*Math. Proc. Camb. Phil. Soc.* (1997), **122**, 45
*Printed in Great Britain*


# On the convex set of completely positive linear maps in matrix algebras

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(*Received* 2 *October* 1995; *revised* 29 *January* 1996)

*Abstract*

Let $P_I$ (respectively $CP_I$) be the convex compact set of all unital positive (respectively completely positive) linear maps from the matrix algebra $M_m(\mathbb{C})$ into $M_n(\mathbb{C})$. We show that maximal faces of $CP_I$ correspond to one dimensional subspaces of the vector space $M_{m,n}(\mathbb{C})$. Furthermore, a maximal face of $CP_I$ lies on the boundary of $P_I$ if and only if the corresponding subspace is generated by a rank one matrix.

## 1. *Introduction*

Let $M_n$ be the $C^*$-algebra of all $n \times n$ matrices over the complex field, and $P_I$ the convex compact set of all unital positive linear maps from $M_m$ into $M_n$, that is, the maps which send the set of positive semi-definite matrices into itself. The convex structures of $P_i$ are highly complicated even in low dimensions, and several authors [**CL**, **KK**, **O**, **R**, **S**, **W**] have considered the possibility of decomposition of $P_I$ into simpler convex subsets. One of them is the convex set $CP_I$ of all unital completely positive linear maps from $M_m$ into $M_n$. Every linear map $\phi: M_m \to M_n$ corresponds to a linear functional $\theta_\phi: M_m \otimes M_n \to \mathbb{C}$ by the formula

$$\theta_\phi: A \mapsto \frac{1}{n} \sum_{i,j=1}^{n} \langle \phi(A_{ij})\, e_j, e_i \rangle, \quad A = \sum_{i,j=1}^{n} A_{ij} \otimes E_{ij} \in M_m \otimes M_n,$$

where $\{E_{ij}\}$ is the usual matrix units for $M_n$ and $\{e_j : j = 1, 2, \ldots, n\}$ is the usual orthonormal basis of $\mathbb{C}^n$. It is easy to see that every projection $P \in M_m \otimes M_n$ gives rise to a face

$$F_P = \{\phi \in CP_I : \theta_\phi(P) = 0\} \tag{1·1}$$

of the convex set $CP_I$. It has been shown in [**SW**] that the map $P \mapsto F_P$ is surjective.

In this note, we find the right inverse of this map which is a lattice isomorphism from the lattice $\mathscr{F}(CP_I)$ of all faces of $CP_I$ into the lattice $\mathscr{E}(M_{m,n})$ of all subspaces of the vector space $M_{m,n}$ of all $m \times n$ complex matrices. With this machinery to hand, we show that maximal faces of $CP_I$ correspond to $(mn-1)$ dimensional subspaces of $M_{m,n}$, or equivalently to one-dimensional subspaces of $M_{m,n}$ with respect to the inner product on $M_{m,n}$ arising from the trace of $M_n$. Because every maximal face of $P_I$ corresponds to a pair of one-dimensional subspaces in $\mathbb{C}^m$ and $\mathbb{C}^n$ [**K**], we see that the

* Partially supported by MOE and GARC.

convex set $CP_I$ has many more maximal faces than $P_I$. We also show that a maximal face of $CP_I$ lies on the boundary of $P_I$ if and only if its corresponding one-dimensional subspace is generated by a rank one matrix.

Throughout this note, we fix the natural numbers $m$ and $n$. By the interior $\operatorname{int} C$ of a convex set $C$ in $\mathbb{R}^\nu$, we mean the relative interior of $C$ with respect to the affine manifold generated by $C$. The boundary $\partial C$ of $C$ is the set difference $C \backslash \operatorname{int} C$.

## 2. *Faces of $CP_I$*

We define the inner product $\langle\, , \rangle_{\mathrm{Tr}}$ on the vector space $M_{m,n}$ of all $m \times n$ complex matrices by

$$\langle V, W \rangle = \operatorname{Tr}(W^* V), \quad V, W \in M_{m,n}(\mathbb{C}),$$

where Tr is the usual trace of $n \times n$ matrices. With this inner product, the vector spaces $M_{m,n}$ and $\mathbb{C}^m \otimes \mathbb{C}^n$ are isometrically isomorphic to each other by the following correspondence

$$V \longleftrightarrow \sum_{j=1}^{n} V(j) \otimes e_j,$$

where $V(j)$ is the $j$th column of $V$. We confuse these two inner product spaces. We also identify a projection in $M_m \otimes M_n$ and its range space in $M_{m,n} = \mathbb{C}^m \otimes \mathbb{C}^n$. For a family $\mathscr{V} = \{V_k : k = 1, 2, \ldots, s\}$ of $m \times n$ matrices, we define the linear map $\phi_{\mathscr{V}} : M_m \to M_n$ by

$$\phi_{\mathscr{V}} : X \mapsto \sum_{k=1}^{s} V_k^* X V_k, \quad X \in M_m.$$

We recall [**C**] that $\phi_{\mathscr{V}}$ is completely positive, and every completely positive linear map arises in this way with a linearly independent family $\mathscr{V}$. In order to compute $\theta_{\phi_{\mathscr{V}}}(Q)$ for a projection $Q \in M_m \otimes M_n$, we choose an orthonormal basis $\{W_l \in \mathbb{C}^m \otimes \mathbb{C}^n = M_{m,n} : l = 1, 2, \ldots, r\}$ of the range space of $Q$. Then we have

$$Q = \sum_{l=1}^{r} \sum_{i,j=1}^{n} W_l(i)\, W_l(j)^* \otimes E_{ij},$$

and so it follows that

$$\begin{aligned}
\theta_{\phi_{\mathscr{V}}}(Q) &= \frac{1}{n} \sum_{l=1}^{r} \sum_{i,j=1}^{n} \langle \psi_{\mathscr{V}}(W_l(i)\, W_l(j)^*)\, e_j, e_i \rangle \\
&= \frac{1}{n} \sum_{l=1}^{r} \sum_{k=1}^{s} \sum_{i,j=1}^{n} \langle V_k^*\, W_l(i)\, W_l(j)^*\, V_k\, e_j, e_i \rangle \\
&= \frac{1}{n} \sum_{l=1}^{r} \sum_{k=1}^{s} \sum_{,j=1}^{n} \langle W_l(j)^*\, V_k\, e_j, W_l(i)^*\, V_k\, e_i \rangle \\
&= \frac{1}{n} \sum_{l=1}^{r} \sum_{k=1}^{s} \left\langle \sum_{j=1}^{n} W_l(j)^*\, V_k(j) \sum_{i=1}^{n} W_l(i)^*\, V_k(i) \right\rangle \\
&= \frac{1}{n} \sum_{l=1}^{r} \sum_{k=1}^{s} |\langle V_l, W_k \rangle_{\mathrm{Tr}}|^2.
\end{aligned}$$

Therefore, we have the following:

PROPOSITION 2·1. *Let $Q$ be a projection in $M_m \otimes M_n$, and $\mathscr{V}$ a finite subset of $M_{m,n}$. Then the following are equivalent:*

(i) $\theta_{\phi_{\mathscr{V}}}(Q) = 0$,
(ii) *The span of $\mathscr{V}$ and the range space of $Q$ are orthogonal to each other.*

For a subspace $E$ of $M_{m,n}$, we define $\Psi(E) = F_{E^\perp}$. Then by Proposition 2·1, we see that

$$\Psi(E) = \{\phi_{\mathscr{V}} \in CP_I : \operatorname{span} \mathscr{V} \subseteq E\}. \tag{2·1}$$

Because $\operatorname{span} \mathscr{V} \subseteq E_1 \wedge E_2$ if and only if $\operatorname{span} \mathscr{V} \subseteq E_i$ for $i = 1, 2$, we have the following:

PROPOSITION 2·2. *The map $E \mapsto \Psi(E)$ is a meet homomorphism from the subspace lattice $\mathscr{E}(M_{m,n})$ of $M_{m,n}$ into the face lattice $\mathscr{F}(CP_I)$ of $CP_I$.*

Now, we proceed to find the right inverse of $\Psi$.

PROPOSITION 2·3. *Let $\mathscr{V} = \{V_k : k = 1, 2, \dots, s\}$ and $\mathscr{W} = \{W_l : l = 1, 2, \dots, r\}$ be linearly independent subsets of $M_{m,n}$ such that $\phi_{\mathscr{V}}, \phi_{\mathscr{W}} \in CP_I$. Then the following are equivalent:*

(i) $\operatorname{span} \mathscr{W} \subseteq \operatorname{span} \mathscr{V}$,
(ii) *There is a real number $t > 1$ such that $(1-t)\,\phi_{\mathscr{W}} + t\phi_{\mathscr{V}} \in CP_I$.*

*Proof.* For the direction (i) $\Rightarrow$ (ii), we write

$$W_l = \sum_{k=1}^{s} a_{lk} V_k, \quad l = 1, 2, \dots, r.$$

Then we have

$$\begin{aligned} \phi_{\mathscr{W}}(X) &= \sum_{l=1}^{r} \left( \sum_{k=1}^{s} a_{lk} V_k \right)^* X \left( \sum_{j=1}^{s} a_{lj} V_j \right) \\ &= \sum_{k,j=1}^{s} \left( \sum_{l=1}^{r} \overline{a_{lk}}\, a_{lj} \right) V_k^* X V_j. \end{aligned}$$

We write $A$ the $r \times s$ matrix whose $(l,k)$-entry is $a_{lk}$. Then there is $t > 1$ such that $(1-t)\,A^*A + tI$ is positive semi-definite, which will be denoted by $B^*B$, with an $s \times s$ matrix $B = [b_{lk}]$. Then we have

$$\begin{aligned} [(1-t)\,\phi_{\mathscr{W}} + t\phi_{\mathscr{V}}](X) &= \sum_{k,j=1}^{s} \left( \sum_{l=1}^{s} \overline{b_{lk}}\, b_{lj} \right) V_k^* X V_j \\ &= \sum_{l=1}^{s} \left( \sum_{k=1}^{s} b_{lk} V_k \right)^* X \left( \sum_{j=1}^{s} b_{lj} V_j \right), \end{aligned}$$

and so it follows that $(1-t)\,\phi_{\mathscr{W}} + t\phi_{\mathscr{V}}$ is completely positive.

For the converse, assume that $(1-t)\,\phi_{\mathscr{W}} + t\phi_{\mathscr{V}}$ is completely positive linear map for a real $t > 1$. We denote this map by $\phi_{\mathscr{W}'}$ with a subset $\mathscr{W}'$ of $M_{m,n}$. Then $\phi_{\mathscr{V}}$ is a convex combination of $\phi_{\mathscr{W}}$ and $\phi_{\mathscr{W}'}$. By [**C**, remark 4], we see that each $W_l$ lies in $\operatorname{span} \mathscr{V}$. ∎

Recall that a point $x$ of a convex set $C$ is an interior point of $C$ if and only if for

each $y \in C$ there is a real number $t > 1$ such that $(1-t)y + tx \in C$. Let $F$ be a non-empty face of $CP_I$. If $\phi_{\mathscr{V}} \in \operatorname{int} F$ and $\phi_{\mathscr{W}} \in F$ then we have $\operatorname{span} \mathscr{W} \subseteq \operatorname{span} \mathscr{V}$ by Proposition 2·3. Therefore, we see that $\operatorname{span} \mathscr{V} = \operatorname{span} \mathscr{W}$ whenever $\phi_{\mathscr{V}}$ and $\phi_{\mathscr{W}}$ are interior points of $F$. For a face $F$ of $CP_I$, we define $\Phi(F) = \operatorname{span} \mathscr{V}$ with an interior point $\phi_{\mathscr{V}}$ of $F$. For the empty face $\varnothing$, we define $\Phi(\varnothing) = \{0\}$.

COROLLARY 2·4. *If $\mathscr{V}$ is a finite subset of $M_{m,n}$ and $F$ is a face of $CP_I$ then the following are equivalent:*

(i) $\phi_{\mathscr{V}} \in F$,
(ii) $\operatorname{span} \mathscr{V} \subseteq \Phi(F)$.

*Proof.* The direction (i) $\Rightarrow$ (ii) is clear. For the converse, assume that $\operatorname{span} \mathscr{V} \subseteq \Phi(F)$, and choose an interior point $\phi_{\mathscr{W}}$ of $F$. Then $\operatorname{span} \mathscr{V} \subseteq \operatorname{span} \mathscr{W}$ implies that there is $t > 1$ such that $\psi = (1-t)\phi_{\mathscr{V}} + t\phi_{\mathscr{W}} \in CP_I$ by Proposition 2·3. Therefore, we see that $\phi_{\mathscr{W}} \in F$ is a non-trivial convex combination of $\phi_{\mathscr{V}}$ and $\psi$. Since $F$ is a face, we conclude that $\phi_{\mathscr{V}} \in F$. ∎

THEOREM 2·5. *For each $F \in \mathscr{F}(CP_I)$, we have $\Psi(\Phi(F)) = F$.*

*Proof.* Apply Corollary 2·4 and relation (2·1). ∎

PROPOSITION 2·6. *The map $F \mapsto \Phi(F)$ is a join homomorphism from the lattice $\mathscr{F}(CP_I)$ into the lattice $\mathscr{E}(M_{m,n})$.*

*Proof.* Let $F_i \in \mathscr{F}(CP_I)$ with an interior point $\phi_{\mathscr{V}_i}$ for $i = 1, 2$. Then $\frac{1}{2}(\phi_{\mathscr{V}_1} + \phi_{\mathscr{V}_2})$ is an interior point of $F_1 \vee F_2$ by [**K**, proposition 2·4]. Therefore, we have

$$\Phi(F_1 \vee F_2) = \operatorname{span}\{\mathscr{V}_1, \mathscr{V}_2\} = \operatorname{span} \mathscr{V}_1 \vee \operatorname{span} \mathscr{V}_2 = \Phi(F_1) \vee \Phi(F_2). \quad ∎$$

If $f\colon L_1 \to L_2$ is a join homomorphism and $g\colon L_2 \to L_1$ is a meet homomorphism between lattices $L_1$ and $L_2$ such that $g \circ f$ is the identity, then both $f\colon L_1 \to f(L_1)$ and $g\colon f(L_1) \to L_1$ are lattice isomorphisms. Indeed, if $x, y \in L_1$ and $f(x) \wedge f(y) = f(z)$ for some $z \in L_1$ then

$$z = g(f(z)) = g(f(x) \wedge f(y)) = g(f(x)) \wedge g(f(y)) = x \wedge y,$$

and so $f\colon L_1 \to f(L_1)$ is a meet homomorphism. Similarly, we have

$$g(f(x) \vee f(y)) = g(f(x \vee y)) = x \vee y = g(f(x)) \vee g(f(y)),$$

which shows that $g\colon f(L_1) \to L_1$ is a join homomorphism. We denote by $\mathscr{E}_{CP_I}$ the range of the map $\Phi\colon \mathscr{F}(CP_I) \to \mathscr{E}(M_{m,n})$. Then we see that $\Phi$ is a lattice isomorphism from $\mathscr{F}(CP_I)$ onto $\mathscr{E}_{CP_I}$ and $\Psi$ is a lattice isomorphism from $\mathscr{E}_{CP_I}$ onto $\mathscr{F}(CP_I)$, by Propositions 2·2 and 2·6.

PROPOSITION 2·7. *Let $E$ be a non-zero subspace of $M_{m,n}$. Then the following are equivalent:*

(i) $E \in \mathscr{E}_{CP_I}$.
(ii) *There is a subset $\mathscr{V} = \{V_k : k = 1, \dots, s\}$ of $M_{m,n}$ such that* $\operatorname{span} \mathscr{V} = E$ *and $\sum_{k=1}^{s} V_k^* V_k$ is a scalar multiple of the identity.*

*Proof.* If there is a face $F$ of $CP_I$ such that $\Phi(F) = E$ then take $\phi_{\mathscr{V}}$ in the interior of $F$. Then $\mathscr{V}$ satisfies the conditions in (ii). Conversely, assume that there is a subset

$\mathscr{V}$ as in (ii). Let $F$ be the face such that $\phi_{\mathscr{V}}$ is an interior point of $F$. Then we have $\Phi(F) = \operatorname{span} \mathscr{V} = E$. ∎

Note that not every subspace of $M_{m,n}$ belongs to $\mathscr{E}_{CP_I}$. Indeed, if $V \in M_{m,n}$ then the one dimensional space spanned by $V$ belongs to $\mathscr{E}_{CP_I}$ if and only if $V^* V$ is a scalar multiple of the identity, that is, $V$ or $V^*$ is a scalar multiple of an isometry.

## 3. *Maximal faces of $CP_I$*

In this section, we show that every $(mn-1)$ dimensional subspace of $M_{m,n}$ belongs to $\mathscr{E}_{CP_I}$. To do this, we construct for each $V \in M_{m,n}$ a family $\{V_\alpha\}$ of $m \times n$ matrices which spans $V^\perp$ and such that $\sum_\alpha V_\alpha^* V_\alpha$ is a scalar multiple of the identity. Our construction depends on the rank of $V$. If $V$ is a rank one matrix then $V$ is of the form $\xi_1 \eta_1^*$, where $\xi_1 \in \mathbb{C}^m$ and $\eta_1 \in \mathbb{C}^n$ may be assume to be unit vectors. We choose orthonormal bases $\{\xi_i : i = 1, \dots, m\}$ and $\{\eta_j : j = 1, \dots, n\}$ including $\xi_1$ and $\eta_1$, respectively. Then

$$(\xi_i \eta_j^*)^* (\xi_k \eta_l^*) = \eta_j \xi_i^* \xi_k \eta_l^* = \delta_{ik} \eta_j \eta_l^*, \tag{3·1}$$

and so $\langle \xi_k \eta_l^*, \xi_i \eta_j^* \rangle = \delta_{ik} \operatorname{Tr}(\eta_j \eta_l^*) = \delta_{ik} \delta_{jl}$. Denote $S = \{(i,j) : i = 2, \dots, m$ or $j = 2, \dots, n\}$, and define

$$V_{i,j} = \begin{cases} \dfrac{1}{\sqrt{m}} \xi_i \eta_j^*, & j = 2, \dots, n, \quad i = 1, \dots, m, \\[2ex] \dfrac{1}{\sqrt{(m-1)}} \xi_i \eta_j^*, & j = 1, \quad i = 2, \dots, m. \end{cases}$$

Then we have

$$\sum_{(i,j) \in S} V_{i,j}^* V_{i,j} = \sum_{i=2}^{m} V_{i,1}^* V_{i,1} + \sum_{j=2}^{n} \sum_{i=1}^{m} V_{i,j}^* V_{i,j} = \eta_1 \eta_1^* + \sum_{j=2}^{n} \eta_j \eta_j^* = I.$$

Therefore, we see that $\{V_{i,j} : (i,j) \in S\}$ is a required family of matrices. For the general cases with arbitrary ranks, we need the following:

Lemma 3·1. *Assume that $\{\xi_1, \xi_2\}$ is linearly independent. Then there exist vectors $x$, $y \in \operatorname{span}\{\xi_1, \xi_2\}$ with the following properties:*

$$\langle x, y \rangle = 0, \quad \langle x, \xi_1 \rangle + \langle y, \xi_2 \rangle = 0, \quad \langle x, \xi_1 \rangle \neq 0, \quad \langle y, \xi_2 \rangle \neq 0.$$

*Proof.* Write $\langle \xi_1, \xi_2 \rangle = re^{i\theta}$ with $r = |\langle \xi_1, \xi_2 \rangle|$. Put

$$\zeta = \frac{\xi_1}{\|\xi_1\|} + e^{i\theta} \frac{\xi_2}{\|\xi_2\|}, \quad \omega = \frac{\xi_1}{\|\xi_1\|} - e^{i\theta} \frac{\xi_2}{\|\xi_2\|}.$$

Then we see that $\langle \zeta, \omega \rangle = 0$ and

$$\|\zeta\| = 2\left(1 + \frac{r}{\|\xi_1\| \, \|\xi_2\|}\right), \quad \|\omega\| = 2\left(1 - \frac{r}{\|\xi_1\| \, \|\xi_2\|}\right) \tag{3·2}$$

are non-zero, because $\{\xi_1, \xi_2\}$ is linearly independent. Define

$$x = \|\xi_2\| \left(\frac{\zeta}{\|\zeta\|} + \frac{\omega}{\|\omega\|}\right), \quad y = e^{-i\theta} \|\xi_1\| \left(-\frac{\zeta}{\|\zeta\|} + \frac{\omega}{\|\omega\|}\right)$$

then it follows that $\langle x, y\rangle = 0$ since $\langle \zeta, \omega\rangle = 0$. From (3·2), we also have $\langle x, \xi_1\rangle = \|\xi_1\|\,\|\xi_2\|$ and $\langle y, \xi_2\rangle = -\|\xi_1\|\,\|\xi_2\|$, from which the conclusion follows.

THEOREM 3·2. *For every $V \in M_{m,n}$, we have $V^\perp \in \mathscr{E}_{CP_I}$.*

*Proof.* If $V$ is of rank one, we have already done. Assume that $V$ is of rank $r$, with $r \geqslant 2$. Take an orthonormal basis $\{\eta_j : j = 1, \dots, r\}$ of $(\operatorname{Ker} V)^\perp$ and put $\xi_j = V\eta_j$ for each $j = 1, \dots, r$. Then $\{\xi_j : j = 1, \dots, r\}$ is a basis of $\operatorname{Im} V$, and we have

$$V = \sum_{j=1}^{r} \xi_j \eta_j^*.$$

Multiply $V$ by a constant, we may assume that $\sum_{j=1}^{r} \|\xi_j\|^2 = r$ without loss of generality. For each fixed $h = 2, \dots, r$, choose $x_h, y_h \in \operatorname{span}\{\xi_1, \xi_h\}$ such that

$$\langle x_h, y_h\rangle = 0, \quad \langle x_h, \xi_1\rangle + \langle y_h, \xi_h\rangle = 0,$$
$$\langle x_h, \xi_1\rangle \neq 0, \quad \langle y_h, \xi_h\rangle \neq 0, \quad \|x_h\| \leqslant 1, \quad \|y_h\| \leqslant 1.$$

For each $j = 1, \dots, r$, we also take an orthogonal basis $\{\zeta_{ij} : i = 1, \dots, r-1\}$ of $(\operatorname{Im} V) \cap \xi_j^\perp$ with

$$\sum_{i=1}^{r-1} \|\zeta_{i1}\|^2 + \sum_{h=2}^{r} \|x_h\|^2 = \sum_{i=1}^{r-1} \|\zeta_{ij}\|^2 + \|y_j\|^2 = r, \quad j = 2, \dots, r.$$

Finally, we take orthonormal bases $\{\eta_l : l = r+1, \dots, n\}$ and $\{\xi_k : k = r+1, \dots, m\}$ of $\operatorname{Ker} V$ and $(\operatorname{Im} V)^\perp$, respectively. Define

$$\begin{aligned} V_{ij} &= \zeta_{ij}\eta_j^*, && i = 1, \dots, r-1, \quad j = 1, \dots, r, \\ W_h &= x_h \eta_1^* + y_h \eta_h^*, && h = 2, \dots, r, \\ U_{kl} &= \xi_k \eta_l^*, && k = r+1, \dots, m \quad \text{or} \quad l = r+1, \dots, n. \end{aligned}$$

Since $\operatorname{Tr}(\eta_j \eta_l^*) = \delta_{jl}$, we have

$$\begin{aligned} \langle V, V_{ij}\rangle &= \sum_{l=1}^{r} \operatorname{Tr}(\eta_j \zeta_{ij}^* \xi_l \eta_l^*) = \operatorname{Tr}(\eta_j \zeta_{ij}^* \xi_j \eta_j) = \langle \xi_j, \zeta_{ij}\rangle = 0, \\ \langle V, W_h\rangle &= \operatorname{Tr}(\eta_1 x_h^* \xi_1 \eta_1^*) + \operatorname{Tr}(\eta_h y_h^* \xi_h \eta_h^*) = \langle \xi_1, x_h\rangle + \langle \xi_h, y_h\rangle = 0, \\ \langle V, U_{kl}\rangle &= \sum_{j=1}^{r} \operatorname{Tr}(\eta_l \xi_k^* \xi_j \eta_j^*) = \sum_{j=1}^{r} \langle \xi_j, \xi_k\rangle \operatorname{Tr}(\eta_l \eta_j^*) = 0. \end{aligned}$$

Therefore, every element in $\mathscr{V} := \{V_{ij}, W_h, U_{kl}\}$ belongs to $V^\perp$. Now, we have

$$\left.\begin{aligned} V_{ij}^* V_{ij} &= \eta_j \zeta_{ij}^* \zeta_{ij} \eta_j^* = \|\zeta_{ij}\|^2 \eta_j \eta_j^*, \\ W_h^* W_h &= (\eta_1 x_h^* + \eta_h y_h^*)(x_h \eta_1^* + y_h \eta_h^*) = \|x_h\|^2 \eta_1 \eta_1^* + \|y_h\|^2 \eta_h \eta_h^*, \\ U_{kl}^* U_{kl} &= \eta_l \xi_k^* \xi_k \eta_l^* = \|\xi_k\|^2 \eta_l \eta_l^*, \end{aligned}\right\} \tag{3·3}$$

and so it follows that

$$\begin{aligned} \sum_{i,j} V_{ij}^* V_{ij} + \sum_h W_h^* W_h &= \left(\sum_{i=1}^{r-1} \|\zeta_{i1}\|^2 + \sum_{h=2}^{r} \|x_h\|^2\right) \eta_1 \eta_1^* + \sum_{j=2}^{r} \left(\sum_{i=1}^{r-1} \|\zeta_{ij}\|^2 + \|y_j\|^2\right) \eta_j \eta_j^* \\ &= r\eta_1 \eta_1^* + r \sum_{j=2}^{r} \eta_j \eta_j^* = r \sum_{j=1}^{r} \eta_j \eta_j^*, \end{aligned}$$

and

$$\sum_{k,l} U_{kl}^* U_{kl} = \sum_{l=1}^{r} \sum_{k=r+1}^{m} \|\xi_k\|^2 \eta_l \eta_l^* + \sum_{l=r+1}^{n} \sum_{k=1}^{m} \|\xi_k\|^2 \eta_l \eta_l^*$$
$$= (m-r) \sum_{l=1}^{r} \eta_l \eta_l^* + m \sum_{l=r+1}^{n} \eta_l \eta_l^*,$$

because $\sum_{k=1}^{r} \|\xi_k\|^2 = r$ and $\|\xi_k\| = 1$ for $k = r+1, \dots, m$. Summing up all of (3·3), it follows that

$$\sum_{V \in \mathscr{V}} V^* V = mI.$$

In order to show that $\operatorname{span} \mathscr{V} = V^\perp$, it remains to show that the family $\mathscr{V}$ is linearly independent since the cardinality of $\mathscr{V}$ is $mn-1$. To do this, assume that

$$\sum_{i,j} a_{ij} V_{ij} + \sum_{h} b_h W_h + \sum_{k,l} c_{kl} U_{kl} = 0.$$

For each fixed $l = r+1, \dots, n$, we apply $\eta_l$ to get $\sum_{k=1}^{m} c_{kl} \xi_k = 0$. Since $\{\xi_k\}$ is linearly independent, we have $c_{kl} = 0$ for each $k = 1, \dots, m$ and $l = r+1, \dots, m$. Applying the vector $\eta_h$ for each fixed $h = 1, \dots, r$, we have

$$\sum_{i=1'}^{r-1} a_{ih} \zeta_{ih} + b_h y_h + \sum_{k=r+1}^{m} c_{kh} \xi_k = 0, \quad h = 2, \dots, r, \tag{3·4}$$

$$\sum_{i=1}^{r-1} a_{i1} \zeta_{i1} + \sum_{h=2}^{r} b_h x_h + \sum_{k=r+1}^{m} c_{k1} \xi_k = 0, \quad h = 1. \tag{3·5}$$

Note that the last terms in (3·4) and (3·5) belong to $(\operatorname{Im} V)^\perp$, but the remaining terms in $\operatorname{Im} V$, and so we see that every $c_{kl}$ is zero. Taking inner product with $\xi_h$ in (3·4), we have $b_h = 0$ for each $h = 2, \dots, r$. From the fact that $\{\zeta_{ih} : i = 1, \dots, r-1\}$ is orthogonal for each $h = 1, \dots, r$, we finally conclude that $a_{ih} = 0$ for each $h = 1, \dots, r$ and $i = 1, \dots, r-1$. ∎

We combine the results in Section 2 and Theorem 3·2, to get the following characterization of maximal faces of $CP_I$.

THEOREM 3·3. *For each $m \times n$ matrix $V \in M_{m,n}$, the set*

$$F_I[V] := \{\phi_{\mathscr{V}} \in CP_I : \operatorname{span} \mathscr{V} \subseteq V^\perp\} \tag{3·6}$$

*is a maximal face of $CP_I$. Conversely, every maximal face of $CP_I$ arises in this way.*

If $P$ is a rank one matrix in $M_m \otimes M_n$ whose range space is generated by $V \in M_{m,n} = \mathbb{C}^m \otimes \mathbb{C}^n$ then $F_I[V]$ is nothing but $F_P$ defined in (1·1).

## 4. *Relations with $P_I$*

In [**K**], we have characterized maximal faces of $P_I$. For each pair $(\xi, \eta) \in \mathbb{C}^m \times \mathbb{C}^n$ of unit vectors, the set

$$F_I[\xi, \eta] = \{\phi \in P_I : \langle \phi(\xi\xi^*) \eta, \eta \rangle = 0\}$$

is a maximal face of $P_I$, and every maximal face of $P_I$ arises in this way. In this

section, we compare facial structures of $P_I$ and $CP_I$. We begin with the following two simple lemmas:

LEMMA 4·1. *Let $\mathscr{V} = \{V_k : k = 1, \dots, s\}$ be a family of $m \times n$ matrices such that $\phi_{\mathscr{V}} \in CP_I$. Then for each unit vectors $\xi \in \mathbb{C}^m$ and $\eta \in \mathbb{C}^n$ the following are equivalent:*

(i) $\phi_{\mathscr{V}} \in F_I[\xi, \eta]$.
(ii) $\langle \xi, V_k \eta \rangle = 0$ *for each* $k = 1, \dots, s$.

*Proof.* We have

$$\phi_{\mathscr{V}} \in F_I[\xi, \eta] \Leftrightarrow \left\langle \sum_{k=1}^{s} V_k^* \xi\xi^* V_k \eta, \eta \right\rangle = 0 \Leftrightarrow \sum_{k=1}^{s} |\xi^* V_k \eta|^2 = 0.$$

LEMMA 4·2. *Let $V = \xi\eta^*$ be a rank one matrix with unit vectors $\xi \in \mathbb{C}^m$ and $\eta \in \mathbb{C}^n$. Then we have the identity*

$$F_I[V] = F_I[\xi, \eta] \bigcap \partial(CP_I).$$

*Proof.* If $W = \xi_1 \eta_1^*$ is a rank one matrix then we have

$$\langle V, W \rangle = \mathrm{Tr}(\eta_1 \xi_1^* \xi\eta^*) = \langle \xi, \xi_1 \rangle \langle \eta, \eta_1 \rangle = \langle \xi, \xi_1 \eta_1^* \eta \rangle = \langle \xi, W\eta \rangle,$$

and so the same relation holds for each matrix $W \in M_{m,n}$. Therefore, for each $\mathscr{V} = \{V_k : k = 1, \dots, s\}$ we have

$$\begin{aligned} \phi_{\mathscr{V}} \in F_I[V] &\Leftrightarrow \langle V, V_k \rangle = 0 \quad \text{for each} \quad k = 1, \dots, s \\ &\Leftrightarrow \langle \xi, V_k \eta \rangle = 0 \quad \text{for each} \quad k = 1, \dots, s \\ &\Leftrightarrow \phi_{\mathscr{V}} \in F_I[\xi, \eta]. \end{aligned}$$

The conclusion follows since $F_I[V] \subseteq \partial(CP_I)$. ∎

We denote by $P$ (respectively $CP$) the convex cone of all positive (respectively completely positive) linear maps from $M_m$ into $M_n$. We also define

$$F[V] = \{\phi_{\mathscr{V}} \in CP : \mathrm{span}\, \mathscr{V} \subseteq V^\perp\}, \quad F[\xi, \eta] = \{\phi \in P : \langle \phi(\xi\xi^*)\eta, \eta \rangle = 0\}.$$

Then every argument in Lemmas 4·1 and 4·2 is still valid if $P_I$, $CP_I$, $F_I[V]$ and $F_I[\xi, \eta]$ are replaced by $P$, $CP$, $F[V]$ and $F[\xi, \eta]$, respectively.

COROLLARY 4·3. *Let $\mathscr{V} = \{V_k : k = 1, \dots, s\}$ be a family of $m \times n$ matrices. Then the following are equivalent:*

(i) *$\phi_{\mathscr{V}}$ is an interior point of $P$.*
(ii) *The set $\{V_k \eta : k = 1, \dots, s\}$ spans $\mathbb{C}^m$ for each non-zero $\eta \in \mathbb{C}^n$.*
(iii) *$\mathscr{V}^\perp$ does not contain a rank one matrix.*

*Proof.* The equivalence of (i) and (ii) follows from Lemma 4·1, because the boundary of a convex set in a Euclidean space is the union of all maximal faces (see [**Rf**, theorem 18·2]). We prove (ii)$\Leftrightarrow$(iii). If $\mathscr{V}^\perp$ contains a rank one matrix $V = \xi\eta^*$ then $\mathrm{span}\, \mathscr{V} \subseteq V^\perp$. Hence, $\phi_{\mathscr{V}} \in F[V] \subseteq F[\xi, \eta]$. This means that $E_\eta := \mathrm{span}\{V_k \eta : k = 1, \dots, s\}$ is contained in $\xi^\perp$ by Lemma 4·1. Conversely, if $E_\eta$ is a proper subspace of $\mathbb{C}^m$ and $\xi \in E_\eta^\perp$ then $\phi_{\mathscr{V}} \in F[\xi, \eta]$. Hence, $\phi_{\mathscr{V}} \in F[V]$ with $V = \xi\eta^*$ by Lemma 4·2, and so it follows that $\mathrm{span}\, \mathscr{V} \subseteq V^\perp$ and $V \in \mathscr{V}^\perp$. ∎

THEOREM 4·4. *The maximal face $F_I[V]$ of $CP_I$ lies on the boundary of $P_I$ if and only if $V$ is of rank one. If this is the case with $V = \xi\eta^*$ then $F_I[V] \subseteq F_I[\xi, \eta]$.*

*Proof.* If $V$ is of rank one then $F_I[V]$ lies in a maximal face of $P_I$ by Lemma 4·2. For the converse, we assume that $V = \sum_{j=1}^{r} \xi_j \eta_j^*$ is a matrix of rank $r \geqslant 2$ where $\{\eta_j : j = 1, \dots, r\}$ is orthonormal, and $F_I[V]$ lies on the boundary of $P_I$. Then $F_I[V]$ lies in a maximal face of $P_I$, say $F_I[\xi, \eta]$. We show that $\xi = 0$ or $\eta = 0$, to get a contradiction. Assume that $\eta \neq 0$.

We retain every notation in the proof of Theorem 3·2, and write $\xi = \sum_{i=1}^{m} \alpha_i \xi_i$ and $\eta = \sum_{j=1}^{n} b_j \eta_j$. Apply Lemma 4·1, to get

$$0 = \langle \xi, U_{kl}\, \eta \rangle = \langle \xi, b_l\, \xi_k \rangle = a_k\, \overline{b_l}$$

for each $k = r+1, \dots, m$ and $l = 1, \dots, n$. Therefore, we have $(a_{r+1}, \dots, a_m) = 0$ or $(b_1, \dots, b_n) = 0$. But, since $\eta \neq 0$, we have $(a_{r+1}, \dots, a_m) = 0$ and $\xi \in \operatorname{span}\{\xi_1, \dots, \xi_r\}$. For each $l = r+1, \dots, n$ and $k = 1, \dots, r$, we also have

$$0 = \langle \xi, U_{kl}\, \eta \rangle = \langle \xi, b_l\, \xi_k \rangle = \sum_{i=1}^{r} a_i\, \overline{b_l}\, \langle \xi_i, \zeta_k \rangle.$$

Since $\{\xi_i : i = 1, \dots, r\}$ is linearly independent, the $r \times r$ matrix $[\langle \xi_i, \xi_k \rangle]_{i,k=1}^{r}$ is non-singular and so $a_i\, \overline{b_l} = 0$ for each $i = 1, \dots, r$ and $l = r+1, \dots, n$. If $(b_{r+1}, \dots, b_n) \neq 0$ then $(a_1, \dots, a_r) = 0$ and so we have $\xi = 0$. We proceed to consider the case $(b_{r+1}, \dots, b_n) = 0$ and $\eta \in \operatorname{span}\{\eta_1, \dots, \eta_r\}$.

Fix $\lambda = 1, 2, \dots, r-1$. In the proof of Theorem 3·2, the family $\{\zeta_{ij}\}$ may be chosen so that the relations

$$\zeta_{ij} \in \{\xi_l : l = 1, \dots, r, \quad l \neq j + \lambda \,(\operatorname{mod} r)\}^{\perp}, \quad j = 1, \dots, r$$

hold. Then it follows that $\langle \zeta_{1j}, \xi_{j+\lambda\,(\operatorname{mod} r)} \rangle \neq 0$, and

$$0 = \langle \xi, V_{1j}\, \eta \rangle = \sum_{l=1}^{r} a_l\, \overline{b_j}\, \langle \xi_l, \zeta_{1j} \rangle = a_{j+\lambda\,(\operatorname{mod} r)}\, \overline{b_j}\, \langle \xi_{j+\lambda\,(\operatorname{mod} r)}, \zeta_{1j} \rangle.$$

Therefore, we have $a_{j+\lambda\,(\operatorname{mod} r)}\, \overline{b_j} = 0$ for each $j = 1, \dots, r$. Since this is true for each $\lambda = 1, \dots, r-1$, we see that

$$a_i\, \overline{b_j} = 0, \quad \text{whenever } i \neq j.$$

For each $h = 2, \dots, r$, we have

$$0 = \langle \xi, W_h\, \eta \rangle = \langle \xi, b_1\, x_h + b_h\, y_h \rangle = a_1\, \overline{b_1}\, \langle \xi_1, x_h \rangle + a_h\, \overline{b_h}\, \langle \xi_h, y_h \rangle.$$

If $b_1 \neq 0$ then $a_h\, \overline{b_1} = 0$ implies that $a_h = 0$. Since $\langle \xi_1, x_h \rangle \neq 0$ we have $a_1\, \overline{b_1} = 0$. Therefore, we have $a_1\, \overline{b_1} = 0$ in any cases, and the relation $\langle \xi_h, y_h \rangle \neq 0$ gives us $a_h\, \overline{b_h} = 0$ for each $h = 2, \dots, r$. Hence, we see that $a_i\, \overline{b_j} = 0$ for each $i, j = 1, 2, \dots, r$. Because $(b_1, \dots, b_r) \neq 0$, we have $(a_1, \dots, a_r) = 0$ and conclude that $\xi = 0$. ∎

By Theorem 4·4, we see that whenever $V$ is a matrix with rank $\geqslant 2$ the interior of the maximal face $F_I[V]$ of $CP_I$ lies in the interior of $P_I$. With this information, one may suspect there might be an extreme point of $CP_I$ which lies in the interior of $P_I$. But this is not the case, whenever $m \geqslant n$.

Proposition 4·5. *If $m \geqslant n$ then every extreme point of $CP_I$ lies on the boundary of $P_I$.*

*Proof.* If $\phi_{\mathscr{V}}$ is an extreme point of $CP_I$ then the cardinality of $\mathscr{V}$ is less than or

equal to $n$ by [**C**, remark 6]. Applying Corollary 4·3, it suffices to show that for any $m \times n$ matrices $V_1, \dots, V_n$ there is $\eta \in \mathbb{C}^n$ such that $\{V_k \eta : k = 1, \dots, n\}$ does not span $\mathbb{C}^m$. If $m > n$ then this is clear, and so it remains to consider the case $m = n$. If one of $V_k$ is singular then this is also clear by taking a non-zero null vector. If each $V_k$ is non-singular, we take an eigenvector $\eta$ of $V_1^{-1} V_2$. Then $\{V_1 \eta, V_2 \eta\}$ is linearly dependent. ∎

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