A CLASS OF ATOMIC POSITIVE LINEAR MAPS IN 3-DIMENSIONAL MATRIX ALGEBRAS

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ABSTRACT
We find a class of positive linear maps from the 3-dimensional matrix algebra into itself which cannot be decomposed into the sums of 2-positive maps and 2-copositive maps.

1. Introduction

Let $M_n$ be the $C^*$-algebra of all $n \times n$ matrices over the complex field. Because the structure of the positive cone $\mathcal{P}(M_n)$ of positive linear maps between $M_n$ is very complicated even in lower dimensions, it would be very useful to find simpler convex cones in $\mathcal{P}(M_n)$ with which every positive linear map can be written as a sum. In this vein, the cones of completely positive maps and completely copositive maps were possible candidates, and this is the case when $n = 2$ [6, 8]. But, in other cases, there are positive maps which are not even the sum of a 2-positive map and a 2-copositive map. Such a map is said to be atomic. Although atomic maps are expected to play a role to understand the structure of $\mathcal{P}(M_n)$, examples of such maps are very rare in the literature [3, 4, 5, 7]. To the best of the author’s knowledge, there is only one known example of atomic maps in the 3-dimensional case [3, 7].

In this note, we use the technique in [4] to produce a large class of atomic maps in 3-dimensional matrix algebra. Such examples are provided as variants of positive linear maps investigated in [1]. For nonnegative real numbers $a, c_1, c_2$ and $c_3$, we define the linear map $\Theta[a; c_1, c_2, c_3]$ (denoted by just $\Theta$ if there is no confusion) from $M_3$ into $M_3$ by

$$\Theta[a; c_1, c_2, c_3](x) = \Delta[a; c_1, c_2, c_3](x) - x,$$

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where 
\[ \Delta[a; c_1, c_2, c_3]((x_{ij})) = \begin{pmatrix} ax_{11} + c_1 x_{33} & 0 & 0 \\ 0 & ax_{22} + c_2 x_{11} & 0 \\ 0 & 0 & ax_{33} + c_3 x_{22} \end{pmatrix} \]
for each \((x_{ij}) \in M_3\). Note that \(\Theta[a; c, c, c] = \Phi[a, 0, c]\) with the notation in [1]. We show that \(\Theta[a; c_1, c_2, c_3]\) is positive if and only if \(a \geq 2\) and \(c_1 c_2 c_3 \geq (3 - a)^3\), and it is 2-positive if and only if \(a \geq 3\). Finally, we see that every positive map \(\Theta[a; c_1, c_2, c_3]\) which is not 2-positive becomes an atomic map. The map \(\Theta[2; 1, 1, 1]\) is just the example of an atom mentioned above [7].

2. Positivity and 2-positivity

**Theorem 2.1.** The linear map \(\Theta[a; c_1, c_2, c_3]\) is positive if and only if the following two conditions are satisfied:

(2.1.i) \(a \geq 2\)

(2.1.ii) \(c_1 c_2 c_3 \geq (3 - a)^3\)

By the same argument as [7, Section 2], it suffices to show the following inequality:

**Lemma 2.2.** Let \(a, b\) and \(c\) be nonnegative real numbers. Then the inequality

\[ \frac{\alpha}{a \alpha + c_1 \gamma} + \frac{\beta}{a \beta + c_2 \alpha} + \frac{\gamma}{a \gamma + c_3 \beta} \leq 1 \]

holds for every positive real numbers \(\alpha, \beta, \gamma\) if and only if the two conditions in (2.1) are satisfied.

**Proof.** For the necessity, we take \(\gamma = 0\) and \(\beta \to \infty\), to get the first condition. Taking

\[ \alpha = c_1^{\frac{3}{2}} c_3^{\frac{1}{2}}, \quad \beta = c_2^{\frac{3}{2}} c_1^{\frac{1}{2}}, \quad \text{and} \quad \gamma = c_3^{\frac{3}{2}} c_2^{\frac{3}{2}}, \]
we get the second condition by a calculation.

In order to prove the sufficiency, put

\[ x = c_1 \frac{\gamma}{\alpha}, \quad y = c_2 \frac{\alpha}{\beta}, \quad z = c_3 \frac{\beta}{\gamma}. \]

Then, it suffices to show

\[ \frac{1}{a + x} + \frac{1}{a + y} + \frac{1}{a + z} \leq 1, \]
or equivalently,
\[
F = xyz + (a - 1)(xy + yz + zx) + (a^2 - 2a)(x + y + z) + (a^3 - 3a^2) \geq 0
\]
under the constraint \(xyz = c_1c_2c_3\).

Using the inequalities \(x + y + z \geq 3(xy)\frac{1}{3}\) and \(xy + yz + zx \geq 3(xyz)\frac{2}{3}\), we have
\[
F \geq (c_1c_2c_3) + 3(a - 1)(c_1c_2c_3)\frac{2}{3} + 3(a^2 - 2a)(c_1c_2c_3)\frac{1}{3} + (a^3 - 3a^2)
\]
\[
= ((c_1c_2c_3)\frac{2}{3} + a)^2 ((c_1c_2c_3)\frac{1}{3} + a - 3)
\]
\[
\geq 0,
\]
from the conditions (2.1). This completes the proof. \(\square\)

We denote by \(M_k(M_n)\) the matrix algebra of order \(k\) over \(M_n\). For a linear map \(\phi : M_n \to M_n\), we define two linear maps \(\phi_k\) and \(\phi^k\) between \(M_k(M_n)\) by
\[
\phi_k([a_{ij}]_{i,j=1}^k) = [\phi(a_{ij})]_{i,j=1}^k
\]
\[
\phi^k([a_{ij}]_{i,j=1}^k) = [\phi(a_{ij})]_{i,j=1}^k,
\]
for \([a_{ij}] \in M_k(M_n)\). Recall that the linear map \(\phi\) is \(k\)-positive (respectively \(k\)-copositive) if \(\phi_k\) (respectively \(\phi^k\)) is positive, and \(\phi\) is completely positive (completely copositive) if \(\phi\) is \(k\)-positive (respectively \(k\)-copositive) for each positive integer \(k = 1, 2, \ldots\). It is well known that \(\phi : M_n \to M_n\) is completely positive if and only if \(\phi\) is \(n\)-positive, and this is equivalent to the positivity of the matrix \(\phi_n([e_{ij}]_{i,j=1}^n)\) in \(M_n(M_n)\) [2], where \(\{e_{ij} ; i, j = 1, 2, \ldots, n\}\) is the usual matrix units. Similarly, \(\phi : M_n \to M_n\) is completely copositive if and only if \(\phi^n([e_{ij}]_{i,j=1}^n)\) is a positive matrix.

**Theorem 2.3.** The linear map \(\Theta[a; c_1, c_2, c_3]\) is completely positive if and only if it is 2-positive if and only if the following condition is satisfied.

\[
(2.2) \quad a \geq 3.
\]

**Proof.** Assume that \(\Theta[a; c_1, c_2, c_3]\) is 2-positive. Let \(\xi = (0, 1, 1, 1, 1, 0) \in \mathbb{C}^6\) and \(P = \xi^*\xi \in M_6\). Then, we have \(\Delta(P) \geq P\), and this is the case if and only if \((\Delta_2(P)^{-1}\xi, \xi) \leq 1\) by [7, Lemma 2], because the matrix \(\Delta_2(P)\) is non-singular. By a direct calculation, we have \((\Delta_2(P)^{-1}\xi, \xi) = \frac{2}{a}\), and so the condition (2.2) follows.

Now, we see that the eigenvalues of the 9x9 matrix \(\Theta_3([e_{ij}])\) are \(0, c_1, c_2, c_3, a\) and \(a - 3\). Hence, if \(a \geq 3\) then the map \(\Theta\) is completely positive. It is clear that the completely positivity implies the 2-positivity. \(\square\)
3. Atomic Positive Maps

For a linear map $\tau : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, we define the real linear map

$$\tilde{\tau}([x_{ij}]) = \frac{1}{2}(\tau([x_{ij}]) + \overline{\tau([x_{ij}])}), \quad [x_{ij}] \in M_3(\mathbb{R})$$

as in [4]. It is clear that if $\tau$ is $k$-positive (respectively $k$-copositive) then so is $\tilde{\tau}$ for $k = 1, 2, \ldots$.

Although the following result is in fact contained in the proof of [4, Theorem], we include a sketch of the proof for the completeness. Note that the linear map $\tau$ between $M_3$ can be identified with the matrix $[\tau(e_{ij})]_{i,j=1}^3 \in M_3(M_3)$.

**Proposition 3.1.** Let $\tau$ be a positive linear map between $M_3(\mathbb{C})$. Assume that the matrix $[\tau(e_{ij})] \in M_3(M_3)$ is of the form

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

(3.1)

If $\tau$ is the sum of a 2-positive linear map $\phi$ and a 2-copositive linear map $\psi$, then $\tilde{\tau}$ is a 2-positive linear map.

**Proof.** From the relation $\tau = \phi + \psi$ and the positivity of $\phi$ and $\psi$, we see that the $3 \times 3$ diagonal submatrices of $[\phi(e_{ij})]$ and $[\psi(e_{ij})]$ have the same form as those of $[\tau(e_{ij})]$. Furthermore, every $2 \times 2$ submatrix of $[\phi(e_{ij})]$ is positive because $\phi$ is 2-positive. Similarly, every $2 \times 2$ submatrix of $[\psi(e_{ij})]$ is also positive. Comparing two matrices $[\tau(e_{ij})]$ and $[\phi(e_{ij})] + [\psi(e_{ij})]$, we see that the matrix $[\psi(e_{ij})]$ is of the form

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & \beta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \gamma \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & \beta & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \gamma \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
where every diagonal $3 \times 3$ submatrix is self-adjoint.

From the above matrix form, it is easy to see that $\tilde{\psi}(A) = \tilde{\psi}(A^t)$ for every matrix $A \in M_3(\mathbb{R})$. If $\begin{pmatrix} A & B \\ B^t & D \end{pmatrix}$ is a positive matrix in $M_2(M_3(\mathbb{R}))$ then we have

$$\left(\tilde{\psi}\right)_2 \begin{pmatrix} A & B \\ B^t & D \end{pmatrix} = \begin{pmatrix} \tilde{\psi}(A) & \tilde{\psi}(B) \\ \tilde{\psi}(B^t) & \tilde{\psi}(D) \end{pmatrix} = \begin{pmatrix} \tilde{\psi}(A) & \tilde{\psi}(B^t) \\ \tilde{\psi}(B) & \tilde{\psi}(D) \end{pmatrix} = \left(\tilde{\psi}^2 \begin{pmatrix} A & B \\ B^t & D \end{pmatrix} \right) \geq 0,$$

because $\tilde{\psi}$ is 2-copositive. Hence, we see that $\tilde{\psi}$ is 2-positive, and it follows that $\tilde{\tau} = \tilde{\phi} + \tilde{\psi}$ is a 2-positive map in $M_3(\mathbb{R})$. □

Note that the associated matrix of the map $\Theta[a; c_1, c_2, c_3]$ is of the form (3.1), and it is easy to see that $\tilde{\Theta}$ is not 2-positive for $2 \leq a < 3$ from the proof of Theorem 2.3. Hence, we have the result:

**Theorem 3.2.** For the positive real numbers $a, c_1, c_2$ and $c_3$ satisfying the conditions:

$$2 \leq a < 3 \quad \text{and} \quad c_1c_2c_3 \geq (3 - a)^3,$$

the maps $\Theta[a; c_1, c_2, c_3]$ are atomic positive linear maps between $M_3(\mathbb{C})$.

It is easy to see that the map $\mathbb{R}^4 \to \mathcal{P}(M_3)$ given by

$$(a, c_1, c_2, c_3) \mapsto \Theta[a; c_1, c_2, c_3]$$

is an affine map. Therefore, the map $\Theta[a; c_1, c_2, c_3]$ is not extremal if $a > 2$ or $c_1c_2c_3 > (3 - a)^3$. It was shown that $\Theta[2; 1, 1, 1]$ is extremal in [3], and H. Osaka recently showed that $\Theta[2; c_1, c_3, c_3]$, with $c_1c_2c_3 = 1$, is extremal using the methods in [3].

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**References**