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On the Local Lifting Property for Operator Spaces

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We study the local lifting property for operator spaces. This is a natural non-commutative analogue of the Banach space local lifting property, but is very different from the local lifting property studied in C^*-algebra theory. We show that an operator space has the λ-local lifting property if and only if it is an $\mathscr{L}\Gamma_{1,\lambda}$ space. These operator space are λ-completely isomorphic to the operator subspaces of the operator preduals of von Neumann algebras, and thus λ-locally reflexive. Moreover, we show that an operator space V has the λ-local lifting property if and only if its operator space dual V^* is λ-injective.

1. INTRODUCTION

The theory of operator spaces is a recently arising area in modern analysis, which is a natural non-commutative quantization of Banach space theory. Many problems in operator spaces are naturally motivated from both Banach space theory and operator algebra theory. Recently, there has been a very important development in the local theory of operator spaces. Some local properties such as local reflexivity, exactness, finite representability, and $\mathscr{O}\mathscr{L}_p$ spaces have been intensively studied in [4, 5, 7, 10–13, 17–20, 23]. In this paper, we study the local lifting property for operator

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spaces. This property is a natural non-commutative analogue of the Banach space local lifting property, but is very different from the local lifting property studied in C^*-algebra theory. In fact, we will show in Section 6 that a C^*-algebra has the local lifting property if and only if it is one dimensional.

Let us recall from the Banach space theory that a Banach space V is said to have the *lifting property* if given any Banach spaces $W \subseteq Y$, a linear contraction $\varphi: V \to Y/W$, and $\varepsilon > 0$, there is a bounded linear map $\tilde{\varphi}: V \to Y$ such that $\|\tilde{\varphi}\| < 1 + \varepsilon$ and $q \circ \tilde{\varphi} = \varphi$, where $q: Y \to Y/W$ denotes the quotient map from Y onto Y/W. Grothendieck [15] showed that a Banach space V has the lifting property if and only if V is isometrically isomorphic to $\ell_1(I)$ for some index set I. It is known from the Grothendieck's result that given a measure space $(X, \mathscr{M}, \mu)$, the Banach space $L_1(X, \mathscr{M}, \mu)$ need not have the lifting property. But, one can show that $L_1(X, \mathscr{M}, \mu)$ has the *local lifting property* in the sense that given any Banach spaces $W \subseteq Y$ and a contraction $\varphi: L_1(X, \mathscr{M}, \mu) \to Y/W$, for every finite dimensional subspace E of $L_1(X, \mathscr{M}, \mu)$ and $\varepsilon > 0$, there exists a bounded linear map $\tilde{\varphi}: E \to Y$ such that $\|\tilde{\varphi}\| < 1 + \varepsilon$ and $q \circ \tilde{\varphi} = \varphi_{|E}$, i.e., we have the commutative diagram

$$\begin{tikzcd} & & Y \arrow[d, "q"] \\ E \arrow[r, hook] \arrow[urr, "\tilde{\varphi}"] & L_1(\mu) \arrow[r, "\varphi"] & Y/W. \end{tikzcd}$$

This is mainly due to the local structure of $L_1(X, \mathscr{M}, \mu)$. A Banach space V is called an $\mathscr{L}_{1,\lambda}$ space for some $\lambda > 1$ if for every finite dimensional subspace E of V, there exists a finite dimensional subspace F of V containing E such that the Banach–Mazur distance $d(F, \ell_1(n)) < \lambda$, where $n = \dim F$. It was shown in [22] that $L_1(X, \mathscr{M}, \mu)$ spaces are exactly $\mathscr{L}_{1,1+\varepsilon}$ spaces for every $\varepsilon > 0$. Therefore, the lifting property of $\ell_1(n)$ spaces implies the local lifting property of $L_1(X, \mathscr{M}, \mu)$.

The operator space version of the lifting property has been studied by Blecher [1] under the name of *projectivity*. For our convenience, we will keep using the name "lifting property" instead of "projectivity" in this paper. We recall from [1] that an operator space V has the *lifting property* (or simply LP) if given any operator spaces $W \subseteq Y$, a complete contraction $\varphi: V \to Y/W$ and $\varepsilon > 0$, there exists a completely bounded linear map $\tilde{\varphi}: V \to Y$ such that $q \circ \tilde{\varphi} = \varphi$ and $\|\tilde{\varphi}\|_{cb} < 1 + \varepsilon$, where $q: Y \to Y/W$ denotes the complete quotient map from Y onto Y/W.

For any index set I, the space $T_I = T(\ell_2(I))$ of all trace class operators on $\ell_2(I)$ is the natural non-commutative analog of $\ell_1(I)$, and T_I becomes an operator space with the canonical operator space matrix norm obtained

by identifying T_I with the operator predual of the operator space $B(\ell_2(I))$ of all bounded linear operators. It was shown in [1] that for each positive $n \in \mathbb{N}$, T_n has the LP since we have the complete isometries

$$\begin{aligned} CB(T_n, Y/W) &= M_n(Y/W) = M_n(Y)/M_n(W) \\ &= CB(T_n, Y)/CB(T_n, W). \end{aligned}$$

However, in contrast to the Banach space theory, T_I does not have the LP for any infinite index set I.

It is known from [11] that the operator spaces T_I have some nice local property. Indeed, given any index set I, T_I is an $\mathscr{T}$ space, i.e., for every finite dimensional subspace E of T_I and $\varepsilon > 0$, there exists a finite dimensional subspace F of T_I containing E such that the completely bounded Banach–Mazur distance $d_{cb}(F, T_n) < 1 + \varepsilon$ for some T_n. Due to this nice local property, we can show that T_I has the operator space version of the local lifting property (see Lemma 3.1). Using this result, we prove in Theorem 3.2 that for some $\lambda \geqslant 1$, an operator space has the λ-local lifting property (λ-LLP) if and only if it is an $\mathscr{L}\Gamma_{1,\lambda}$ space introduced in Junge [18]. Moreover, we show that every $\mathcal{O}\mathscr{L}_{1,\lambda^+}$ space has the λ-LLP, and every operator space with the λ-LLP must be λ-finitely representable in $\{T_n\}_{n \in \mathbb{N}}$, λ-completely isomorphic to an operator subspace of the operator predual of a von Neumann algebra, and λ-locally reflexive.

The λ-LLP is closely related to the λ-injectivity. In Section 4 we study some equivalent conditions of λ-injectivity for dual operator spaces and show in Section 5 that an operator space V has the λ-LLP if and only if its operator dual V^* is λ-injective. In Section 6, we show that the only C^*-algebra having the LLP is the one-dimensional C^*-algebra. We also show that if an operator space V has the λ-weak expectation property and is locally reflexive, then V^* has the λ-LLP. Therefore, the operator dual of a separable nuclear operator space has the LLP.

2. SOME PRELIMINARY NOTIONS

For the convenience of the readers, we recall some of basic notations and terminologies in operator spaces. the details can be found in [1, 2, 7–10, 23, 26]. Given a Hilbert space H, we let $B(H)$ denote the space of all bounded linear operators on H. For each natural number $n \in \mathbb{N}$, there is a canonical norm $\|\cdot\|_n$ on the $n \times n$ matrix space $M_n(B(H))$ given by identifying $M_n(B(H))$ with $B(H^n)$. We call this family of norms $\{\|\cdot\|_n\}$ an *operator space matrix norm* on $B(H)$. An *operator space* V is a norm closed subspace of some $B(H)$ equipped with the distinguished operator space matrix norm inherited from $B(H)$. An abstract matrix norm characterization

of operator spaces was given in [25] (see [9] for a simple proof). The morphisms in the category of operator spaces are the completely bounded linear maps. Given operators spaces V and W, a linear map $\varphi: V \to W$ is *completely bounded* if the corresponding linear mappings $\varphi_n: M_n(V) \to M_n(W)$ defined by $\varphi_n([x_{ij}]) = [\varphi(x_{ij})]$ are uniformly bounded, i.e.

$$\|\varphi\|_{cb} = \sup\{\|\varphi_n\|: n \in \mathbb{N}\} < \infty.$$

A map φ is *completely contractive* (respectively, *completely isometric*, *a complete quotient*) if $\|\varphi\|_{cb} \leqslant 1$ (respectively, for each $n \in \mathbb{N}$, φ_n is an isometry, a quotient map), and we say that φ is a *completely bounded isomorphism* if it is a completely bounded linear isomorphism with a completely bounded inverse. We denote by $CB(V, W)$ the space of all completely bounded maps from V into W. It is known that $CB(V, W)$ is an operator space with the operator space matrix norm given by identifying $M_n(CB(V, W)) = CB(V, M_n(W))$. In particular, if V is an operator space, then its dual space V^* is an operator space with operator space matrix norm given by the identification $M_n(V^*) = CB(V, M_n)$ (see [2, 8]).

Given operator spaces V and W, and a completely bounded map φ: $V \to W$, the corresponding dual map φ^*: $W^* \to V^*$ is completely bounded with $\|\varphi^*\|_{cb} = \|\varphi\|_{cb}$. Furthermore, φ: $V \to W$ is a completely isometric injection if and only if φ^* is a complete quotient map. On the other hand, if φ: $V \to W$ is a surjection, then φ is a compete quotient map if and only if φ^* is a completely isometric injection. These facts may be proved by imitating the Banach space proofs, using the *matrix parings*

$$M_n(V) \times M_n(V^*) \to M_{n \times n}: (v, f) \mapsto \langle\langle v, f \rangle\rangle = [f_{k,l}(v_{i,j})],$$

and the relations

$$\|f\| = \sup\{\|\langle\langle v, f \rangle\rangle\| : \|v\| \leqslant 1, v \in M_n(V^*)\}$$

(which coincides with the usual definition of the operator norm on V^*) and

$$\|v\| = \sup\{\|\langle\langle v, f \rangle\rangle\| : \|f\| \leqslant 1, f \in M_n(V^*)\}$$

(see [9]).

We use the notations $V \check{\otimes} W$ and $V \hat{\otimes} W$ for the injective and projective operator space tensor products (see [2, 8]). The operator space tensor products share many of the properties of the Banach space analogues. For example, we have the natural complete isometries

$$CB(V, W^*) = (V \hat{\otimes} W)^*, \qquad CB(W, V^*) = (V \hat{\otimes} W)^*,$$

and the completely isometric injection

$$V^* \check{\otimes} W \hookrightarrow CB(V, W).$$

The tensor product $\check{\otimes}$ is *injective* in the sense that if $\varphi: W \to Y$ is a completely isometric injection, then so is

$$\mathrm{id}_V \otimes \varphi : V \check{\otimes} W \to V \check{\otimes} Y.$$

On the other hand, the tensor product $\hat{\otimes}$ is *projective* in the sense that if $\varphi: W \to Y$ is a completely quotient map, then so is

$$\mathrm{id}_V \otimes \varphi : V \hat{\otimes} W \to V \hat{\otimes} Y.$$

However, the tensor product $\hat{\otimes}$ is not injective. In general, given operator spaces $W \subseteq Y$ with the inclusion map $\iota: W \hookrightarrow Y$, we get a complete contraction

$$\mathrm{id}_V \otimes \iota : V \hat{\otimes} W \to V \hat{\otimes} Y. \tag{2.1}$$

Local reflexivity for operator spaces has been studied in [4, 5, 7, 10, 12, 17]. An operator space V is called *λ-locally reflexive* (for some $\lambda \geqslant 1$) if for every finite dimensional operator space E and every complete contraction $\varphi: E \to V^{**}$, there exists a net of completely bounded maps $\varphi_\alpha: E \to V$ such that $\|\varphi_\alpha\|_{cb} \leqslant \lambda$ and $\varphi_\alpha \to \varphi$ in the point-weak* topology. An operator space is *locally reflexive* if it is 1-locally reflexive. It is known that for every finite dimensional operator space E, there is a canonical completely contractive linear isomorphism

$$\Phi: E \hat{\otimes} V^* \to (E^* \check{\otimes} V)^* \tag{2.2}$$

given by $\Phi(x \otimes f)(g \otimes y) = g(x)\, f(y)$. Taking the adjoint, we have the complete contraction

$$\Phi^*: (E^* \check{\otimes} V)^{**} \to E^* \check{\otimes} V^{**}.$$

Since $E^* \check{\otimes} V = CB(E, V)$ and $E^* \check{\otimes} V^{**} = CB(E, V^{**})$, we conclude that V is λ-locally reflexive if and only if

$$\|\Phi^{-1}\| = \|(\Phi^{-1})^*\| = \|(\Phi^*)^{-1}\| \leqslant \lambda.$$

It has been shown in [5] that operator predual of an injective von Neumann algebra is locally reflexive. Therefore, T_I and its operator subspaces are locally reflexive.

Pisier [23] introduced the operator space analogue of the Banach–Mazur distance. Given operator spaces V and W, if V is completely isomorphic to W then we define

$$d_{cb}(V, W) = \inf\{\|t\|_{cb}\,\|t^{-1}\|_{cb}\},$$

where the infimum is taken over all completely bounded isomorphisms $t\colon V \to W$, and we define $d_{cb}(V, W) = \infty$ if V is not completely isomorphic to W.

An operator space V is called λ-finitely representable in $\{T_n\}_{n\in\mathbb{N}}$ if for every finite dimensional subspace E of V and $\varepsilon > 0$, there exists a subspace F of some T_n such that $d_{cb}(E, F) < \lambda + \varepsilon$, and V is called *finitely representable* in $\{T_n\}_{n\in\mathbb{N}}$ if it is 1-finitely representable in $\{T_n\}_{n\in\mathbb{N}}$ (cf. [5]). An operator space V is called an $\mathcal{OL}_{1,\lambda}$ *space* in [13] if for every finite dimensional subspace E of V, there exists a finite dimensional subspace F of V containing E such that $d_{cb}(F, \ell_1(I, T_{n(i)})) < \lambda$, where I is a finite index set and $\ell_1(I, T_{n(i)})$ is the operator dual of the finite dimensional C^*-algebra $\ell_\infty(I, M_{n(i)})$. We say that V is an $\mathcal{OL}_{1,\lambda^+}$ *space* if it is an $\mathcal{OL}_{1,\lambda+\varepsilon}$ space for every $\varepsilon > 0$. It is clear that if V is an $\mathcal{OL}_{1,\lambda^+}$ space then V is λ-finitely representable in $\{T_n\}_{n\in\mathbb{N}}$. We will show in the next section that

$$\begin{aligned}\mathcal{OL}_{1,\lambda^+}\text{ space} &\Rightarrow \lambda\text{-local lifting property}\\ &\Rightarrow \lambda\text{-finitely representability in } \{T_n\}_{n\in\mathbb{N}}.\end{aligned}$$

3. λ-LOCAL LIFTING PROPERTY

We say that an operator space V has the *λ-local lifting property* (λ-*LLP*) for some $\lambda \geqslant 1$ if given any operator spaces $W \subseteq Y$ and a complete contraction $\varphi\colon V \to Y/W$, for every finite dimensional subspace E of V and $\varepsilon > 0$, there exists a completely bounded linear map $\tilde{\varphi}\colon E \to Y$ such that $\|\tilde{\varphi}\|_{cb} < \lambda + \varepsilon$ and $q \circ \tilde{\varphi} = \varphi_{|E}$, i.e., we have the commutative diagram

$$\begin{tikzcd} & & Y \arrow[d, "q"] \\ E \arrow[r, hook] \arrow[urr, "\tilde{\varphi}"] & V \arrow[r, "\varphi"] & Y/W, \end{tikzcd}$$

where $q\colon Y \to Y/W$ denotes the complete quotient map from Y onto Y/W. We say that an operator space V has the *local lifting property* (*LLP*) if V has the 1-local lifting property. It is clear that the LP implies the LLP. The

following lemma shows that the LLP does not imply the LP since T_I does not have the LP for infinite index set I (see discussion in Section 1).

LEMMA 3.1. *For any index set I, T_I has the LLP.*

Proof. Since T_I is a $\mathscr{T}$ space, for every finite dimensional subspace E of T_I and $\varepsilon > 0$, there exists a finite dimensional subspace F of T_I such that $E \subseteq F \subseteq T_I$ and $d_{cb}(F, T_n) < 1 + \varepsilon$ for some T_n, i.e., there exists a completely bounded isomorphism $t: F \to T_n$ with $\|t\|_{cb} \|t^{-1}\|_{cb} < 1 + \varepsilon$. Without loss of generality, we may assume that $\|t\|_{cb} \leqslant 1$ and $\|t^{-1}\|_{cb} < 1 + \varepsilon$.

Given any operator spaces $W \subseteq Y$ and a complete contraction $\varphi: T_I \to Y/W$, the map $\varphi \circ t^{-1}: T_n \to Y/W$ is completely bounded with $\|\varphi \circ t^{-1}\|_{cb} < 1 + \varepsilon$. Since T_n has the LP, there exists a completely bounded linear map $\psi: T_n \to Y$ such that $\|\psi\|_{cb} < 1 + \varepsilon$ and $\varphi \circ t^{-1} = q \circ \psi$. It follows that $\psi \circ t_{|E}: E \to Y$ is a completely bounded linear map such that

$$\|\psi \circ t_{|E}\|_{cb} \leqslant \|\psi\|_{cb} \|t\|_{cb} < 1 + \varepsilon$$

and $\varphi_{|E} = \varphi \circ t^{-1} \circ t_{|E} = q \circ \psi \circ t_{|E}$, that is, we have the commutative diagram

$$\begin{tikzcd}
 & T_n \arrow[r, "\psi"] \arrow[d, shift left, "t^{-1}"] & Y \arrow[d, "q"] \\
E \arrow[ur] \arrow[r, hook] & F \arrow[r, hook] \arrow[u, shift left, "t"] & T_I \arrow[r, "\varphi"] & Y/W.
\end{tikzcd}$$

This shows that T_I has the LLP. ∎

It is known (see [1, Corollary 3.2]) that every operator space V is a complete quotient space of some T_I space. Let $q_V: T_I \to V$ denote the complete quotient map from T_I onto V and $W = \ker q_V$. Then we have the complete isometry $V = T_I/W$.

THEOREM 3.2. *Let V be an operator space with $V = T_I/W$. Then the following are equivalent*:

(i) *V has the λ-LLP.*

(ii) *For every finite dimensional operator space E, every complete contraction $\varphi: E \to V$ and $\varepsilon > 0$, there exists a completely bounded linear map $\tilde{\varphi}: E \to T_I$ such that $\|\tilde{\varphi}\|_{cb} < \lambda + \varepsilon$ and the diagram*

$$\begin{tikzcd} & T_I \arrow[d, "q_V"] \\ E \arrow[ur, "\widetilde{\varphi}"] \arrow[r, "\varphi"] & V \end{tikzcd}$$

commutes.

(iii) *For every finite dimensional subspace E of V and $\varepsilon > 0$, there exists completely bounded linear maps $s: E \to T_n$ and $t: T_n \to V$ such that $\|s\|_{cb} \|t\|_{cb} < \lambda + \varepsilon$ and the diagram*

$$\begin{tikzcd} & T_n \arrow[dr, "t"] & \\ E \arrow[ur, "s"] \arrow[rr, "\iota"] & & V \end{tikzcd} \tag{3.1}$$

commutes, where $\iota: E \to V$ denotes the inclusion map.

Proof. (i) $\Rightarrow$ (ii). This is obvious by applying the definition of λ-LLP to the finite dimensional subspace $\varphi(E)$ of V.

(ii) $\Rightarrow$ (iii). For any finite dimensional subspace E of $V = T_I/W$ and $0 < \varepsilon < 1$, it follows from (ii) that there exists a completely bounded linear map $\varphi: E \to T_I$ such that $\|\varphi\|_{cb} < \lambda + \varepsilon/3$ and the diagram

$$\begin{tikzcd} & T_I \arrow[d, "q_V"] \\ E \arrow[ur, "\varphi"] \arrow[r, "\iota"] & V \end{tikzcd}$$

commutes. Since T_I is a $\mathscr{T}$ space, there exists a finite dimensional subspace F of T_I containing $\varphi(E)$ such that $d_{cb}(F, T_n) < 1 + \varepsilon/(3\lambda)$ for some T_n. Let $\psi: F \to T_n$ be a completely bounded isomorphism such that $\|\psi\|_{cb} \|\psi^{-1}\|_{cb} < 1 + \varepsilon/(3\lambda)$. We get the commutative diagram

$$\begin{tikzcd} \varphi(E) \arrow[r, hook] & F \arrow[r, hook] \arrow[d, shift right, "\psi"'] & T_I \arrow[dd, "q_V"] \\ & T_n \arrow[u, shift right, "\psi^{-1}"'] \arrow[dr, "t"] & \\ E \arrow[uu, "\varphi"] \arrow[ur, "s"] \arrow[rr, "\iota"] & & V. \end{tikzcd}$$

Then $s = \psi \circ \varphi$: $E \to T_n$ and $t = q_V \circ \psi^{-1}$: $T_n \to V$ are completely bounded maps such that $t \circ s = \iota$ and

$$\begin{aligned} \|s\|_{cb}\, \|t\|_{cb} &\leqslant \|\varphi\|_{cb}\, \|\psi\|_{cb}\, \|\psi^{-1}\|_{cb} \\ &< \|\varphi\|_{cb}\, (1 + \varepsilon/(3\lambda)) \\ &< (\lambda + \varepsilon/3)(1 + \varepsilon/(3\lambda)) < \lambda + \varepsilon. \end{aligned}$$

(iii) $\Rightarrow$ (i). For every finite dimensional operator space E of V and $0 < \varepsilon < 1$, there exists a commutative diagram (3.1) of completely bounded maps such that $\|s\|_{cb}\, \|t\|_{cb} < \lambda + \varepsilon/3$. Given operator spaces $W \subseteq Y$, we let q: $Y \to Y/W$ denote the complete quotient map from Y onto Y/W. If φ: $V \to Y/W$ is a complete contraction, then $\varphi \circ t$: $T_n \to Y/W$ has a completely bounded lifting ψ: $T_n \to Y$ such that

$$\|\psi\|_{cb} \leqslant \|t\|_{cb}\, (1 + \varepsilon/(3\lambda)).$$

Therefore, we have the commutative diagram

$$\begin{tikzcd} & T_n \arrow[rr, "\psi"] \arrow[dr, "t"] & & Y \arrow[d, "q"] \\ E \arrow[ur, "s"] \arrow[rr, "\iota"] & & V \arrow[r, "\varphi"] & Y/W. \end{tikzcd}$$

Then $\tilde{\varphi} = \psi \circ s : E \to Y$ is a completely bounded map such that $\varphi_{|E} = q \circ \tilde{\varphi}$ and

$$\begin{aligned} \|\tilde{\varphi}\|_{cb} &\leqslant \|\psi\|_{cb}\, \|s\|_{cb} \\ &\leqslant \|s\|_{cb}\, \|t\|_{cb}\, (1 + \varepsilon/(3\lambda)) \\ &< (\lambda + \varepsilon/3)(1 + \varepsilon/(3\lambda)) < \lambda + \varepsilon. \quad \blacksquare \end{aligned}$$

We note that an operator space V satisfying the condition (iii) in Theorem 3.2 is called an $\mathscr{L}\Gamma_{1,\lambda}$ *space* by Junge [18], where he showed that the operator predual R_* of a von Neumann algebra R is an $\mathscr{L}\Gamma_{1,\lambda}$ space for some $\lambda \geqslant 1$ if and only if R is an injective von Neumann algebra. We will show in Section 5 that the corresponding result is true for operator spaces, that is, an operator space V has the λ-LLP for some $\lambda \geqslant 1$ if and only if V^* is λ-injective.

COROLLARY 3.3. *If V is an $\mathcal{O}\mathscr{L}_{1,\lambda^+}$ space, then V has the λ-LLP.*

Proof. If V is an $\mathcal{OL}_{1,\lambda^+}$ space, then for every finite dimensional subspace E of V and $\varepsilon > 0$, there exists a finite dimensional subspace F of V containing E such that $d_{cb}(F, \ell_1(I, T_{n(i)})) < \lambda + \varepsilon$ for some $\ell_1(I, T_{n(i)})$ space with a finite index set I. If we put $n = \sum_{i \in I} n_i$, then it is clear that $\ell_\infty(I, M_{n(i)})$ is a block-diagonal C^*-subalgebra of M_n, and the block-diagonal projection from M_n onto $\ell_\infty(I, M_{n(i)})$ is completely contractive. Taking the adjoint, we can identify $\ell_1(I, T_{n(i)})$ with a block-diagonal operator subspace of T_n, which is completely contractively complemented in T_n. Therefore, we can conclude that V satisfies the condition (iii) in Theorem 3.2, and thus has the λ-LLP. ∎

It is worth to note that one can study a generalized notion of $\mathcal{OL}_{1,\lambda}$ spaces by considering rectangular $\ell_1(I, T_{m(i), n(i)})$ spaces, which are the operator duals of rectangular block-diagonal operator spaces $\ell_\infty(I, M_{m(i), n(i)})$ (cf. [13]). One can prove that every such generalized $\mathcal{OL}_{1,\lambda^+}$ space also has the λ-LLP. The proof is the same as that in Corollary 3.3. Therefore, we may get many examples of operator spaces having the LLP. Some typical examples are: The operator preduals of injective von Neumann algebras, the column (respectively, row) Hilbert spaces, the operator space inductive limits of $\ell_1(I, T_{m(i), n(i)})$ spaces.

COROLLARY 3.4. *If V has the λ-LLP, then V is λ-finitely representable in $\{T_n\}_{n \in \mathbb{N}}$.*

Proof. It follow from Theorem 3.2(iii) that for every finite dimensional subspace E of V and $\varepsilon > 0$, there exists a commutative diagram (3.1) such that $\|s\|_{cb} \|t\|_{cb} < \lambda + \varepsilon$. Then $F = s(E)$ is a subspace of T_n and $s: E \to F$ is a completely bounded isomorphism such that $s^{-1} = t_{|F}$ and $\|s\|_{cb} \|s^{-1}\|_{cb} < \lambda + \varepsilon$. Therefore, V is λ-finitely representable in $\{T_n\}_{n \in \mathbb{N}}$. ∎

It was shown in [5, Proposition 7.4] that if an operator space V is finitely representable in $\{T_n\}_{n \in \mathbb{N}}$, then there exists an (infinite) index set Γ, a free ultrafilter $\mathcal{U}$ on Γ and a family of $T_{n(\alpha)}$ spaces ($\alpha \in \Gamma$) such that V is completely isometric to an operator subspace of $\prod_{\mathcal{U}} T_{n(\alpha)}$. Using a similar argument as that given in [5], we can prove this for general $\lambda \geqslant 1$ case. It is also known that $\prod_{\mathcal{U}} T_{n(\alpha)}$ is completely isometric to the operator predual of a von Neumann algebra (see [5, 14, 24]). Therefore, we can conclude that if V is λ-finitely representable in $\{T_n\}_{n \in \mathbb{N}}$ then V is λ-locally reflexive. We state this in the following proposition.

PROPOSITION 3.5. *Let V be λ-finitely representable in $\{T_n\}_{n \in \mathbb{N}}$. Then there exist an index set Γ, a free ultrafilter $\mathcal{U}$ on Γ and a family of $T_{n(\alpha)}$ spaces ($\alpha \in \Gamma$) such that V is completely isomorphic to an operator subspace*

$\tilde{V}$ of $\prod_{\mathscr{U}} T_{n(\alpha)}$ with $d_{cb}(V, \tilde{V}) \leqslant \lambda$. Therefore, we conclude that V is λ-locally reflexive.

If V has the λ-LLP, we can obtain the following slightly stronger result.

PROPOSITION 3.6. *If V has the λ-LLP, then there exist an index set Γ, a free ultrafilter $\mathscr{U}$ on Γ and a family of $T_{n(\alpha)}$ spaces $(\alpha \in \Gamma)$ such that we have the commutative diagram of completely bounded maps*

$$\begin{tikzcd} & \prod_{\mathcal{U}} T_{n(\alpha)} \arrow[dr, "\psi"] & \\ V \arrow[ur, "\varphi"] \arrow[rr, "\iota_V"] & & V^{**} \end{tikzcd} \tag{3.2}$$

with $\|\varphi\|_{cb} \|\psi\|_{cb} \leqslant \lambda$.

Proof. Let Γ be the collection of all pairs (E, ε) of finite dimensional subspaces E of V and $\varepsilon > 0$. There is a canonical partial order on Γ given by $(E, \varepsilon) \preccurlyeq (E', \varepsilon')$ if and only if $E \subseteq E'$ and $\varepsilon' \leqslant \varepsilon$. For our convenience, we denote $\alpha = (E_\alpha, \varepsilon_\alpha) \in \Gamma$. For each $\alpha \in \Gamma$, put $\Gamma_\alpha = \{\alpha' \in \Gamma : \alpha \preccurlyeq \alpha'\}$, and let $\mathscr{F}_{\preccurlyeq}$ denote the filter on Γ given by

$$\mathscr{F}_{\preccurlyeq} = \{\Lambda \subseteq \Gamma : \Gamma_\alpha \subseteq \Lambda \text{ for some } \alpha \in \Gamma\}.$$

It is clear that $\bigcap_{\alpha \in I} \Gamma_\alpha = \varnothing$. Then there is a free ultrafilter $\mathscr{U}$ on Γ containing $\mathscr{F}_{\preccurlyeq}$.

Since V has the λ-LLP, it follows from Theorem 3.2 that for each $\alpha = (E_\alpha, \varepsilon_\alpha) \in \Gamma$, there exist completely bounded linear maps $s_\alpha : E_\alpha \to T_{n(\alpha)}$ and $t_\alpha : T_{n(\alpha)} \to V$ such that $\|s_\alpha\|_{cb} \|t_\alpha\|_{cb} < \lambda + \varepsilon_\alpha$ and the diagram

$$\begin{tikzcd} & T_{n(\alpha)} \arrow[dr, "t_\alpha"] & \\ E_\alpha \arrow[ur, "s_\alpha"] \arrow[rr, "\iota_\alpha"] & & V \end{tikzcd}$$

commutes, where $\iota_\alpha : E_\alpha \to V$ denotes the inclusion map. Without loss of generality, we may assume that s_α is a complete contraction and $\|t_\alpha\|_{cb} < \lambda + \varepsilon_\alpha$.

We may define the map

$$\varphi : V \to \prod_{\mathscr{U}} T_{n(\alpha)};\ v \mapsto \varphi(v) = [(w_\alpha)],$$

where $w_\alpha = s_\alpha(v)$ if $v \in E_\alpha$ and $w_\alpha = 0$ if $v \notin E_\alpha$, and the map ψ: $\prod_{\mathscr{U}} T_{n(\alpha)} \to V^{**}$ by letting

$$\psi([(w_\alpha)]) = \text{weak}^* \lim_{\mathscr{U}} t_\alpha(w_\alpha) \in V^{**}$$

for every $[(w_\alpha)] \in \prod_{\mathscr{U}} T_{n(\alpha)}$. It is easy to verify that φ and ψ are well defined completely bounded linear maps with $\|\varphi\|_{cb} \leqslant 1$ and $\|\psi\|_{cb} \leqslant \lambda$ such that the diagram (3.2) commutes. ∎

In general, the λ-LLP does not pass to operator subspaces. However, we may have the following result for completely complemented subspaces.

PROPOSITON 3.7. *Suppose that V has the λ-LLP and Z is an operator subspace of V. If there is a completely bounded projection P: $V \to Z$ with $\|P\|_{cb} \leqslant \mu$, then Z also has the $\lambda\mu$-LLP.*

Proof. Given operator spaces $W \subseteq Y$ and a complete contraction φ: $Z \to Y/W$, the map $\varphi \circ P : V \to Y/W$ is completely bounded with $\|\varphi \circ P\|_{cb} \leqslant \|P\|_{cb} \leqslant \mu$. Let E be a finite dimensional subspace of Z and $\varepsilon > 0$ given. Since V has the λ-LLP and E is a finite dimensional subspace of V, there exists a completely bounded map ψ: $E \to Y$ such that $\|\psi\|_{cb} < \lambda\mu + \varepsilon$ and $q \circ \psi(x) = \varphi \circ P(x) = \varphi(x)$ for all $x \in E$, where q denotes the complete quotient from Y onto Y/W. This shows that Z has the $\lambda\mu$-LLP. ∎

4. λ-INJECTIVITY

Given operator spaces V and $W \subseteq Y$, we may define an operator space matrix norm on $V \otimes W$ obtained by identifying $V \otimes W$ with an operator subspace of $V \hat{\otimes} Y$, and let $V \hat{\otimes}_Y W$ denote the norm closure of $V \otimes W$ in $V \hat{\otimes} Y$. It is known from (2.1) that the map $\mathrm{id}_V \otimes \iota$ on $V \otimes W$ determines a complete contraction

$$S\colon V \hat{\otimes} W \to V \hat{\otimes}_Y W, \tag{4.1}$$

where ι: $W \to Y$ is the inclusion map. If S is a bounded isomorphism from $V \hat{\otimes} W$ onto $V \hat{\otimes}_Y W$ we define

$$\mathrm{proj}(V, W \subseteq Y) = \|S^{-1}\|,$$

and we put $\mathrm{proj}(V, W \subseteq Y) = \infty$ otherwise. We can also consider the completely bounded case by letting

$$\mathrm{proj}_{cb}(V, W \subseteq Y) = \|S^{-1}\|_{cb}$$

if S is a completely bounded isomorphism, and $\mathrm{proj}_{cb}(V, W \subseteq Y) = \infty$ otherwise. It is clear that

$$1 \leqslant \mathrm{proj}(V, W \subseteq Y) \leqslant \mathrm{proj}_{cb}(V, W \subseteq Y) \leqslant \infty.$$

In this paper, we only need to consider $\mathrm{proj}(V, W \subseteq Y)$.

PROPOSITION 4.1. *Given operator spaces V and $W \subseteq Y$, the following are equivalent*:

(i) $\mathrm{proj}(V, W \subseteq Y) \leqslant \lambda$.

(ii) *For every complete contraction φ: $W \to V^*$, there is a completely bounded linear map $\tilde{\varphi}$: $Y \to V^*$ such that $\tilde{\varphi}_{|W} = \varphi$ and $\|\tilde{\varphi}\|_{cb} \leqslant \lambda$.*

Proof. (i) $\Rightarrow$ (ii). Given any complete contraction φ: $W \to V^*$, since we have the isometry $CB(W, V^*) = (V \hat{\otimes} W)^*$, we may identify φ with a contractive linear functional F_φ on $V \hat{\otimes} W$ given by

$$\langle F_\varphi, v \otimes w \rangle = \langle \varphi(w), v \rangle$$

for all $v \in V$ and $w \in W$. Since $\mathrm{proj}(V, W \subseteq Y) = \|S^{-1}\| \leqslant \lambda$, we see that $F_\varphi \circ S^{-1}$ is a bounded linear functional on $V \hat{\otimes}_Y W$ with $\|F_\varphi \circ S^{-1}\| \leqslant \lambda$. It follows from the Hahn–Banach theorem that $F_\varphi \circ S^{-1}$ has a bounded linear extension $\tilde{F}$: $V \hat{\otimes} Y \to \mathbb{C}$ such that $\|\tilde{F}\| \leqslant \lambda$. Then $\tilde{F}$ corresponds to a completely bounded linear map $\tilde{\varphi}$: $Y \to V^*$ such that $\tilde{\varphi}_{|W} = \varphi$ and $\|\tilde{\varphi}\|_{cb} = \|\tilde{F}\| \leqslant \lambda$.

(ii) $\Rightarrow$ (i). The statement (ii) tells us that the closed unit ball of $(V \hat{\otimes} W)^* = CB(W, V^*)$ is contained in the image of the restriction map of the closed λ-ball of $(V \hat{\otimes} Y)^* = CB(Y, V^*)$. For every $u \in V \otimes W \subseteq V \otimes Y$, we have

$$\begin{aligned}
\|u\|_{V \hat{\otimes} W} &= \sup\{|\langle f, u \rangle| : f \in (V \hat{\otimes} W)^* \ \|f\| \leqslant 1\} \\
&\leqslant \sup\{|\langle f, u \rangle| : f \in (V \hat{\otimes} Y)^* \ \|f\| \leqslant \lambda\} \\
&= \sup\{|\lambda \langle f/\lambda, u \rangle| : f \in (V \hat{\otimes} Y)^* \ \|f\| \leqslant \lambda\} \\
&= \lambda \ \|u\|_{V \hat{\otimes} Y} = \lambda \ \|u\|_{V \hat{\otimes}_Y W}.
\end{aligned}$$

This shows that the identity map on $W \otimes V$ can be extended to a bounded linear isomorphism from $V \hat{\otimes} W$ onto $V \hat{\otimes}_Y W$ with $\|S^{-1}\| \leqslant \lambda$. Therefore, we have $\mathrm{proj}(V, W \subseteq Y) \leqslant \lambda$. ∎

An operator space Z is called *λ-injective* if given any operator spaces $W \subseteq Y$, every completely bounded map φ: $W \to Z$ has an extension

$\tilde{\varphi}$: $Y \to Z$ such that $\|\tilde{\varphi}\|_{cb} \leqslant \lambda \|\varphi\|_{cb}$. An operator space is said to be *injective* if it is 1-injective. It was shown in [8] that a dual operator space V^* is injective if and only if for any completely isometric inclusion ι: $W \hookrightarrow Y$,

$$\mathrm{id}_V \otimes \iota : V \hat{\otimes} W \hookrightarrow V \hat{\otimes}_Y W$$

is a (completely) isometric isomorphism, i.e.,

$$\mathrm{proj}(V, W \subseteq Y) = 1.$$

Generalizing this to λ-injectivity case, we have the following corollary, which is an immediate consequence of Proposition 4.1.

COROLLARY 4.2. *Let V be an operator space. Then V^* is λ-injective if and only if* $\mathrm{proj}(V, W \subseteq Y) \leqslant \lambda$ *for all operator spaces $W \subseteq Y$.*

We note that if E is a finite dimensional operator space with $\dim E = n$, then E^* must be n-injective. To see this, we take an Auerbach basis $\{f_1, ..., f_n\}$ of E^* with $\|f_i\| = 1$. Given operator spaces $W \subseteq Y$, every completely contractive linear map φ: $W \to E^*$ can be written as

$$\varphi = \varphi_1 \otimes f_1 + \cdots + \varphi_n \otimes f_n$$

with $\varphi_i \in W^*$ and $\|\varphi_i\| \leqslant 1$. It follows from the Hahn–Banach theorem that each φ_i has a contractive extension $\tilde{\varphi}_i \in Y^*$. Then

$$\tilde{\varphi} = \tilde{\varphi}_1 \otimes f_1 + \cdots + \tilde{\varphi}_n \otimes f_n : Y \to E^*$$

is a completely bounded extension of φ such that $\|\tilde{\varphi}\|_{cb} \leqslant \sum_{i=1}^n \|\tilde{\varphi}_i\|_{cb} \leqslant n$. Therefore, E^* must be n-injective. This also shows that if E is an n-dimensional operator space then $\mathrm{proj}(E, W \subseteq Y) \leqslant n$ for any operator spaces $W \subseteq Y$.

PROPOSITION 4.3. *Given an operator space V, we have* $\mathrm{proj}(V, W \subseteq Y) \leqslant \lambda$ *for all operator spaces $W \subseteq Y$ if and only if* $\mathrm{proj}(V, E \subseteq F) \leqslant \lambda$ *for all finite dimensional operator spaces $E \subseteq F$.*

Proof. The direction ($\Rightarrow$) is obvious,

($\Leftarrow$) Given operator spaces $W \subseteq Y$ and an element $u \in V \otimes W$, we can write $u = v_1 \otimes w_1 + \cdots + v_n \otimes w_n$ for some $v_i \in V$ and $w_i \in W (i = 1, ..., n)$. We let $E = \mathrm{span}\{w_1, ..., w_n\}$ be the finite dimensional subspace of W spanned by $w_1, ..., w_n$. Then we have $u \in V \otimes E$.

Assume (without loss of generality) that $\|u\|_{V \hat{\otimes}_Y W} < 1$. Then there exist contractive elements $\alpha \in M_{1,pq}$, $v = [v_{ij}] \in M_p(V)$, $y = [y_{k\ell}] \in M_q(Y)$ and $\beta \in M_{pq,1}$ such that $u = \alpha(v \otimes y)\beta$. Let F be the linear subspace of Y

spanned by $\{y_{k\ell}, w_i: 1 \leqslant k, \ell \leqslant q, 1 \leqslant i \leqslant n\}$. Then F is a finite dimensional subspace of Y containing E and we have $\|u\|_{V \hat{\otimes}_F E} = \|u\|_{V \hat{\otimes} F} < 1$. Since $\operatorname{proj}(V, E \subseteq F) \leqslant \lambda$, we have

$$\|u\|_{V \hat{\otimes} W} \leqslant \|u\|_{V \hat{\otimes} E} \leqslant \lambda \, \|u\|_{V \hat{\otimes}_F E} < \lambda.$$

Therefore, we have $\operatorname{proj}(V, W \subseteq Y) \leqslant \lambda$. ∎

An operator space V is said to be *λ-finitely injective* if for any finite dimensional operator spaces $E \subseteq F$ and a complete contraction $\varphi: E \to V$ there is a completely bounded extension $\tilde{\varphi}: F \to V$ such that $\|\tilde{\varphi}\|_{cb} \leqslant \lambda$. It follows from Proposition 4.1 that a dual operator space V^* is λ-finitely injective if and only if $\operatorname{proj}(V, E \subseteq F) \leqslant \lambda$ for any finite dimensional operator spaces $E \subseteq F$. Therefore, we can get the following corollary, which is an operator space analogue of the corresponding result of Effros and Choi [3] for dual operator systems.

COROLLARY 4.4. *Given an operator space V, the dual space V^* is λ-injective if an only if it is λ-finitely injective.*

On the other hand, we may fix operator spaces $W \subseteq Y$ in $\operatorname{proj}(V, W \subseteq Y)$ and consider different operator spaces V.

PROPOSITION 4.5. *Let V and $W \subseteq Y$ be operator spaces. If* $\operatorname{proj}(E, W \subseteq Y) \leqslant \lambda$ *for every finite dimensional subspace E of V, we have* $\operatorname{proj}(V, W \subseteq Y) \leqslant \lambda$.

Proof. Let u be an element in $V \otimes W$. If $\|u\|_{V \hat{\otimes}_Y W} < 1$, then there exists a representation $u = \alpha(v \otimes y)\beta$, where $\alpha \in M_{1,pq}$, $v = [v_{i,j}] \in M_p(V)$, $y = [y_{k,\ell}] \in M_q(Y)$ and $\beta \in M_{pq,1}$ with norm less than 1. Let $E = \operatorname{span}\{v_{i,j}\}$ be the finite dimensional subspace of V spanned by $\{v_{ij}: i, j = 1, ..., p\}$. Then we can regard u as an element in $E \otimes Y$ with $\|u\|_{E \hat{\otimes} Y} < 1$. It is easy to see (by choosing a basis for E) that u is in fact contained in $E \otimes W$. It follows from the hypothesis that

$$\|u\|_{V \hat{\otimes} W} \leqslant \|u\|_{E \hat{\otimes} W} \leqslant \lambda \, \|u\|_{E \hat{\otimes}_Y W} < \lambda.$$

This shows that $S: V \hat{\otimes} W \to V \hat{\otimes}_Y W$ is a bounded linear isomorphism with $\|S^{-1}\|_{cb} \leqslant \lambda$. ∎

Assume that W is a λ-completely complemented subspace of Y and $P: Y \to W$ is a completely bounded projection with $\|P\|_{cb} \leqslant \lambda$. Then for any operator space V the map $S: V \hat{\otimes} W \to V \hat{\otimes}_Y W$ in (4.1) is a completely bounded isomorphism with $S^{-1} = (\mathrm{id}_V \otimes P)_{|V \hat{\otimes}_Y W}$ and

$$\operatorname{proj}(V, W \subseteq Y) = \|S^{-1}\| \leqslant \|P\|_{cb} \leqslant \lambda.$$

In particular, if W is λ-injective then W is λ-completely complemented in every operator space Y containing it as an operator subspace. In this case, we have $\operatorname{proj}(V, W \subseteq Y) \leqslant \lambda$ for all V and Y containing W.

Considering a slightly weaker condition, we may assume that W is λ-relatively weakly injective in Y. Given operator spaces $W \subseteq Y$, we say that W is *λ-relatively weakly injective* in Y for some $\lambda \geqslant 1$ if there is a complete bounded linear map π: $Y \to W^{**} = \bar{W}^{\text{weak}^*} \subseteq Y^{**}$ such that $\|\pi\|_{cb} \leqslant \lambda$ and $\pi(w) = w$ for all $w \in W$. This definition is a natural operator space analogue of Kirchberg's definition of relatively weak injectivity for C^*-algebras introduced in [19].

THEOREM 4.6. *Let W be a subspace of an operator space Y. Then the following are equivalent*:

(i) *W is λ-relatively weakly injective in Y.*

(ii) $\operatorname{proj}(V, W \subseteq Y) \leqslant \lambda$ *for every operator space V.*

(iii) $\operatorname{proj}(E, W \subseteq Y) \leqslant \lambda$ *for every finite dimensional operator space E.*

(iv) *For every operator space V and a complete contraction ϕ: $W \to V^*$, there is a completely bounded map $\tilde{\phi}$: $Y \to V^*$ such that $\|\tilde{\phi}\|_{cb} \leqslant \lambda$ and $\tilde{\phi}|_W = \phi$.*

Proof. (i) $\Rightarrow$ (ii). Le π: $Y \to W^{**}$ be a completely bounded map such that $\|\pi\|_{cb} \leqslant \lambda$ and $\pi(w) = w$ for all $w \in W$. Given $u \in V \otimes W$, $u = (\mathrm{id}_V \otimes \pi)(u)$ can be regarded as an element in $V \hat{\otimes} W^{**}$. Since $\operatorname{proj}(V, W \subseteq W^{**}) = 1$ (see [7, p. 184, 5, Lemma 3.1]), we have

$$\|u\|_{V \hat{\otimes} W} = \|(\mathrm{id}_V \otimes \pi)(u)\|_{V \hat{\otimes} W^{**}} \leqslant \lambda \, \|u\|_{V \hat{\otimes}_Y W}.$$

This shows that $\operatorname{proj}(V, W \subseteq Y) = \|S^{-1}\| \leqslant \lambda$.

The equivalences (ii) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iv) follow from Proposition 4.5 and Proposition 4.1, respectively.

(vi) $\Rightarrow$ (i). If we let $V = W^*$, then $V^* = W^{**}$ and the identity mapping $\mathrm{id}_W \in CB(W, W) \subseteq CB(W, V^*)$ has a completely bounded extension $\pi \in CB(Y, W^{**}) = CB(Y, V^*)$ with $\|\pi\|_{cb} \leqslant \lambda$. It is clear that $\pi(w) = w$ for all $w \in W$. Therefore, W is λ-relatively weakly injective in Y. ∎

If $W \subseteq B(H)$, it follows from the Arveson–Wittstock Hahn–Banach completely bounded extension theorem that W is λ-relatively weakly injective in $B(H)$ if and only if W is λ-relatively weakly injective in every operator space Y containing W as an operator subspace. In this case, we say that W has the *operator space λ-weak expectation property* (or simply λ-WEP). If $W = A$ is a C^*-algebra, then A has the operator space 1-WEP if and only

if A has the Lance's WEP defined for C^*-algebras in [21]. A proof of this equivalence can be found in [19, p. 495].

It is clear that if an operator space V is λ-injective (respectively, V^{**} is λ-injective), then V has the λ-WEP. It is worth to note that for dual operator spaces, the λ-WEP implies the λ-injectivity. Even though this is probably well known to the experts, we provide a proof for the convenience of the readers.

PROPOSITION 4.7. *Let $W = Z^*$ be a dual operator space. If $W \subseteq Y$ is λ-relatively weakly injective in Y, then W is λ-completely complemented in Y. Therefore, a dual operator space W has the λ-WEP if and only if W is λ-injective.*

Proof. Assume that W is λ-relatively weakly injective in Y. Then there is a completely bounded map $\pi\colon Y \to W^{**}$ such that $\pi(w) = w$ for every $w \in W$ and $\|\pi\|_{cb} \leqslant \lambda$. We let $\iota_Z\colon Z \hookrightarrow Z^{**} = W^*$ denote the canonical inclusion of Z into its second dual. Then the adjoint map $(\iota_Z)^*\colon W^{**} \to W$ is a complete contraction from W^{**} onto W, and thus $P = (\iota_Z)^* \circ \pi$ is a completely bounded projection from Y onto W since

$$\langle P(w), f \rangle = \langle ((\iota_Z)^* \circ \pi)(w), f \rangle = \langle \pi(w), \iota_Z(f) \rangle = \langle w, f \rangle$$

for every $w \in W$ and $f \in Z$. This shows that W is λ-completely complemented in Y. ∎

5. DUALITY BETWEEN LLP AND INJECTIVITY

It was shown in [18] that the operator predual R_* of a von Neumann algebra R is an $\mathscr{L}\Gamma_{1,\lambda}$ space (i.e., has the λ-LLP by Theorem 3.2) if and only if R is injective. The main purpose of this section is to show that this is also true for general operator spaces. Let us first prove the easy part, which was given in [18]. We are indebted to M. Junge to allow us to include the proof here.

PROPOSITION 5.1. *If V has the λ-LLP, then V^* is λ-injective.*

Proof. We only need to show that $\operatorname{proj}(V, W \subseteq Y) \leqslant \lambda$ for any operator spaces $W \subseteq Y$, that is,

$$\|u\|_{V \hat{\otimes} W} \leqslant \lambda \, \|u\|_{V \hat{\otimes}_Y W}$$

for $u \in V \otimes W$. Let us assume that $\|u\|_{V \hat{\otimes}_Y} < 1$. Then there exist contractive elements $\alpha \in M_{1,k\ell}$, $v = [v_{ij}] \in M_\ell(V)$, $y \in M_k(Y)$ and $\beta \in M_{k\ell,1}$ such that $u = \alpha(v \otimes y)\beta$. Let $E = \operatorname{span}\{v_{ij}\}$ be the finite dimensional subspace of V

spanned by $\{v_{ij}: i, j=1, ..., l\}$, and let $\iota: E \hookrightarrow V$ denote the inclusion map. Then we have $u \in E \otimes W$. Since V has the λ-LLP, we have from Theorem 3.2 that for every $\varepsilon > 0$, there exist $n \in \mathbb{N}$ and completely bounded maps $s: E \to T_n$ and $t: T_n \to V$ such that $t \circ s = \iota$ and $\|t\|_{cb} \|s\|_{sb} < \lambda + \varepsilon$. It follows that

$$\begin{aligned} \|u\|_{V \hat{\otimes} W} &\leqslant \|(t \circ s \otimes \mathrm{id}_W)(u)\|_{E \hat{\otimes} W} \\ &\leqslant \|t\|_{cb} \|(s \otimes \mathrm{id}_W)(u)\|_{T_n \hat{\otimes} W} \\ &= \|t\|_{cb} \|(s \otimes \mathrm{id}_Y)(u)\|_{T_n \hat{\otimes} Y} \\ &\leqslant \|t\|_{cb} \|s\|_{cb} \|u\|_{V \hat{\otimes} Y} < \lambda + \varepsilon, \end{aligned}$$

where the equality in the third place follows from Corollary 4.2 and the injectivity of $(T_n)^* = M_n$. Letting $\varepsilon \to 0$, we get $\|u\|_{V \hat{\otimes} W} \leqslant \lambda$, and this completes the proof. ∎

Proposition 5.1 shows that the converse of Corollary 3.4 is not true. Indeed, while $B(\ell^2)^*$ is finitely representable in $\{T_n\}_{n \in \mathbb{N}}$ (see [5]), it does not have the λ-LLP since its operator dual $B(\ell^2)^{**}$ is not λ-injective for any $\lambda \geqslant 1$. However, the following proposition shows that if, in addition, V is injective then λ-finite representability in $\{T_n\}_{n \in \mathbb{N}}$ implies the λ-LLP.

PROPOSITION 5.2. *If an injective operator space V is λ-finitely representable in $\{T_n\}_{n \in \mathbb{N}}$, then V has the λ-LLP.*

Proof. Since V is λ-finitely representable in $\{T_n\}_{n \in \mathbb{N}}$, for every finite dimensional subspace E of V and $\varepsilon > 0$, there is an operator subspace F of some T_n and a completely bounded isomorphism $s: E \to F$ such that $\|s\|_{cb} \|s^{-1}\|_{cb} < \lambda + \varepsilon$. Without loss of generality, we let $\|s\|_{cb} \leqslant 1$ and $\|s^{-1}\|_{cb} < \lambda + \varepsilon$. Since V is injective, the map $s^{-1}: F \to E \hookrightarrow V$ has a completely bounded extension $t = T_n \to V$ with $\|t\|_{cb} < \lambda + \varepsilon$. It is clear that $t \circ s = \iota$, and thus V has the λ-LLP by Theorem 3.2. ∎

The converse of Proposition 5.1 is easy to prove for the operator predual R_* of a von Neumann algebra R. Indeed, if R is an injective von Neumann algebra, it is semidiscrete and thus there exist completely contractive maps $\varphi_\alpha: R_* \to T_{n(\alpha)}$ and $\psi_\alpha: T_{n(\alpha)} \to R_*$ such that $\psi_\alpha \circ \varphi_\alpha \to \mathrm{id}_{R_*}$ in the point-norm topology. This implies that R_* satisfies the condition (iii) in Theorem 3.2. However, this technique is not available for general operator spaces. We have to prove the converse of Proposition 5.1 in a different way.

Let W be an operator subspace of Y and let $q: Y \to Y/W$ denote the complete quotient map. For any finite dimensional operator space F, the short sequence of operator spaces

$$0 \to F \check{\otimes} W \to F \check{\otimes} Y \to F \check{\otimes} Y/W \to 0 \tag{5.1}$$

is exact (in the algebraic sense), and $\mathrm{id}_F \otimes q$ induces a completely contractive linear isomorphism

$$T: F \check{\otimes} Y/F \check{\otimes} W \to F \check{\otimes} Y/W. \tag{5.2}$$

In general, the inverse map T^{-1} need not be a contraction. Using a similar notion introduces in Pisier [23], we use

$$\mathrm{ex}(F, W \subseteq Y) = \|T^{-1}\|$$

to denote the degree of exactness of the short sequence (5.1).

It is known from (i) $\Leftrightarrow$ (ii) of Theorem 3.2 that an operator space $V = T_I/W$ has the λ-LLP if and only if for every finite dimensional operator space E, every complete contraction $\varphi: E \to V = T_I/W$ and $\varepsilon > 0$, there exists a completely bounded linear map $\tilde{\varphi}: E \to T_I$ such that $q_V \circ \tilde{\varphi} = \varphi$ and $\|\tilde{\varphi}\|_{cb} < \lambda + \varepsilon$. Letting $F = E^*$, this is equivalent to saying that the map

$$\begin{aligned} T^{-1}: F \check{\otimes} T_I/W = CB(E, T_I/W) &\to F \check{\otimes} T_I/F \check{\otimes} W \\ &= CB(E, T_I)/CB(E, W) \end{aligned}$$

is bounded by λ. Therefore, we get the following proposition.

PROPOSITION 5.3. *Let us assume that $V = T_I/W$. Then V has the λ-LLP if and only if we have* $\mathrm{ex}(F, W \subseteq T_I) \leqslant \lambda$ *for every finite dimensional operator space F.*

In particular, V has the LLP if and only if $\mathrm{ex}(F, W \subseteq T_I) = 1$ *for every finite dimensional operator space F.*

Using the notion of $\mathrm{ex}(F, W \subseteq Y)$, we may characterize in the following proposition the pair $W \subseteq Y$ of operator spaces for which $W^\perp$ is λ-relatively weakly injective in Y^*, under the suitable assumption of local reflexivity. We recall from Theorem 4.6 and Proposition 4.7 that $W^\perp$ is λ-relatively weakly injective in Y^* if and only if $W^\perp$ is λ-completely complemented in Y^* if and only if

$$\mathrm{proj}(F^*, W^\perp \subseteq Y^*) \leqslant \lambda$$

for any finite dimensional operator space F.

PROPOSITION 5.4. *Assume that Y is a locally reflexive operator space and W is an operator subspace of Y. If $W^\perp$ is λ-completely complemented in Y^*, then we have the following*:

(i) *Y/W is λ-locally reflexive.*

(ii) $\mathrm{ex}(F, W \subseteq Y) \leqslant \lambda$ *for every finite dimensional operators space F.*

Conversely, if Y/W is locally reflexive then the condition (ii) *implies that $W^{\perp}$ is λ-completely complemented in Y^*.*

Proof. Let F be a finite dimensional operator space. We consider the following two maps

$$S\colon F^* \hat{\otimes} W^{\perp} \to F^* \hat{\otimes}_{Y^*} W^{\perp} \qquad \text{and} \qquad T\colon F \check{\otimes} Y/F \check{\otimes} W \to F \check{\otimes} Y/W$$

arising from (4.1) and (5.2), respectively. If we denote the quotient maps by

$$q\colon Y \to Y/W, \qquad \text{and} \qquad Q\colon F \check{\otimes} Y \to F \check{\otimes} Y/F \check{\otimes} W,$$

respectively, then we have $\mathrm{id}_F \otimes q = T \circ Q$. We show that the following diagram

$$\begin{tikzcd}
(F\check{\otimes} Y/W)^* \arrow[rr, "(\mathrm{id}_F \otimes q)^*"] \arrow[dr, "T^*"] & & (F\check{\otimes} Y)^* \arrow[dd, "\Phi^{-1}"] \\
& (F\check{\otimes} W)^{\perp} \arrow[ur, "Q^*"] & \\
F^*\hat{\otimes} W^{\perp} \arrow[uu, "\Phi_1"] \arrow[r, "S"] & F^*\hat{\otimes}_{Y^*} W^{\perp} \arrow[uu, "\Phi_2"] \arrow[r, hook] & F^*\hat{\otimes} Y^*
\end{tikzcd} \tag{5.3}$$

commutes. Here, Φ and Φ_1 are the completely contractive linear isomorphisms arising from (2.2), and Φ_2 is the restriction of Φ to $F^* \hat{\otimes}_{Y^*} W^{\perp}$. For each $x \in F^*$, $g \in W^{\perp} \subset Y^*$, $f \in F$ and $y \in Y$, we have $q^*(g) = g$ and

$$\begin{aligned}
\langle (\mathrm{id}_F \otimes q)^* \circ \Phi_1(x \otimes g), f \otimes y \rangle &= \langle \Phi_1(x \otimes g), f \otimes q(y) \rangle \\
&= \langle x, f \rangle \langle g, q(y) \rangle \\
&= \langle x, f \rangle \langle g, y \rangle \\
&= \langle \Phi(x \otimes g), f \otimes y \rangle \\
&= \langle \Phi \circ S(x \otimes g), f \otimes y \rangle.
\end{aligned}$$

Therefore, we see that the big rectangle in (5.3) commutes. Note that Q^* is nothing but the inclusion map, and T^* is the same map as $(\mathrm{id}_F \otimes q)^*$ with different ranges. Furthermore, it is easy to see that the image of Φ_2 is just $(F \check{\otimes} W)^{\perp}$ by the linear isomorphism $F^* \otimes W^{\perp} \cong (F \check{\otimes} W)^{\perp}$. Hence, we see that the diagram (5.3) commutes. From the assumption that Y is locally reflexive, it follows that Φ and Φ_2 are complete isometries.

If $W^\perp$ is λ-completely complemented in Y^*, then

$$\|S^{-1}\| = \operatorname{proj}(F^*,\ W^\perp \subseteq Y^*) \leqslant \lambda$$

and thus

$$\|\Phi_1^{-1}\| = \|S^{-1} \circ \Phi_2^{-1} \circ T^*\| \leqslant \|S^{-1}\| \leqslant \lambda.$$

Hence, Y/W is λ-locally reflexive. Furthermore, we have

$$\begin{aligned}\operatorname{ex}(F,\ W \subseteq Y) &= \|T^{-1}\| = \|(T^*)^{-1}\| \\ &= \|\Phi_1 \circ S^{-1} \circ \Phi_2^{-1}\| \leqslant \|S^{-1}\| \leqslant \lambda.\end{aligned}$$

Conversely, if Y/W is locally reflexive and $\operatorname{ex}(F,\ W \subseteq Y) \leqslant \lambda$ then Φ_1 is a complete isometry and $\|T^{-1}\| \leqslant \lambda$. Therefore, we have

$$\begin{aligned}\operatorname{proj}(F^*,\ W^\perp \subseteq Y^*) &= \|S^{-1}\| = \|\Phi_1^{-1} \circ (T^*)^{-1} \circ \Phi_2\| \\ &= \|(T^*)^{-1}\| = \|T^{-1}\| \leqslant \lambda.\end{aligned}$$

This shows that $W^\perp$ is λ-completely complemented in Y^* by Theorem 4.6 and Proposition 4.7. ∎

Note that local reflexivity does not pass to the complete quotients, since every operator space is the complete quotient of some T_I, which is locally reflexive. It is known in [10] that the complete quotient of a locally reflexive operator space by a complete M-ideal is again locally reflexive. Proposition 5.4(i) extends this result.

Now, we combine Proposition 5.3 and Proposition 5.4 to get the following theorem.

THEOREM 5.5. *Let V be an operator space. Then V^* is λ-injective if and only if V has the λ-LPP.*

Proof. Assume that V^* is λ-injective. If W is an operator subspace of T_I such that $V = T_I/W$ then $W^\perp = V^*$ is λ-completely complemented in $T_I^* = B(\ell^2)$. Since T_I is locally reflexive, we have from Proposition 5.4 that $\operatorname{ex}(F,\ W \subseteq T_I) \leqslant \lambda$ for all finite dimensional operator space F. Then V has the λ-LLP by Proposition 5.3. The other direction was proved in Proposition 5.1. ∎

We say that an operator space V has the *λ-finitely lifting property* (*λ-FLP*) if for any finite dimensional operator spaces $E \subseteq F$, any complete contraction $\varphi\colon V \to F/E$ and $\varepsilon > 0$, there exists a complete bounded map $\tilde{\varphi}\colon V \to F$ such that $\|\tilde{\varphi}\|_{cb} \leqslant \lambda + \varepsilon$ and $q \circ \tilde{\varphi} = \varphi$, where $q\colon F \to F/E$ denotes the complete quotient map. We also say that an operator space V has the *λ-weak* lifting property* (*λ-weak* LP*) if for any operator spaces $W \subseteq Y$ and any complete contraction $\varphi\colon V \to Y^*/W^\perp$, there exists a completely bounded map $\tilde{\varphi}\colon V \to Y^*$ such that $\tilde{q} \circ \tilde{\varphi} = \varphi$ and $\|\tilde{\varphi}\|_{cb} \leqslant \lambda$, where $\tilde{q}\colon Y^* \to Y^*/W^\perp$ denotes the complete quotient map from Y^* onto $Y^*/W^\perp$.

PROPOSITION 5.6. *Let V be an operator space. Then the following are equivalent*:

(i) *V has the λ-LLP.*

(ii) *V is λ-weak* LP.*

(iii) *V has the λ-FLP.*

Proof. (i) $\Rightarrow$ (ii). Let us first recall from Corollary 4.1 and Theorem 5.5 that V has the λ-LLP if and only if $\operatorname{proj}(V, W \subseteq Y) \leqslant \lambda$ for all operator spaces $W \subseteq Y$. Given operator spaces $W \subseteq Y$, if $\operatorname{proj}(V, W \subseteq Y) \leqslant \lambda$, then the map

$$S^{-1} = V \hat{\otimes}_Y W \to V \hat{\otimes} W$$

and thus its adjoint map

$$(S^{-1})^*: (V \hat{\otimes} W)^* \to (V \hat{\otimes}_Y W)^*$$

are bounded linear isomorphisms with $\|(S^{-1})^*\| = \|S^{-1}\| \leqslant \lambda$. Given any completely contractive map $\varphi \in CB(V, Y^*/W^\perp) = CB(V, W^*) = (V \hat{\otimes} W)^*$, we can identity φ with a contractive linear functional F_φ on $V \hat{\otimes} W$, and thus $F_\varphi \circ S^{-1}$ is a bounded linear functional on $V \hat{\otimes}_Y W$ with $\|F_\varphi \circ S^{-1}\| \leqslant \|S^{-1}\| \leqslant \lambda$. Since $V \hat{\otimes}_Y W$ is an operator subspace of $V \hat{\otimes} Y$, the functional $F_\varphi \circ S^{-1}$ has a norm preserving extension $\tilde{F}$ on $V \hat{\otimes} Y$. Then $\tilde{F}$ corresponds to a completely bounded map $\tilde{\varphi} \in CB(V, Y^*)$ such that $\|\tilde{\varphi}\|_{cb} = \|\tilde{F}\| \leqslant \lambda$ and $\tilde{q} \circ \tilde{\varphi} = \varphi$, where $\tilde{q}$: $Y^* \to Y^*/W^\perp$ denotes the complete quotient map. This shows (i) $\Rightarrow$ (ii). Reserving the above calculation, we can get the direction (ii) $\Rightarrow$ (i).

The proof for (i) $\Rightarrow$ (iii) is similar. In this case, we only need to use the fact that $\operatorname{proj}(V, W \subseteq Y) \leqslant \lambda$ for all operator spaces $W \subseteq Y$ if and only if $\operatorname{proj}(V, E \subseteq F) \leqslant \lambda$ for all finite dimensional operator spaces $E \subseteq F$ (see Proposition 4.3). ∎

In the case of Banach spaces, it was shown in [15] that a Banach space X has the Banach space version of 1-weak* LP if and only if X^* is a 1-injective Banach space. Moreover, a Banach space satisfies these conditions if and only if it is isometrically isomorphic to $L_1(\mu)$ for a measure μ.

6. SOME RELATED PROPERTIES

It was shown in [1] that the only C^*-algebra with the LP is the trivial one dimensional C^*-algebra. It is also known from Section 3 that

$$\text{LP} \Rightarrow \text{LLP} \Rightarrow \text{finite representability in} \{T_n\}.$$

The following proposition shows that these are all equivalent for C^*-algebras, and in fact, the one dimensional C^*-algebra is the only C^*-algebra which satisfies these conditions. This indicates that our notion of local lifting property is very different from the local lifting property studied in C^*-algebra theory (see [19]), which is described in terms of local lifting of completely positive maps with respect to the quotient maps onto C^*-quotients by norm-closed two-sided ideals.

PROPOSITION 6.1. *For a C^*-algebra A, the following are equivalent*:

(i) *A has the LP.*

(ii) *A has the LLP.*

(iii) *A is finitely representable in $\{T_n\}_{n\in\mathbb{N}}$.*

(iv) *A is the one-dimensional trivial C^*-algebra.*

Proof. It is clear that (iv) $\Rightarrow$ (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). In order to show that (iii) implies (iv). Assume that A is a C^*-algebra with $\dim A \geqslant 2$. Then the maximal abelian $*$-subalgebra B of A must have $\dim B \geqslant 2$. Using the argument in Haagerup [16], Lemma 2.7, it is easy to see that there is a subspace V of B, which is completely isometric to ℓ_2^∞. If A is finitely representable in $\{T_n\}_{n\in\mathbb{N}}$, then V is also finitely representable in $\{T_n\}_{n\in\mathbb{N}}$. Since $V \cong \ell_2^\infty$ is an injective operator space, we see that V has the LLP by Proposition 5.2. This implies that ℓ_2^∞ has the LLP and thus the LP, which gives a contradiction.

PROPOSITION 6.2. *If V has the λ-WEP and is locally reflexive, then V^{**} is λ-injective, and thus V^* has the λ-LLP.*

Proof. The argument here is similar to the proof given in [4, Proposition 5.4]. In order to show that V^{**} is λ-finitely injective, let $E \subseteq F$ be finite dimensional operator spaces and $\varphi\colon E \to V^{**}$ a complete contraction. Since V is locally reflexive, there exists a net of complete contractions $\varphi_\alpha\colon E \to V$ such that $\varphi_\alpha \to \varphi$ in the point-weak* topology. Let V be an operator subspace of $B(H)$ for some Hilbert space H and let $\iota\colon V \hookrightarrow B(H)$ denote the completely isometric inclusion map. For each α, the complete contraction $\iota \circ \varphi_\alpha\colon E \to B(H)$ has a completely contractive extension $\psi_\alpha\colon F \to B(H)$. Since V has the λ-WEP, there exists a completely bounded map $\pi\colon B(H) \to V^{**}$ such that $\|\pi\|_{cb} \leqslant \lambda$ and $\pi(v) = v$ for all $v \in V$. Then we get a net of completely bounded maps $\pi \circ \psi_\alpha\colon F \to V^{**}$ with $\|\pi \circ \psi_\alpha\|_{cb} \leqslant \lambda$. Since the closed λ-ball of $CB(F, V^{**}) = (F \hat{\otimes} V^*)^*$ is weak*-compact, $\{\pi \circ \psi_\alpha\}$ has a cluster point $\psi \in CB(F, V^{**})$ with $\|\psi\|_{cb} \leqslant \lambda$. It is easy to see that ψ is a completely bounded extension of φ. This shows that V^{**} if λ-finitely injective, and thus λ-injective. ∎

We recall that an operator space V is *λ-nuclear* if there exist completely bounded linear maps $\varphi_\alpha: V \to M_{n(\alpha)}$ and $\psi_\alpha: M_{n(\alpha)} \to V$ such that $\|\psi_\alpha\|_{cb} \|\varphi_\alpha\|_{cb} \leqslant \lambda$ and $\psi_\alpha \circ \varphi_\alpha \to \mathrm{id}_V$ in the point-norm topology. An operator space is said to be *nuclear* if it is 1-nuclear.

It is known from Kirchberg [20] that if V is a separable nuclear operator space, then there exist a left ideal L and a right ideal R of the CAR algebra $B = M_2 \otimes M_2 \ldots$ such that V is completely isometric to $B/(L+R)$. If we let l and r be the supporting projections of L and R, then $p = 1 - r$ and $q = 1 - l$ are closed projections in B^{**} such that V^{**} is completely isometrically and weak *-continuously isomorphic to $pB^{**}q$. Since B^{**} is an injective von Neumann algebra, V^{**} is also injective and thus V^* has the LLP. One can conclude that V is, in fact, locally reflexive since B is nuclear and V is completely isometrically isomorphic to pBq. On the other hand, it would be interesting to know that if the local reflexivity of V and the LLP of V^* would imply the nuclearity of V. This is equivalent (by Proposition 6.2) to ask if locally reflexivity and WEP would imply nuclearity for general operator spaces. This is true for C^*-algebras (see [4, Proposition 5.4]).

Note added in proof (June, 1999). After we submitted this paper, Effros, Ozawa, and the second author [6] proved recently that the above question is true for general operator spaces, i.e., an operator space V is nuclear if and only if it is locally reflexive and has the WEP. They also proved in [6] that an operator space V has the LLP if and only if it can be identified with an off-diagonal corner of the operator predual of an injective von Neumann algebra, i.e., there is a complete isometry

$$V = (1-e)\, R_* e,$$

where R is an injective von Neumann algebra and e is a projection in R. It follows that an operator space V has the LLP if and only if there exist complete contractions $s_\alpha = V \to T_{n(\alpha)}$ and $t_\alpha = T_{n(\alpha)} \to V$ such that $t_\alpha \circ s_\alpha \to \mathrm{id}_V$ in the point-norm topology.

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