A CLASS OF ATOMIC POSITIVE LINEAR MAPS IN 3–DIMENSIONAL MATRIX ALGEBRAS

SEUNG-HYEOK KYE

Department of Mathematics, Seoul National University
Seoul 151-742, Korea
E-mail: kye@krsnucc1.bitnet

ABSTRACT
We find a class of positive linear maps from the 3-dimensional matrix algebra into itself which cannot be decomposed into the sums of 2-positive maps and 2-copositive maps.

1. Introduction

Let M_n be the C^*-algebra of all $n\times n$ matrices over the complex field. Because the structure of the positive cone $\mathcal{P}(M_n)$ of positive linear maps between M_n is very complicated even in lower dimensions, it would be very useful to find simpler convex cones in $\mathcal{P}(M_n)$ with which every positive linear map can be written as a sum. In this vein, the cones of completely positive maps and completely copositive maps were possible candidates, and this is the case when $n = 2$ [**6**, **8**]. But, in other cases, there are positive maps which are not even the sum of a 2-positive map and a 2-copositive map. Such a map is said to be *atomic*. Although atomic maps are expected to play a role to understand the structure of $\mathcal{P}(M_n)$, examples of such maps are very rare in the literature [**3**, **4**, **5**, **7**]. To the best of the author's knowledge, there is only one known example of atomic maps in the 3-dimensional case [**3**, **7**].

In this note, we use the technique in [**4**] to produce a large class of atomic maps in 3-dimensional matrix algebra. Such examples are provided as variants of positive linear maps investigated in [**1**]. For nonnegative real numbers a, c_1, c_2 and c_3, we define the linear map $\Theta[a; c_1, c_2, c_3]$ (denoted by just Θ if there is no confusion) from M_3 into M_3 by

$$\Theta[a; c_1, c_2, c_3](x) = \Delta[a; c_1, c_2, c_3](x) - x,$$

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where

$$\Delta[a;c_1,c_2,c_3]((x_{ij})) = \begin{pmatrix} ax_{11}+c_1x_{33} & 0 & 0 \\ 0 & ax_{22}+c_2x_{11} & 0 \\ 0 & 0 & ax_{33}+c_3x_{22} \end{pmatrix}$$

for each $(x_{ij}) \in M_3$. Note that $\Theta[a;c,c,c] = \Phi[a,0,c]$ with the notation in [**1**]. We show that $\Theta[a;c_1,c_2,c_3]$ is positive if and only if $a \geq 2$ and $c_1c_2c_3 \geq (3-a)^3$, and it is 2-positive if and only if $a \geq 3$. Finally, we see that every positive map $\Theta[a;c_1,c_2,c_3]$ which is not 2-positive becomes an atomic map. The map $\Theta[2;1,1,1]$ is just the example of an atom mentioned above [**7**].

2. Positivity and 2-positivity

Theorem 2.1. *The linear map $\Theta[a;c_1,c_2,c_3]$ is positive if and only if the following two conditions are satisfied:*

(2.1.i) $$a \geq 2$$

(2.1.ii) $$c_1c_2c_3 \geq (3-a)^3$$

By the same argument as [**7**, Section 2], it suffices to show the following inequality:

Lemma 2.2. *Let a, b and c be nonnegative real numbers. Then the inequality*

$$\frac{\alpha}{a\alpha+c_1\gamma}+\frac{\beta}{a\beta+c_2\alpha}+\frac{\gamma}{a\gamma+c_3\beta} \leq 1$$

holds for every positive real numbers α, β, γ if and only if the two conditions in (2.1) are satisfied.

Proof. For the necessity, we take $\gamma = 0$ and $\beta \to \infty$, to get the first condition. Taking

$$\alpha = c_1^{\frac{2}{3}}c_3^{\frac{1}{3}}, \qquad \beta = c_2^{\frac{2}{3}}c_1^{\frac{1}{3}}, \qquad \text{and} \qquad \gamma = c_3^{\frac{2}{3}}c_2^{\frac{1}{3}},$$

we get the second condition by a calculation.

In order to prove the sufficiency, put

$$x = c_1\frac{\gamma}{\alpha}, \qquad y = c_2\frac{\alpha}{\beta}, \qquad z = c_3\frac{\beta}{\gamma}.$$

Then, it suffuices to show

$$\frac{1}{a+x}+\frac{1}{a+y}+\frac{1}{a+z} \leq 1,$$

or equivalently,

$$F = xyz + (a-1)(xy + yz + zx) + (a^2 - 2a)(x + y + z) + (a^3 - 3a^2) \geq 0$$

under the constraint $xyz = c_1c_2c_3$.

Using the inequalities $x + y + z \geq 3(xyz)^{\frac{1}{3}}$ and $xy + yz + xz \geq 3(xyz)^{\frac{2}{3}}$, we have

$$\begin{aligned} F &\geq (c_1c_2c_3) + 3(a-1)(c_1c_2c_3)^{\frac{2}{3}} + 3(a^2 - 2a)(c_1c_2c_3)^{\frac{1}{3}} + (a^3 - 3a^2) \\ &= ((c_1c_2c_3)^{\frac{1}{3}} + a)^2 ((c_1c_2c_3)^{\frac{1}{3}} + a - 3) \\ &\geq 0, \end{aligned}$$

from the conditions (2.1). This completes the proof. □

We denote by $M_k(M_n)$ the matrix algebra of order k over M_n. For a linear map $\phi : M_n \to M_n$, we define two linear maps ϕ_k and ϕ^k between $M_k(M_n)$ by

$$\begin{aligned} \phi_k([a_{ij}]_{i.j=1}^k) &= [\phi(a_{ij})]_{i.j=1}^k \\ \phi^k([a_{ij}]_{i.j=1}^k) &= [\phi(a_{ji})]_{i.j=1}^k, \end{aligned}$$

for $[a_{ij}] \in M_k(M_n)$. Recall that the linear map ϕ is *k-positive* (respectively *k-copositive*) if ϕ_k (respectively ϕ^k) is positive, and ϕ is *completely positive* (*completely copositive*) if ϕ is k-positive (respectively k-copositive) for each positive integer $k = 1, 2, \ldots$. It is well known that $\phi : M_n \to M_n$ is completely positive if and only if ϕ is n-positive, and this is equivalent to the positivity of the matrix $\phi_n([e_{ij}]_{i,j=1}^n)$ in $M_n(M_n)$ [**2**], where $\{e_{ij}\ ;\ i, j = 1, 2, \ldots, n\}$ is the usual matrix units. Similarly, $\phi : M_n \to M_n$ is completely copositive if and only if $\phi^n([e_{ij}]_{i,j=1}^n)$ is a positive matrix.

Theorem 2.3. *The linear map $\Theta[a; c_1, c_2, c_3]$ is completely positive if and only if it is 2-positive if and only if the following condition is satisfied.*

$$a \geq 3. \tag{2.2}$$

Proof. Assume that $\Theta[a; c_1, c_2, c_3]$ is 2-positive. Let $\xi = (0, 1, 1, 1, 1, 0) \in \mathbb{C}^6$ and $P = \xi^*\xi \in M_6$. Then, we have $\Delta(P) \geq P$, and this is the case if and only if $(\Delta_2(P)^{-1}\xi, \xi) \leq 1$ by [**7**, Lemma 2], because the matrix $\Delta_2(P)$ is non-singular. By a direct calculation, we have $(\Delta_2(P)^{-1}\xi, \xi) = \frac{3}{a}$, and so the condition (2.2) follows.

Now, we see that the eigenvalues of the 9×9 matrix $\Theta_3([e_{ij}])$ are $0, c_1, c_2, c_3, a$ and $a - 3$. Hence, if $a \geq 3$ then the map Θ is completely positive. It is clear that the completely positivity implies the 2-positivity. □

3. Atomic Positive Maps

For a linear map $\tau : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, we define the real linear map

$$\widetilde{\tau}([x_{ij}]) = \frac{1}{2}(\tau([x_{ij}]) + \overline{\tau([x_{ij}])}), \qquad [x_{ij}] \in M_3(\mathbb{R})$$

as in [4]. It is clear that if τ is k-positive (respectively k-copositive) then so is $\widetilde{\tau}$ for $k = 1, 2, \ldots$.

Although the following result is in fact contained in the proof of [4, Theorem], we include a sketch of the proof for the completeness. Note that the linear map τ between M_3 can be identified with the matrix $[\tau(e_{ij})]_{i,j=1}^3 \in M_3(M_3)$.

Proposition 3.1. *Let τ be a positive linear map between $M_3(\mathbb{C})$. Assume that the matrix $[\tau(e_{ij})] \in M_3(M_3)$ is of the form*

$$\begin{pmatrix}
\cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdot & 0 & \cdot \\
\cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & \cdot \\[2ex]
\cdot & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\
\cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot & 0 & \cdot \\[2ex]
\cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdot & 0 & \cdot \\
0 & \cdot & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot & 0 & \cdot
\end{pmatrix}. \tag{3.1}$$

If τ is the sum of a 2-positive linear map ϕ and a 2-copositive linear map ψ, then $\widetilde{\tau}$ is a 2-positive linear map.

Proof. From the relation $\tau = \phi + \psi$ and the positivity of ϕ and ψ, we see that the 3×3 diagonal submatrices of $[\phi(e_{ij})]$ and $[\psi(e_{ij})]$ have the same form as those of $[\tau(e_{ij})]$. Furthermore, every 2×2 submatrix of $[\phi(e_{ij})]$ is positive because ϕ is 2-positive. Similarly, every 2×2 submatrix of $[\psi(e_{ji})]$ is also positive. Comparing two matrices $[\tau(e_{ij})]$ and $[\phi(e_{ij})] + [\psi(e_{ij})]$, we see that the matrix $[\psi(e_{ij})]$ is of the form

$$\begin{pmatrix}
\cdot & \cdot & 0 & 0 & 0 & 0 & \beta & 0 & 0 \\
\cdot & \cdot & 0 & 0 & \alpha & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\[2ex]
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \overline{\alpha} & 0 & 0 & \cdot & \cdot & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 & \gamma \\[2ex]
\overline{\beta} & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & \cdot \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \overline{\gamma} & \cdot & 0 & \cdot
\end{pmatrix},$$

where every diagonal 3×3 submatrix is self-adjoint.

From the above matrix form, it is easy to see that $\widetilde{\psi}(A) = \widetilde{\psi}(A^t)$ for every matrix $A \in M_3(\mathbb{R})$. If $\begin{pmatrix} A & B \\ B^t & D \end{pmatrix}$ is a positive matrix in $M_2(M_3(\mathbb{R}))$ then we have

$$(\widetilde{\psi})_2 \begin{pmatrix} A & B \\ B^t & D \end{pmatrix} = \begin{pmatrix} \widetilde{\psi}(A) & \widetilde{\psi}(B) \\ \widetilde{\psi}(B^t) & \widetilde{\psi}(D) \end{pmatrix} = \begin{pmatrix} \widetilde{\psi}(A) & \widetilde{\psi}(B^t) \\ \widetilde{\psi}(B) & \widetilde{\psi}(D) \end{pmatrix} = (\widetilde{\psi})^2 \begin{pmatrix} A & B \\ B^t & D \end{pmatrix} \geq 0,$$

because $\widetilde{\psi}$ is 2-copositive. Hence, we see that $\widetilde{\psi}$ is 2-positive, and it follows that $\widetilde{\tau} = \widetilde{\phi} + \widetilde{\psi}$ is a 2-positive map in $M_3(\mathbb{R})$. □

Note that the associated matrix of the map $\Theta[a; c_1, c_2, c_3]$ is of the form (3.1), and it is easy to see that $\widetilde{\Theta}$ is not 2-positive for $2 \leq a < 3$ from the proof of Theorem 2.3. Hence, we have the result:

Theorem 3.2. *For the positive real numbers a, c_1, c_2 and c_3 satisfying the conditions:*

$$2 \leq a < 3 \qquad \text{and} \qquad c_1c_2c_3 \geq (3-a)^3,$$

the maps $\Theta[a; c_1, c_2, c_3]$ are atomic positive linear maps between $M_3(\mathbb{C})$.

It is easy to see that the map $\mathbb{R}^4 \to \mathcal{P}(M_3)$ given by

$$(a, c_1, c_2, c_3) \mapsto \Theta[a; c_1, c_2, c_3]$$

is an affine map. Therefore, the map $\Theta[a; c_1, c_2, c_3]$ is not extremal if $a > 2$ or $c_1c_2c_3 > (3-a)^3$. It was shown that $\Theta[2; 1, 1, 1]$ is extremal in [**3**], and H. Osaka recently showed that $\Theta[2; c_1, c_2, c_3]$, with $c_1c_2c_3 = 1$, is extremal using the methods in [**3**].

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