we conclude that there is no unital map in the interior of $\sigma(D, E)$, and this completes the proof of (iii) $\Longleftrightarrow$ (iv). The equivalences (i) $\Longleftrightarrow$ (iv) and (ii) $\Longleftrightarrow$ (iv) are easier with the same methods. $\square$

Next, we consider the face of type VI. This face is a two dimensional convex cone generated by ϕ_V and ϕ^W, where V and W are nonsingular matrices satisfying conditions in [1], Proposition 3.6. Note that every interior point of this face is of the form $\phi_V + \phi^W$. Take a unit vector y_0 such that $Vy_0 \parallel \overline{Wy_0}$. Then $Vy_0 = ax_0$ and $Wy_0 = b\overline{x_0}$ with scalars a, b and a unit vector x_0. Take unitaries S and T such that $Se_1 = y_0$ and $Te_1 = x_0$. Then we have

$$T^*VSe_1 = ae_1, \qquad \overline{T}^*WSe_1 = be_1.$$

We may write

$$V_0 = \begin{pmatrix} a & x \\ 0 & y \end{pmatrix}, \qquad W_0 = \begin{pmatrix} b & z \\ 0 & w \end{pmatrix},$$

so that the relations

$$V = TV_0S^*, \qquad W = \overline{T}W_0S^*$$

hold, where we may assume that a and b are positive numbers. Note that an interior point $\phi_V + \phi^W$ is unital if and only if $V^*V + W^*W = I$ if and only if $V_0^*V_0 + W_0^*W_0 = I$ if and only if the following relations hold:

$$\begin{aligned} &a^2 + b^2 = 1 \\ &|x|^2 + |y|^2 + |z|^2 + |w|^2 = 1 \\ &ax + bz = 0 \end{aligned} \tag{8}$$

Now, we note that $Vy_i \parallel \overline{Wy_i}$ if and only if $TV_0S^*y_i \parallel T\overline{W_0S^*y_i}$ if and only if $V_0(S^*y_i) \parallel \overline{W_0(S^*y_i)}$. Therefore, two matrices V_0, W_0 also satisfy the conditions in [1], Proposition 3.6. Since

$$\overline{W_0}^{-1}V_0 = \frac{1}{b\overline{w}}\begin{pmatrix} a\overline{w} & x\overline{w} - y\overline{z} \\ 0 & by \end{pmatrix},$$

we have also the relations

$$\begin{aligned} &a|w| = b|y| \\ &a\overline{w}(\overline{x}w - \overline{y}z) + (x\overline{w} - y\overline{z})b\overline{y} = 0, \end{aligned} \tag{9}$$

by the equation (5) in [1]. Write

$$\gamma = |x|^2 + |z|^2, \qquad \delta = |y|^2 + |w|^2. \tag{10}$$

From the relations (8), (9) and (10), we have

$$a = \frac{1}{\sqrt{\delta}}|y|, \qquad b = \frac{1}{\sqrt{\delta}}|w|, \qquad |x| = \frac{\sqrt{\gamma}}{\sqrt{\delta}}|w|, \qquad |z| = \frac{\sqrt{\gamma}}{\sqrt{\delta}}|y|.$$

If we write $y = |y|e^{i\theta}, w = |w|e^{i\phi}$, then we have

$$V_0 = \begin{pmatrix} \frac{1}{\sqrt{\delta}}|y| & \frac{\sqrt{\gamma}}{\sqrt{\delta}}|w|e^{i\psi} \\ 0 & |y|e^{i\theta} \end{pmatrix}, \qquad W_0 = \begin{pmatrix} \frac{1}{\sqrt{\delta}}|w| & -\frac{\sqrt{\gamma}}{\sqrt{\delta}}|y|e^{i\psi} \\ 0 & |w|e^{i\phi} \end{pmatrix}, \tag{11}$$

for a $\psi \in \mathbb{R}$, by the third equation of (8). Finally, we see by a direct calculation that the second equation of (9) holds if and only if

$$\gamma = 0 \qquad \text{or} \qquad e^{2i\psi} = -e^{i(\theta+\phi)}.$$

This shows the following:

Proposition 3.3. *Assume that V, W are non-singular matrices and convex cone generated by ϕ_V and ϕ^W is a face of $\mathbb{P}[M_2, M_2]$. Then this face has a unital map in the interior if and only if V and W are of the forms*

$$V^{\pm} = T \begin{pmatrix} \frac{1}{\sqrt{\delta}}|y| & \frac{\sqrt{\gamma}}{\sqrt{\delta}}|w|e^{i\psi^{\pm}} \\ 0 & |y|e^{i\theta} \end{pmatrix} S^*, \qquad W^{\pm} = \overline{T} \begin{pmatrix} \frac{1}{\sqrt{\delta}}|w| & -\frac{\sqrt{\gamma}}{\sqrt{\delta}}|y|e^{i\psi^{\pm}} \\ 0 & |w|e^{i\phi} \end{pmatrix} S^*,$$

respectively, where

$$0 < |y|^2 + |w|^2 = \delta \leq 1, \qquad \gamma + \delta = 1, \qquad (e^{i\psi^{\pm}})^2 = -e^{i(\theta+\phi)}, \tag{12}$$

and S and T are unitaries.

We use the convention

$$\phi + \frac{\pi}{2} \leq \psi^+ < \phi + \frac{3}{2}\pi, \qquad \phi - \frac{\pi}{2} \leq \psi^- < \phi + \frac{\pi}{2}.$$

Then we have

$$\psi^{\pm} = \begin{cases} \dfrac{\theta+\phi}{2} \pm \dfrac{\pi}{2}, & \text{if } 0 \leq \dfrac{\theta-\phi}{2} < \pi, \\ \dfrac{\theta+\phi}{2} \mp \dfrac{\pi}{2}, & \text{if } -\pi \leq \dfrac{\theta-\phi}{2} < 0. \end{cases} \tag{13}$$

It is clear that a face of type VII does not have a unital map at all. The remaining cases of types VIII and IX are extreme rays generated by ϕ_U or ϕ^U, and these faces have unital maps in the interiors if and only if U is a unitary.

4. Relations between faces

We recall [5] that every maximal face of $\mathbb{P}_I$ is of the form $F[x, y] \cap \mathbb{P}_I$ for a $(x, y) \in \mathbb{CP}^1 \times \mathbb{CP}^1$, where $\mathbb{CP}^1$ is the one dimensional complex projective plane, and any two maximal faces are affine isomorphic each other. Therefore, we consider the

single maximal face $\mathbb{F} = F[e_2, e_1] \cap \mathbb{P}_I$, which is associated with the pair (D, E) of subspaces

$$D = E = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{C} \right\}.$$

Note that D and E are orthogonal to the matrix $e_2 e_1^* = e_{21}$. Throughout this section, we identify a linear map $\phi : M_2 \to M_2$ with the the 4×4 matrix;

$$\phi = \begin{pmatrix} \phi(e_{11}) & \phi(e_{12}) \\ \phi(e_{21}) & \phi(e_{22}) \end{pmatrix}.$$

We begin with the faces of types VI. Note that a face of type VI in $F[e_2, e_1]$ is two dimensional convex cone generated by ϕ_{V_0} and ϕ^{W_0}, where V_0 and W_0 are defined in (11) with the condition (12). Hence, we see that this face gives rise to an extreme point if and only if V_0 is not an unitary if and only if W_0 is not an unitary if and only if $\gamma \neq 0$. In this case, we see by a direct calculation that the corresponding extreme point $\phi_{V_0} + \phi^{W_0}$ is given by

$$\phi_{V_0} + \phi^{W_0} = \begin{pmatrix} 1 & 0 & 0 & \alpha \\ 0 & \gamma & \overline{\beta} & \varepsilon^{\pm} \\ 0 & \beta & 0 & 0 \\ \overline{\alpha} & \overline{\varepsilon^{\pm}} & 0 & \delta \end{pmatrix} =: \mathbb{E}^{\pm}_{\alpha,\beta},$$

with

$$\gamma = 1 - \delta, \qquad \alpha = \frac{|y|}{\sqrt{\delta}} y, \qquad \beta = \frac{|w|}{\sqrt{\delta}} w, \qquad \varepsilon^{\pm} = 2\frac{\sqrt{\gamma}}{\sqrt{\delta}} |yw| e^{i(\theta - \psi^{\pm})}. \tag{14}$$

Note that the number δ is given by

$$|\alpha| + |\beta| = \frac{1}{\sqrt{\delta}}(|y|^2 + |w|^2) = \sqrt{\delta}, \qquad 0 < \delta < 1$$

by (10), and the relation (14) defines a one-to-one correspondence between (y, w) with $0 < |y|^2 + |w|^2 < 1$ and (α, β) with $0 < |\alpha| + |\beta| < 1$. The reverse relations are given by

$$y = \delta^{1/4} |\alpha|^{-1/2} \alpha, \qquad w = \delta^{1/4} |\beta|^{-1/2} \beta.$$

We also note that

$$|\varepsilon^{\pm}|^2 = 4\frac{\gamma}{\delta}|yw|^2 = 2\frac{\gamma}{\delta}\left[(|y|^2 + |w|^2)^2 - (|y|^4 + |w|^4)\right] = 2\gamma(\delta - |\alpha|^2 - |\beta|^2),$$

and

$$\arg \varepsilon^{\pm} = \begin{cases} \dfrac{\theta - \phi}{2} \mp \dfrac{\pi}{2}, & \text{if } 0 \leq \dfrac{\theta - \phi}{2} < \pi, \\ \dfrac{\theta - \phi}{2} \pm \dfrac{\pi}{2}, & \text{if } -\pi \leq \dfrac{\theta - \phi}{2} < 0 \end{cases} \tag{15}$$

by (13).

When $\alpha = 0$ or $\beta = 0$, we define

$$E_{\alpha,\beta} = \begin{pmatrix} 1 & 0 & 0 & \alpha \\ 0 & \gamma & \overline{\beta} & 0 \\ 0 & \beta & 0 & 0 \\ \overline{\alpha} & 0 & 0 & \delta \end{pmatrix}$$

with

$$\sqrt{\delta} = |\alpha| + |\beta|, \qquad \gamma = 1 - \delta,$$

and continue to consider the faces of type IV with $\gamma = 0$. Recall that $\gamma = 0$ if and only if V_0 and W_0 in (11) are unitaries. In this case, it is clear that thses faces give rise to the line segments;

$$\mathbb{L}_{\alpha,\beta} = \operatorname{conv}\{\mathbb{E}_{\alpha,0}, \mathbb{E}_{0,\beta}\} \qquad \text{with} \quad |\alpha| = |\beta| = 1.$$

It is also clear that the map ϕ_U and ϕ^U in faces of type VIII and IX also give rise to extreme points

$$\mathbb{E}_{\alpha,0} \quad \text{and} \quad \mathbb{E}_{0,\beta} \qquad \text{with} \quad |\alpha| = |\beta| = 1,$$

respectively.

We proceed to consider the faces of types III, IV and V. By Proposition 3.2, we see that every face of this type is associated with a pair $(D_{\lambda,\mu}, E_{\lambda,\mu})$ of subspaces

$$D_{\lambda,\mu} = \operatorname{span}\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right\}, \qquad E_{\lambda,\mu} = \operatorname{span}\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \mu \end{pmatrix} \right\},$$

where $\lambda, \mu \in \mathbb{C}$ with $0 < |\lambda| < 1$ and $|\mu| = 1$. Note that

$$(D_{\lambda,\mu}, E_{\lambda,\mu}) = (D_{\lambda',\mu'}, E_{\lambda',\mu'}) \iff (\lambda, \mu) = (\lambda', \mu').$$

We write this face by $\mathbb{F}_{\lambda,\mu}$:

$$\mathbb{F}_{\lambda,\mu} = \sigma(D_{\lambda,\mu}, E_{\lambda,\mu}) \cap \mathbb{P}_I = \operatorname{conv}\{\Phi_{D_{\lambda,\mu}}, \Phi^{E_{\lambda,\mu}}\} \cap \mathbb{P}_I.$$

If $\phi = \sum_i \phi_{A_i} \in \Phi_{D_{\lambda,\mu}}$ is unital with $A_i = \begin{pmatrix} a_i\lambda^{-1} & b_i \\ 0 & a_i \end{pmatrix}$ then we have

$$\sum_i |a_i|^2 = |\lambda|^2, \qquad \sum_i |b_i|^2 = 1 - |\lambda|^2, \qquad \sum_i a_i\overline{b_i} = 0,$$

from the relation $\sum A_i^* A_i = I$. It follows that $\phi = \mathbb{E}_{\lambda,0}$ is a unique unital map in the face $\Phi_{D_{\lambda,\mu}}$, and this gives rise to a unique extreme point. By the same argument, we also have a unique extreme point $\mathbb{E}_{0,\lambda\mu}$ in the face $\Phi^{E_{\lambda,\mu}}$. We conclude that faces of types IV and V give rise to extreme points

$$\mathbb{E}_{\alpha,0} \quad \text{and} \quad \mathbb{E}_{0,\beta} \qquad \text{with} \quad 0 < |\alpha|, |\beta| < 1,$$

respectively.

We have shown that the face $\mathbb{F}_{\lambda,\mu}$ has extreme points $\mathbb{E}_{\lambda,0}$ and $\mathbb{E}_{0,\lambda\mu}$. Because the corresponding spaces $D_{\lambda,\mu}$ and $E_{\lambda,\mu}$ do not have an unitary, it is now clear that every proper face $\mathbb{F}_{\lambda,\mu}$ which is neither $\mathbb{E}_{\lambda,0}$ nor $\mathbb{E}_{0,\lambda\mu}$ with $0 < |\alpha| < 1$ comes out

from the faces of types VI with $\gamma \neq 0$. Note that $\mathbb{E}^{\pm}_{\alpha,\beta} \in \mathbb{F}_{\lambda,\mu}$ if and only if $V_0 \in D_{\lambda,\mu}$ and $W_0 \in E_{\lambda,\mu}$, where V_0, W_0 are given in (11). This is the case if and only if

$$\frac{1}{\sqrt{\delta}} = \lambda^{-1} e^{i\theta} = \overline{\mu}\lambda^{-1} e^{i\phi}.$$

Since $\alpha = |\alpha| e^{i\theta}$, $\beta = |\beta| e^{i\phi}$, we see that $\mathbb{E}^{\pm}_{\alpha,\beta}$ is an extreme point of $\mathbb{F}_{\lambda,\mu}$ if and only if the following relations

$$|\alpha| + |\beta| = |\lambda|, \quad \arg\alpha = \arg\lambda, \quad \arg\beta = \arg\mu + \arg\lambda, \quad \alpha, \beta \neq 0 \tag{16}$$

hold. We conclude that every proper face of $\mathbb{F}_{\lambda,\mu}$ is an extreme point. For each fixed (α, β) with the relations (16) there are exactly two extreme points $\mathbb{E}^{+}_{\alpha,\beta}$ and $\mathbb{E}^{-}_{\alpha,\beta}$ in the face $\mathbb{F}_{\lambda,\mu}$, and we have

$$\frac{1}{2}\mathbb{E}^{+}_{\alpha,\beta} + \frac{1}{2}\mathbb{E}^{-}_{\alpha,\beta} = \frac{|\alpha|}{|\lambda|}\mathbb{E}_{\lambda,0} + \frac{|\beta|}{|\lambda|}\mathbb{E}_{0,\lambda\mu}, \qquad \mathbb{E}_{\lambda,0} = \lim_{\beta\to 0} \mathbb{E}^{\pm}_{\alpha,\beta}, \qquad \mathbb{E}_{0,\lambda\mu} = \lim_{\alpha\to 0} \mathbb{E}^{\pm}_{\alpha,\beta}.$$

Finally, we consider the faces of types II. By Proposition 3.1, we see that a face of this type corresponds to the pair $(D_{a,b}, E_{a,b})$ of subspaces

$$D_{a,b} = \operatorname{span}\left\{\begin{pmatrix} a & -r \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -s \\ 0 & b \end{pmatrix}\right\} = \{e_2 e_1^*, \xi\eta^*\}^{\perp},$$

$$E_{a,b} = \operatorname{span}\left\{\begin{pmatrix} a & -r \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -s \\ 0 & \overline{b} \end{pmatrix}\right\} = \{e_2 e_1^*, \overline{\xi}\eta^*\}^{\perp},$$

where $\xi = \begin{pmatrix} \overline{b} \\ s \end{pmatrix}$ and $\eta = \begin{pmatrix} r \\ a \end{pmatrix}$, with the restrictions

$$r^2 + |a|^2 = s^2 + |b|^2 = 1$$
$$0 \leq |b| < |a| \leq 1 \quad \text{or} \quad |a| = |b| = 0 \quad \text{or} \quad |a| = |b| = 1.$$

We denote this face by $\mathbb{G}_{a,b}$:

$$\mathbb{G}_{a,b} = \sigma(D_{a,b}, E_{a,b}) \cap \mathbb{P}_I.$$

Recall that if $|a| = |b| = 0$ then $F[e_2, e_1] = F[\xi, \eta]$ and $\mathbb{G}_{a,b} = F[e_2, e_1] \cap \mathbb{P}_I = \mathbb{F}$. Therefore, we may ignore this case. Note that any proper face of $\mathbb{G}_{a,b}$ comes out from faces of types VI, VIII and IX. We see that $\mathbb{G}_{a,b}$ has an extreme point arising from a face of type VIII (respectively of type IX) if and only if $D_{a,b}$ (respectively $E_{a,b}$) has an unitary if and only if there are $p, q \in \mathbb{C}$ such that

$$|ap| = 1, \qquad |bq| = 1, \qquad rp + sq = 0$$

if and only if $r = s$ if and only if $|a| = |b| = 1$.

In the case of $|a| = |b| = 1$, we see that $D_{a,b} = E_{a,b} = \operatorname{span}\{e_{11}, e_{22}\}$ is independent of the choice of a and b. We denote this face by $\mathbb{H}$:

$$\mathbb{H} = \sigma(D, D) \cap \mathbb{P}_I,$$

with $D = \operatorname{span}\{e_{11}, e_{22}\}$. Since $e_2 e_1^*$ and $e_1 e_2^*$ are orthogonal to D, we see that

$$\mathbb{H} = F[e_2, e_1] \cap F[e_1, e_2] \cap \mathbb{P}_I.$$

Since $\phi(e_{22}) = e_{22}$ for every $\phi \in \mathbb{H}$, we also see that no extreme points of the forms $\mathbb{E}^{\pm}_{\alpha,\beta}$ belong to $\mathbb{H}$, and every maximal face of $\mathbb{H}$ is of the form $\mathbb{L}_{\alpha,\beta}$. Note that $\mathbb{L}_{\alpha,\beta} \subset \mathbb{H}$ if and only if

$$\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \in D, \qquad \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} \in D.$$

Since this is always the case, we see that every $\mathbb{L}_{\alpha,\beta}$ is a maximal face of $\mathbb{H}$. We conclude that

$$\mathbb{H} = \left\{ \begin{pmatrix} 1 & 0 & 0 & \alpha \\ 0 & 0 & \overline{\beta} & 0 \\ 0 & \beta & 0 & 0 \\ \overline{\alpha} & 0 & 0 & 1 \end{pmatrix} : |\alpha| + |\beta| \le 1 \right\},$$

and $\mathbb{H}$ is affine isomorphic to the unit ball of the two dimensional ℓ^1-space over the complex field.

Now, we consider the case of $|b| = 0$ (and $a \neq 0$ of course). In this case, $\mathbb{E}^{\pm}_{\alpha,\beta} \in \mathbb{G}_{a,0}$ implies that $y = w = 0$ in (11), which is absurd. This shows that $\mathbb{G}_{a,0}$ has no proper face and is an extreme point in itself. It is easy to see that this extreme point is independent of the choice of a with $0 < |a| \le 1$, and $\mathbb{G}_{a,0} = \mathbb{E}_{0,0}$.

Next, we assume that $0 < |b| < |a| \le 1$, and proceed to determine which extreme point $\mathbb{E}^{\pm}_{\alpha,\beta}$ belongs to the face $\mathbb{G}_{a,b}$. First of all, it is easy to see that

$$(D_{a,b}, E_{a,b}) = (D_{a',b'}, E_{a',b'}) \iff \begin{cases} (a,b) = (a',b'), & \text{if } 0 < |b| < |a| < 1, \\ b = b' & \text{if } 0 < |b| < |a| = 1. \end{cases}$$

Note that $\mathbb{E}^{\pm}_{\alpha,\beta} \in \mathbb{G}_{a,b}$ if and only if $V_0 \in D_{a,b}$ and $W_0 \in E_{a,b}$ with V_0 and W_0 in (11) if and only if the equations

$$\begin{aligned} &\frac{1}{\sqrt{\delta}}\frac{r}{a}|\alpha|^{1/2}e^{-i\theta/2} + \frac{s}{b}|\alpha|^{1/2}e^{i\theta/2} = -\frac{\sqrt{\gamma}}{\sqrt{\delta}}|\beta|^{1/2}e^{i(\psi^{\pm}-\theta/2)} = \mp i\frac{\sqrt{\gamma}}{\sqrt{\delta}}|\beta|^{1/2}e^{i\phi/2} \\ &\frac{1}{\sqrt{\delta}}\frac{r}{a}|\beta|^{1/2}e^{-i\phi/2} + \frac{s}{\overline{b}}|\beta|^{1/2}e^{i\phi/2} = \frac{\sqrt{\gamma}}{\sqrt{\delta}}|\alpha|^{1/2}e^{i(\psi^{\pm}-\phi/2)} = \pm i\frac{\sqrt{\gamma}}{\sqrt{\delta}}|\alpha|^{1/2}e^{i\theta/2} \end{aligned} \tag{17}$$

hold by (13) when $0 \le \dfrac{\theta-\phi}{2} < \pi$. If $-\pi \le \dfrac{\theta-\phi}{2} < 0$ then $\mathbb{E}^{\pm}_{\alpha,\beta} \in \mathbb{G}_{a,b}$ if and only if $\mp$ is replaced by $\pm$ and vice versa in (17). We put $|\alpha|^{1/2}e^{i\theta/2}$ in the second equation into the first equation to get

$$\frac{r^2}{|a|^2} - \delta\frac{s^2}{|b|^2} + \gamma = 0, \qquad \text{or equivalently} \qquad \frac{|b|}{|a|} = \sqrt{\delta} = |\alpha| + |\beta|, \tag{18}$$

since $\gamma+\delta = 1$. We also note that the first and second equations in (17) are equivalent under the relations in (18).

We consider the case $0 < |b| < |a| = 1$ or equivalently $r = 0$. In this case, we have $\sqrt{\delta} = |\alpha| + |\beta| = |b|$ and $\gamma = 1 - |b|^2 = s^2$, and so it follows from the first

equation of (17) that

$$\frac{s^2}{|b|^2}|\alpha| = \frac{\gamma}{\delta}|\beta| = \frac{s^2}{|b|^2}|\beta|, \qquad \frac{\theta-\phi}{2} = \begin{cases} \arg b \mp \dfrac{\pi}{2}, & \text{if } \ 0 \le \dfrac{\theta-\phi}{2} < \pi, \\ \arg b \pm \dfrac{\pi}{2}, & \text{if } \ -\pi \le \dfrac{\theta-\phi}{2} < 0. \end{cases}$$

If $0 \le \dfrac{\theta-\phi}{2} < \pi$ then we have $\mathbb{E}^+_{\alpha,\beta} \in \mathbb{G}_{1,b}$ if and only if $0 \le \arg b - \dfrac{\pi}{2} < \pi$ if and only if $\dfrac{\pi}{2} \le \arg b < \dfrac{3}{2}\pi$, and the same conclusion holds when $-\pi \le \dfrac{\theta-\phi}{2} < 0$. Therefore, we see that

$$\mathbb{E}^+_{\alpha,\beta} \in \mathbb{G}_{1,b} \iff \frac{\pi}{2} \le \arg b < \frac{3}{2}\pi, \qquad \mathbb{E}^-_{\alpha,\beta} \in \mathbb{G}_{1,b} \iff -\frac{\pi}{2} \le \arg b < \frac{\pi}{2}.$$

in any cases, and we have

$$|\alpha| = |\beta| = \frac{1}{2}|b|, \qquad \arg\alpha - \arg\beta = 2\arg b + \pi. \tag{19}$$

Now, assume that $0 < |b| < |a| < 1$. Taking the square of the absolute value of the first equation in (17), we have

$$r^2 + s^2 + 2\,\mathrm{Re}\,\frac{|ab|}{\overline{a}b} rs\, e^{i\theta} = (|a|^2 - |b|^2)\frac{|\beta|}{|\alpha|}. \tag{20}$$

Therefore, there exist θ satisfying (20) if and only if $r^2 + s^2 + 2rs \ge (|a|^2 - |b|^2)\dfrac{|\beta|}{|\alpha|}$ if and only if the relation $\dfrac{|\beta|}{|\alpha|} \le \dfrac{(r+s)^2}{s^2 - r^2} = \dfrac{s+r}{s-r}$ holds. By the same argument for the second equation of (17), we also have $\dfrac{|\alpha|}{|\beta|} \le \dfrac{s+r}{s-r}$. Hence, we conclude that if $\mathbb{E}^{\pm}_{\alpha,\beta}$ belongs to $\mathbb{G}_{a,b}$ then

$$|\alpha| + |\beta| = \frac{|b|}{|a|}, \qquad \frac{s-r}{s+r} \le \frac{|\beta|}{|\alpha|} \le \frac{s+r}{s-r}. \tag{21}$$

Let $(|\alpha|, |\beta|)$ be fixed with the relations in (18) and (21), and write

$$A(a,b,\alpha) = \frac{1}{\sqrt{\delta}}\frac{r}{a}|\alpha|^{1/2}e^{-i\theta/2} + \frac{s}{b}|\alpha|^{1/2}e^{i\theta/2}.$$

Then there exists a θ (recall again that we wrote $\theta = \arg\alpha$ and $\phi = \arg\beta$) such that $|A(a,b,\alpha)| = \dfrac{\sqrt{\gamma}}{\sqrt{\delta}}|\beta|^{1/2}$. Therefore, there exist a unique ϕ such that

$$A(a,b,\alpha) = \begin{cases} -i\dfrac{\sqrt{\gamma}}{\sqrt{\delta}}|\beta|^{1/2}e^{i\phi/2}, & \text{if } 0 \le \dfrac{\theta-\phi}{2} < \pi, \\ +i\dfrac{\sqrt{\gamma}}{\sqrt{\delta}}|\beta|^{1/2}e^{i\phi/2}, & \text{if } \ -\pi \le \dfrac{\theta-\phi}{2} < 0 \end{cases}$$

or there exists a unique ϕ such that

$$A(a,b,\alpha)=\begin{cases}+i\dfrac{\sqrt{\gamma}}{\sqrt{\delta}}|\beta|^{1/2}e^{i\phi/2}, & \text{if } 0\le\dfrac{\theta-\phi}{2}<\pi,\\ -i\dfrac{\sqrt{\gamma}}{\sqrt{\delta}}|\beta|^{1/2}e^{i\phi/2}, & \text{if } -\pi\le\dfrac{\theta-\phi}{2}<0,\end{cases}$$

and so we see that there is β with $\mathbb{E}^+_{\alpha,\beta}\in\mathbb{G}_{a,b}$ or there is β with $\mathbb{E}^-_{\alpha,\beta}\in\mathbb{G}_{a,b}$. Furthermore, both cases do not occur simultaneously. By the continuity, we see that the extreme points of $\mathbb{G}_{a,b}$ consist of $\mathbb{E}^+_{\alpha,\beta}$, or consists of $\mathbb{E}^-_{\alpha,\beta}$. If the both strict inequalities hold in (21) then two pairs (α,β) satisfy $\mathbb{E}^\pm_{\alpha,\beta}\in\mathbb{G}_{a,b}$. On the other hand, if the one equality holds in (21) then there is a unique pair (α,β) satisfying $\mathbb{E}^\pm_{\alpha,\beta}\in\mathbb{G}_{a,b}$.

Now, we summarize the above discussion as follows:

Theorem 4.1. *Every proper face of* $\mathbb{F}=F[e_2,e_1]\cap\mathbb{P}_I[M_2,M_2]$ *is one of the following:*

$$\begin{aligned}&\mathbb{F}_{\lambda,\mu}\ (0<|\lambda|<1,\ |\mu|=1)\\ &\mathbb{G}_{a,b}\ (0<|b|<|a|<1\quad\text{or}\quad 0<|b|<1,\ a=1)\\ &\mathbb{H}\\ &\mathbb{L}_{\alpha,\beta}\ (|\alpha|=|\beta|=1)\\ &\mathbb{E}^\pm_{\alpha,\beta}\ (0<|\alpha|+|\beta|<1)\\ &\mathbb{E}_{\alpha,\beta}\ (0\le|\alpha|+|\beta|\le1,\quad \alpha=0\text{ or }\beta=0)\end{aligned}$$

We also have the following:

(i) *Maximal faces of* $\mathbb{F}$ *consist of* $\mathbb{F}_{\lambda,\mu},\mathbb{G}_{a,b},\mathbb{H}$ *and* $\mathbb{E}_{0,0}$.
(ii) *Maximal faces of* $\mathbb{F}_{\lambda,\mu}$ *consist of extreme points* $\mathbb{E}_{\lambda,0},\mathbb{E}_{0,\lambda\mu}$ *and* $\mathbb{E}^\pm_{\alpha,\beta}$ *with* (16).
(iii) *Maximal faces of* $\mathbb{G}_{1,b}$ *consist of extreme points* $\mathbb{E}^+_{\alpha,\beta}$ *or* $\mathbb{E}^-_{\alpha,\beta}$ *with* (19).
(iv) *Maximal faces of* $\mathbb{G}_{a,b}$ $(|a|<1)$ *consist of extreme points* $\mathbb{E}^+_{\alpha,\beta}$ *or* $\mathbb{E}^-_{\alpha,\beta}$ *with* (21) *and* (18).
(v) *Maximal faces of* $\mathbb{H}$ *consist of* $\mathbb{L}_{\alpha,\beta}$.
(vi) *Maximal faces of* $\mathbb{L}_{\alpha,\beta}$ *consist of two extreme points* $\mathbb{E}_{\alpha,0}$ *and* $\mathbb{E}_{0,\beta}$.
(vii) $\mathbb{E}^\pm_{\alpha,\beta}$ *and* $\mathbb{E}_{\alpha,\beta}$ *are extreme points of* $\mathbb{F}$.

We remark here that extreme points $\mathbb{E}^\pm_{\alpha,\beta}$ and $\mathbb{E}_{\alpha,\beta}$ exhaust all extreme points discussed in [11], Theorem 8.2. Since $\phi(e_{22})=\delta e_{22}$ for any $\phi\in\mathbb{F}$, we may slice $\mathbb{F}$ by $\delta\in[0,1]$. For a fixed $\delta\in[0,1]$, put

$$\mathbb{S}_\delta=\{\phi\in\mathbb{F}:\phi(e_{22})=\delta e_{22}\}.$$

It is clear that

$$\mathbb{S}_0=\mathbb{E}_{0,0},\qquad \mathbb{S}_1=\mathbb{H}\supset\mathbb{L}_{\alpha,\beta}.$$

We also have

$$\mathbb{F}_{\lambda,\mu} \subset \mathbb{S}_\delta \iff |\lambda| = \sqrt{\delta}, \qquad \mathbb{G}_{a,b} \subset \mathbb{S}_\delta \iff \frac{|b|}{|a|} = \sqrt{\delta},$$

$$\mathbb{E}^{\pm}_{\alpha,\beta}, \mathbb{E}_{\alpha,\beta} \subset \mathbb{S}_\delta \iff |\alpha| + |\beta| = \sqrt{\delta}$$

For a fixed $\delta \in (0,1)$, we see that maximal faces of $\mathbb{S}_\delta$ consist of unexposed faces $\mathbb{F}_{\lambda,\mu}$'s and exposed faces $\mathbb{G}_{a,b}$'s, which are parametrized by

$$I = \{(\lambda,\mu) \in \mathbb{C}^2 : |\lambda| = \sqrt{\delta},\ |\mu| = 1\}$$
$$J = \{(a,b) \in \mathbb{C}^2 : 0 < |a| < 1, |b| = \sqrt{\delta}|a| \quad \text{or} \quad a = 1, |b| = \sqrt{\delta}\},$$

respectively. Note that I is the torus and J is the solid torus without boundary. It is also apparent that $\mathbb{F}_{\lambda,\mu}$ and $\mathbb{G}_{a,b}$ are two dimensional convex bodies and $\mathbb{S}_\delta$ is a six dimensional convex body for $\delta \in (0,1)$. Finally, we see that $\mathbb{F}$ is a seven dimensional convex body.

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