On the convex set of completely positive linear maps in matrix algebras

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Abstract

Let $P_i$ (respectively $CP_i$) be the convex compact set of all unital positive (respectively completely positive) linear maps from the matrix algebra $M_m(\mathbb{C})$ into $M_n(\mathbb{C})$. We show that maximal faces of $CP_i$ correspond to one dimensional subspaces of the vector space $M_{m,n}(\mathbb{C})$. Furthermore, a maximal face of $CP_i$ lies on the boundary of $P_i$ if and only if the corresponding subspace is generated by a rank one matrix.

1. Introduction

Let $M_n$ be the $C^*$-algebra of all $n \times n$ matrices over the complex field, and $P_i$ the convex compact set of all unital positive linear maps from $M_m$ into $M_n$, that is, the maps which send the set of positive semi-definite matrices into itself. The convex structures of $P_i$ are highly complicated even in low dimensions, and several authors [CL, KK, O, R, S, W] have considered the possibility of decomposition of $P_i$ into simpler convex subsets. One of them is the convex set $CP_i$ of all unital completely positive linear maps from $M_m$ into $M_n$. Every linear map $\phi : M_m \rightarrow M_n$ corresponds to a linear functional $\theta_\phi : M_m \otimes M_n \rightarrow \mathbb{C}$ by the formula

$$\theta_\phi : A \mapsto \frac{1}{n} \sum_{i,j=1}^{n} \langle \phi(A_{ij}) e_j, e_i \rangle, \quad A = \sum_{i,j=1}^{n} A_{ij} \otimes E_{ij} \in M_m \otimes M_n,$$

where $\{E_{ij}\}$ is the usual matrix units for $M_n$ and $\{e_j : j = 1, 2, \ldots, n\}$ is the usual orthonormal basis of $\mathbb{C}^n$. It is easy to see that every projection $P \in M_m \otimes M_n$ gives rise to a face

$$F_P = \{ \phi \in CP_i : \theta_\phi(P) = 0 \}$$

of the convex set $CP_i$. It has been shown in [SW] that the map $P \mapsto F_P$ is surjective.

In this note, we find the right inverse of this map which is a lattice isomorphism from the lattice $\mathcal{F}(CP_i)$ of all faces of $CP_i$ into the lattice $\mathcal{E}(M_{m,n})$ of all subspaces of the vector space $M_{m,n}$ of all $m \times n$ complex matrices. With this machinery to hand, we show that maximal faces of $CP_i$ correspond to $(mn-1)$ dimensional subspaces of $M_{m,n}$, or equivalently to one-dimensional subspaces of $M_{m,n}$ with respect to the inner product on $M_{m,n}$ arising from the trace of $M_n$. Because every maximal face of $P_i$ corresponds to a pair of one-dimensional subspaces in $\mathbb{C}^m$ and $\mathbb{C}^n$ [K], we see that the

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convex set $CP_I$ has many more maximal faces than $P_I$. We also show that a maximal face of $CP_I$ lies on the boundary of $P_I$ if and only if its corresponding one-dimensional subspace is generated by a rank one matrix.

Throughout this note, we fix the natural numbers $m$ and $n$. By the interior $\text{int} C$ of a convex set $C$ in $\mathbb{R}^r$, we mean the relative interior of $C$ with respect to the affine manifold generated by $C$. The boundary $\partial C$ of $C$ is the set difference $C \setminus \text{int} C$.

2. Faces of $CP_I$

We define the inner product $\langle , \rangle_{\text{Tr}}$ on the vector space $M_{m,n}$ of all $m \times n$ complex matrices by

$$\langle V, W \rangle = \text{Tr}(W^*V), \quad V, W \in M_{m,n}(\mathbb{C}),$$

where $\text{Tr}$ is the usual trace of $n \times n$ matrices. With this inner product, the vector spaces $M_{m,n}$ and $\mathbb{C}^m \otimes \mathbb{C}^n$ are isometrically isomorphic to each other by the following correspondence

$$V \mapsto \sum_{j=1}^n V(j) \otimes e_j,$$

where $V(j)$ is the $j$th column of $V$. We confuse these two inner product spaces. We also identify a projection in $M_m \otimes M_n$ and its range space in $M_{m,n} = \mathbb{C}^m \otimes \mathbb{C}^n$.

For a family $\mathscr{F} = \{V_k : k = 1, 2, \ldots, s\}$ of $m \times n$ matrices, we define the linear map $\phi_{\mathscr{F}} : M_m \to M_n$ by

$$\phi_{\mathscr{F}} : X \mapsto \sum_{k=1}^s V_k^* XV_k, \quad X \in M_m.$$

We recall that $\phi_{\mathscr{F}}$ is completely positive, and every completely positive linear map arises in this way with a linearly independent family $\mathscr{F}$. In order to compute $\theta_{\phi_{\mathscr{F}}}(Q)$ for a projection $Q \in M_m \otimes M_n$, we choose an orthonormal basis $\{W_l \in \mathbb{C}^m \otimes \mathbb{C}^n = M_{m,n} : l = 1, 2, \ldots, r\}$ of the range space of $Q$. Then we have

$$Q = \sum_{l=1}^r \sum_{i,j=1}^n W_l(i) W_l(j)^* \otimes E_{ij},$$

and so it follows that

$$\theta_{\phi_{\mathscr{F}}}(Q) = \frac{1}{n} \sum_{l=1}^r \sum_{i,j=1}^n \|\phi_{\mathscr{F}}(W_l(i) W_l(j)^*) e_j e_i\|^2$$

$$= \frac{1}{n} \sum_{l=1}^r \sum_{i,j=1}^s \sum_{k=1}^n \langle V_k^* W_l(i) W_l(j)^* V_k e_j, e_i \rangle$$

$$= \frac{1}{n} \sum_{l=1}^r \sum_{i,j=1}^s \sum_{k=1}^n \langle W_l(j)^* V_k e_j, W_l(i)^* V_k e_i \rangle$$

$$= \frac{1}{n} \sum_{l=1}^r \sum_{i,j=1}^s \langle \sum_{j=1}^n W_l(j)^* V_k(j) \sum_{i=1}^n W_l(i)^* V_k(i) \rangle$$

$$= \frac{1}{n} \sum_{l=1}^r \sum_{i,j=1}^s \langle V_l, W_k \rangle_{\text{Tr}}^2.$$

Therefore, we have the following:
Proposition 2.1. Let $Q$ be a projection in $M_m \otimes M_n$, and $\mathcal{V}$ a finite subset of $M_{m,n}$. Then the following are equivalent:

(i) $\theta_{\phi_{\mathcal{V}}}(Q) = 0$,

(ii) The span of $\mathcal{V}$ and the range space of $Q$ are orthogonal to each other.

For a subspace $E$ of $M_{m,n}$, we define $\Psi(E) = F_{E^{\perp}}$. Then by Proposition 2.1, we see that

\[ \Psi(E) = \{ \phi_\mathcal{V} \in CP_1 : \text{span } \mathcal{V} \subseteq E \}. \]  

(2.1)

Because span $\mathcal{V} \subseteq E_1 \wedge E_2$ if and only if span $\mathcal{V} \subseteq E_i$ for $i = 1, 2$, we have the following:

Proposition 2.2. The map $E \mapsto \Psi(E)$ is a meet homomorphism from the subspace lattice $\mathcal{E}(M_{m,n})$ of $M_{m,n}$ into the face lattice $\mathcal{F}(CP_1)$ of $CP_1$.

Now, we proceed to find the right inverse of $\Psi$.

Proposition 2.3. Let $\mathcal{W} = \{ V_k : k = 1, 2, \ldots, s \}$ and $\mathcal{U} = \{ W_l : l = 1, 2, \ldots, r \}$ be linearly independent subsets of $M_{m,n}$ such that $\phi_\mathcal{W}, \phi_\mathcal{U} \in CP_1$. Then the following are equivalent:

(i) span $\mathcal{U} \subseteq$ span $\mathcal{W}$,

(ii) There is a real number $t > 1$ such that $(1-t)\phi_\mathcal{W} + t\phi_\mathcal{U} \in CP_1$.

Proof. For the direction (i) $\Rightarrow$ (ii), we write

\[ W_l = \sum_{k=1}^{s} a_{lk} V_k, \quad l = 1, 2, \ldots, r. \]

Then we have

\[ \phi_\mathcal{U}(X) = \sum_{l=1}^{r} \left( \sum_{k=1}^{s} a_{lk} V_k \right)^* \left( \sum_{j=1}^{s} a_{lj} V_j \right) \]

\[ = \sum_{k,j=1}^{s} \left( \sum_{l=1}^{r} a_{lk} a_{lj} \right) V_k^* V_j. \]

We write $A$ the $r \times s$ matrix whose $(l,k)$-entry is $a_{lk}$. Then there is $t > 1$ such that $(1-t)A^*A + tI$ is positive semi-definite, which will be denoted by $B^*B$, with an $s \times s$ matrix $B = [b_{lk}]$. Then we have

\[ [(1-t)\phi_\mathcal{W} + t\phi_\mathcal{U}](X) = \sum_{k,j=1}^{s} \left( \sum_{l=1}^{r} b_{lk} b_{lj} \right) V_k^* V_j \]

\[ = \sum_{l=1}^{r} \left( \sum_{k=1}^{s} b_{lk} V_k \right)^* X \left( \sum_{j=1}^{s} b_{lj} V_j \right), \]

and so it follows that $(1-t)\phi_\mathcal{W} + t\phi_\mathcal{U}$ is completely positive.

For the converse, assume that $(1-t)\phi_\mathcal{W} + t\phi_\mathcal{U}$ is completely positive linear map for a real $t > 1$. We denote this map by $\phi_{\mathcal{V}'}$ with a subset $\mathcal{V}'$ of $M_{m,n}$. Then $\phi_{\mathcal{V}'}$ is a convex combination of $\phi_\mathcal{W}$ and $\phi_\mathcal{U}$. By [C, remark 4], we see that each $W_l$ lies in span $\mathcal{V}'$.

Recall that a point $x$ of a convex set $C$ is an interior point of $C$ if and only if for
each $y \in C$ there is a real number $t > 1$ such that $(1-t)y + tx \in C$. Let $F$ be a non-empty face of $CP_1$. If $\phi_\psi \in \text{int} F$ and $\phi_x \in F$ then we have $\text{span} \psi \subseteq \text{span} \phi$ by Proposition 2.3. Therefore, we see that $\text{span} \phi = \text{span} \psi$ whenever $\phi_\psi$ and $\phi_x$ are interior points of $F$. For a face $F$ of $CP_1$, we define $\Phi(F) = \text{span} \phi$ with an interior point $\phi_\psi$ of $F$. For the empty face $\emptyset$, we define $\Phi(\emptyset) = \{0\}$.

**Corollary 2.4.** If $\nu$ is a finite subset of $M_{m,n}$ and $F$ is a face of $CP_1$ then the following are equivalent:

(i) $\phi_\psi \in F$.

(ii) $\text{span} \psi \subseteq \Phi(F)$.

**Proof.** The direction (i) $\Rightarrow$ (ii) is clear. For the converse, assume that $\text{span} \psi \subseteq \Phi(F)$, and choose an interior point $\phi_\psi$ of $F$. Then $\text{span} \psi \subseteq \text{span} \phi$ implies that there is $t > 1$ such that $\psi = (1-t) \phi_\psi + t \phi_y \in CP_1$ by Proposition 2.3. Therefore, we see that $\phi_y \in F$ is a non-trivial convex combination of $\phi_\psi$ and $\phi$. Since $F$ is a face, we conclude that $\phi_\psi \in F$.

**Theorem 2.5.** For each $F \in \mathcal{F}(CP_1)$, we have $\Psi(\Phi(F)) = F$.

**Proof.** Apply Corollary 2.4 and relation (2.1).

**Proposition 2.6.** The map $F \mapsto \Phi(F)$ is a join homomorphism from the lattice $\mathcal{F}(CP_1)$ into the lattice $\mathcal{E}(M_{m,n})$.

**Proof.** Let $F_i \in \mathcal{F}(CP_1)$ with an interior point $\phi_\psi$ for $i = 1, 2$. Then $\frac{1}{2}(\phi_\psi + \phi_\psi)$ is an interior point of $F_1 \lor F_2$ by [K, proposition 2.4]. Therefore, we have

$$\Phi(F_1 \lor F_2) = \text{span} \{\psi_1, \psi_2\} = \text{span} \psi_1 \lor \text{span} \psi_2 = \Phi(F_1) \lor \Phi(F_2).$$

If $f: L_1 \rightarrow L_2$ is a join homomorphism and $g: L_2 \rightarrow L_1$ is a meet homomorphism between lattices $L_1$ and $L_2$ such that $g \circ f$ is the identity, then both $f: L_1 \rightarrow f(L_1)$ and $g: f(L_1) \rightarrow L_1$ are lattice isomorphisms. Indeed, if $x, y \in L_1$ and $f(x) \land f(y) = f(z)$ for some $z \in L_1$ then $z = g(f(z)) = g(f(x) \land f(y)) = g(f(x)) \land g(f(y)) = x \land y$, and so $f: L_1 \rightarrow f(L_1)$ is a meet homomorphism. Similarly, we have

$$g(f(x) \lor f(y)) = g(f(x) \lor y) = x \lor y = g(f(x)) \lor f(f(y)),$$

which shows that $g: f(L_1) \rightarrow L_1$ is a join homomorphism. We denote by $\mathcal{E}_{CP_1}$ the range of the map $\Phi: \mathcal{F}(CP_1) \rightarrow \mathcal{E}(M_{m,n})$. Then we see that $\Phi$ is a lattice isomorphism from $\mathcal{F}(CP_1)$ onto $\mathcal{E}_{CP_1}$ and $\Psi$ is a lattice isomorphism from $\mathcal{E}_{CP_1}$ onto $\mathcal{F}(CP_1)$, by Propositions 2.2 and 2.6.

**Proposition 2.7.** Let $E$ be a non-zero subspace of $M_{m,n}$. Then the following are equivalent:

(i) $E \in \mathcal{E}_{CP_1}$

(ii) There is a subset $\nu = \{V_k: k = 1, \ldots, s\}$ of $M_{m,n}$ such that $\text{span} \nu = E$ and $\sum_{k=1}^s V_k = 0$ is a scalar multiple of the identity.

**Proof.** If there is a face $F$ of $CP_1$ such that $\Phi(F) = E$ then take $\phi_\psi$ in the interior of $F$. Then $\nu$ satisfies the conditions in (ii). Conversely, assume that there is a subset
Then we have \( \Phi(F) = \text{span} \, \mathcal{V} = E \).

Note that not every subspace of \( M_{m,n} \) belongs to \( \mathcal{E}_{CP} \). Indeed, if \( V \in M_{m,n} \) then the one dimensional space spanned by \( V \) belongs to \( \mathcal{E}_{CP} \) if and only if \( V^* \) is a scalar multiple of the identity, that is, \( V \) or \( V^* \) is a scalar multiple of an isometry.

### 3. Maximal faces of \( CP \)

In this section, we show that every \((mn-1)\) dimensional subspace of \( M_{m,n} \) belongs to \( \mathcal{E}_{CP} \). To do this, we construct for each \( V \in M_{m,n} \) a family \( \{V_i\}_i \) of \( m \times n \) matrices which spans \( V^\perp \) and such that \( \sum_i V_i^* \) is a scalar multiple of the identity. Our construction depends on the rank of \( V \). If \( V \) is a rank one matrix then \( V \) is of the form \( \eta \xi^* \), where \( \xi \in \mathbb{C}^m \) and \( \eta \in \mathbb{C}^n \) may be assumed to be unit vectors. We choose orthonormal bases \( \{\xi_i : i = 1, \ldots, m\} \) and \( \{\eta_j : j = 1, \ldots, n\} \) including \( \xi \) and \( \eta \), respectively. Then

\[
(\xi_i \eta_i^*)(\xi_j \eta_j^*) = \eta_j \xi_i^* \xi_j \eta_i^* = \delta_{ij} \eta_j \eta_i^*,
\]

and so \( \langle \xi_i \eta_i^*, \xi_j \eta_j^* \rangle = \delta_{ij} \text{Tr}(\eta_j \eta_i^*) = \delta_{ij} \delta_{ji} \). Denote \( S = \{(i,j) : i = 2, \ldots, m \} \) or \( j = 2, \ldots, n\}, \) and define

\[
V_{i,j} = \begin{cases} 
\frac{1}{\sqrt{m}} \xi_i \eta_i^*, & j = 2, \ldots, n, \quad i = 1, \ldots, m, \\
\frac{1}{\sqrt{(m-1)} \xi_i \eta_i^*}, & j = 1, \quad i = 2, \ldots, m.
\end{cases}
\]

Then we have

\[
\sum_{(i,j) \in S} V_{i,j}^* V_{i,j} = \sum_{i=2}^m V_{i,1}^* V_{i,1} + \sum_{j=2}^n \sum_{i=1}^m V_{i,j}^* V_{i,j} = \eta \eta^* + \sum_{j=2}^n \eta_j \eta_j^* = I.
\]

Therefore, we see that \( \{V_{i,j} : (i,j) \in S\} \) is a required family of matrices. For the general cases with arbitrary ranks, we need the following:

**Lemma 3.1.** Assume that \( \{\xi_1, \xi_2\} \) is linearly independent. Then there exist vectors \( x, y \in \text{span} \{\xi_1, \xi_2\} \) with the following properties:

\[
\langle x, y \rangle = 0, \quad \langle x, \xi_1 \rangle + \langle y, \xi_2 \rangle = 0, \quad \langle x, \xi_1 \rangle \neq 0, \quad \langle y, \xi_2 \rangle \neq 0.
\]

**Proof.** Write \( \langle \xi_1, \xi_2 \rangle = re^{i\theta} \) with \( r = |\langle \xi_1, \xi_2 \rangle| \). Put

\[
\xi = \frac{\xi_1}{\langle \xi_1 \rangle} + e^{i\theta} \frac{\xi_2}{\langle \xi_2 \rangle} \quad \omega = \frac{\xi_1}{\langle \xi_1 \rangle} - e^{i\theta} \frac{\xi_2}{\langle \xi_2 \rangle}.
\]

Then we see that \( \langle \xi, \omega \rangle = 0 \) and

\[
\|\xi\| = 2 \left( 1 + \frac{r}{\langle \xi_1 \rangle \langle \xi_2 \rangle} \right), \quad \|\omega\| = 2 \left( 1 - \frac{r}{\langle \xi_1 \rangle \langle \xi_2 \rangle} \right)
\]

are non-zero, because \( \{\xi_1, \xi_2\} \) is linearly independent. Define

\[
x = \langle \xi_2 \rangle \left( \frac{\xi}{\|\xi\|} + \frac{\omega}{\|\omega\|} \right), \quad y = e^{-i\theta} \langle \xi_1 \rangle \left( -\frac{\xi}{\|\xi\|} + \frac{\omega}{\|\omega\|} \right)
\]
then it follows that \( \langle x, y \rangle = 0 \) since \( \langle \xi, \omega \rangle = 0 \). From (3.2), we also have \( \langle x, \xi_1 \rangle = \| \xi_1 \| \| \xi_2 \| \) and \( \langle y, \xi_2 \rangle = -\| \xi_2 \| \| \xi_2 \| \), from which the conclusion follows.

**Theorem 3.2.** For every \( V \in \mathcal{M}_{m,n} \), we have \( V^\perp \in \mathcal{E}_{CP} \).

**Proof.** If \( V \) is of rank one, we have already done. Assume that \( V \) is of rank \( r \), with \( r \geq 2 \). Take an orthonormal basis \( \{ \eta_j : j = 1, \ldots, r \} \) of (\( \text{Ker} \ V \)^\perp and put \( \xi_j = V \eta_j \) for each \( j = 1, \ldots, r \). Then \( \{ \xi_j : j = 1, \ldots, r \} \) is a basis of \( \text{Im} \ V \), and we have

\[
V = \sum_{j=1}^{r} \xi_j \eta_j^*.
\]

Multiply \( V \) by a constant, we may assume that \( \sum_{j=1}^{r} \| \xi_j \|^2 = r \) without loss of generality. For each fixed \( h = 2, \ldots, r \), choose \( x_h, y_h \in \text{span} \{ \xi_1, \xi_h \} \) such that

\[
\langle x_h, y_h \rangle = 0, \quad \langle x_h, \xi_1 \rangle + \langle y_h, \xi_h \rangle = 0,
\]

\[
\langle x_h, \xi_1 \rangle \neq 0, \quad \langle y_h, \xi_2 \rangle \neq 0, \quad \| x_h \| \leq 1, \quad \| y_h \| \leq 1.
\]

For each \( j = 1, \ldots, r \), we also take an orthogonal basis \( \{ \xi_{ij} : i = 1, \ldots, r-1 \} \) of (\( \text{Im} \ V \) ∩ \( \xi_j^\perp \)) with

\[
\sum_{i=1}^{r-1} \| \xi_{ij} \|^2 + \sum_{h=2}^{r} \| x_h \|^2 = \sum_{i=1}^{r-1} \| \xi_{ij} \|^2 + \| y_j \|^2 = r, \quad j = 2, \ldots, r.
\]

Finally, we take orthonormal bases \( \{ \eta_l : l = r+1, \ldots, n \} \) and \( \{ \xi_k : k = r+1, \ldots, m \} \) of \( \text{Ker} \ V \) and (\( \text{Im} \ V \)^\perp), respectively. Define

\[
V_{ij} = \xi_{ij} \eta_j^*, \quad i = 1, \ldots, r-1, \quad j = 1, \ldots, r,
\]

\[
W_h = x_h \eta_1^* + y_h \eta_h^*, \quad h = 2, \ldots, r,
\]

\[
U_{kl} = \xi_k \eta_l^*, \quad k = r+1, \ldots, m \quad \text{or} \quad l = r+1, \ldots, n.
\]

Since \( \text{Tr}(\eta_j \eta_j^*) = \delta_{jj} \), we have

\[
\langle V, V_{ij} \rangle = \sum_{j=1}^{r} \text{Tr}(\eta_j \xi_{ij}^* \xi_j) = \langle \xi_j, \xi_{ij} \rangle = 0,
\]

\[
\langle V, W_h \rangle = \langle \xi_j, x_h \eta_1^* + y_h \eta_h^* \rangle + \langle \eta_j, \xi_k \eta_l^* \rangle = \langle \xi_j, x_h \rangle + \langle \xi_k, y_h \rangle = 0,
\]

\[
\langle V, U_{kl} \rangle = \sum_{j=1}^{r} \text{Tr}(\eta_j \xi_{ij}^* \xi_k) = \sum_{j=1}^{r} \langle \xi_j, \xi_k \rangle \text{Tr}(\eta_j \eta_j^*) = 0.
\]

Therefore, every element in \( \mathcal{F} := \{ V_{ij}, W_h, U_{kl} \} \) belongs to \( V^\perp \). Now, we have

\[
V_{ij}^* V_{ij} = \eta_j \xi_{ij}^* \xi_j \eta_j^* = \| \xi_{ij} \|^2 \eta_j \eta_j^*,
\]

\[
W_h^* W_h = (x_h \eta_1^* + y_h \eta_h^*) (x_h \eta_1^* + y_h \eta_h^*) = \| x_h \|^2 \eta_1 \eta_1^* + \| y_h \|^2 \eta_h \eta_h^*,
\]

\[
U_{kl}^* U_{kl} = \xi_k \eta_l^* \xi_k \eta_l^* = \| \xi_k \|^2 \eta_l \eta_l^*,
\]

and so it follows that

\[
\sum_{i,j} V_{ij}^* V_{ij} + \sum_{h} W_h^* W_h = \left( \sum_{i=1}^{r-1} \| \xi_{ij} \|^2 + \sum_{h=2}^{r} \| x_h \|^2 \right) \eta_j \eta_j^* + \sum_{j=2}^{r} \left( \sum_{i=1}^{r-1} \| \xi_{ij} \|^2 + \| y_j \|^2 \right) \eta_j \eta_j^*
\]

\[
= r \eta_1 \eta_1^* + r \sum_{j=2}^{r} \eta_j \eta_j^* = r \sum_{j=1}^{r} \eta_j \eta_j^*.
\]
and

\[ \sum_{k,l} U_{kl}^* U_{kl} = \sum_{l=1}^{r} \sum_{k=r+1}^{m} \| \xi_k \|^2 \eta_l \eta_l^* + \sum_{l=r+1}^{n} \sum_{k=1}^{m} \| \xi_k \|^2 \eta_l \eta_l^* \]

\[ = (m-r) \sum_{l=1}^{r} \eta_l \eta_l^* + m \sum_{l=r+1}^{n} \eta_l \eta_l^*, \]

because \( \sum_{k=1}^{r} \| \xi_k \|^2 = r \) and \( \| \xi_k \| = 1 \) for \( k = r+1, \ldots, m \). Summing up all of (3.3), it follows that

\[ \sum_{V \in \mathcal{F}} V^* V = mI. \]

In order to show that \( \text{span} \mathcal{F} = V^\perp \), it remains to show that the family \( \mathcal{F} \) is linearly independent since the cardinality of \( \mathcal{F} \) is \( mn - 1 \). To do this, assume that

\[ \sum_{i,j} a_{ij} V_{ij} + \sum_{h} b_h W_h + \sum_{k,l} c_{kl} U_{kl} = 0. \]

For each fixed \( l = r+1, \ldots, n \), we apply \( \eta_l \) to get \( \sum_{k=1}^{m} c_{kl} \xi_k = 0 \). Since \( \{ \xi_k \} \) is linearly independent, we have \( c_{kl} = 0 \) for each \( k = 1, \ldots, m \) and \( l = r+1, \ldots, m \). Applying the vector \( \eta_h \) for each fixed \( h = 1, \ldots, r \), we have

\[ \sum_{i=1}^{r-1} a_{ih} \xi_{ih} + b_h y_h + \sum_{k=r+1}^{m} c_{kh} \xi_k = 0, \quad h = 2, \ldots, r, \quad (3.4) \]

\[ \sum_{i=1}^{r-1} a_{ih} \xi_{ih} + b_h x_h + \sum_{k=r+1}^{m} c_{kh} \xi_k = 0, \quad h = 1, \quad (3.5) \]

Note that the last terms in (3.4) and (3.5) belong to \( (\text{Im} V)^\perp \), but the remaining terms in \( \text{Im} V \), and so we see that every \( c_{kl} \) is zero. Taking inner product with \( \xi_h \) in (3.4), we have \( b_h = 0 \) for each \( h = 2, \ldots, r \). From the fact that \( \{ \xi_{ih} : i = 1, \ldots, r-1 \} \) is orthogonal for each \( h = 1, \ldots, r \), we finally conclude that \( a_{ih} = 0 \) for each \( h = 1, \ldots, r \) and \( i = 1, \ldots, r-1 \).

We combine the results in Section 2 and Theorem 3.2, to get the following characterization of maximal faces of \( CP_i \).

**Theorem 3.3.** For each \( m \times n \) matrix \( V \in M_{m,n} \), the set

\[ F_i[V] := \{ \phi \in CP_i : \text{span} \mathcal{F} \subseteq V^\perp \} \]

is a maximal face of \( CP_i \). Conversely, every maximal face of \( CP_i \) arises in this way.

If \( P \) is a rank one matrix in \( M_m \otimes M_n \) whose range space is generated by \( V \in M_{m,n} = \mathbb{C}^m \otimes \mathbb{C}^n \) then \( F_i[V] \) is nothing but \( F_i \) defined in (1.1).

### 4. Relations with \( P_i \)

In [K], we have characterized maximal faces of \( P_i \). For each pair \( (\xi, \eta) \in \mathbb{C}^m \times \mathbb{C}^n \) of unit vectors, the set

\[ F_i[\xi, \eta] = \{ \phi \in P_i : \langle \phi(\xi^*) \eta, \eta \rangle = 0 \} \]

is a maximal face of \( P_i \), and every maximal face of \( P_i \) arises in this way. In this
section, we compare facial structures of $P_I$ and $CP_I$. We begin with the following two simple lemmas:

**Lemma 4.1.** Let $\mathcal{F}^* = \{V_k: k = 1, \ldots, s\}$ be a family of $m \times n$ matrices such that $\phi_v \in CP_I$. Then for each unit vectors $\xi \in \mathbb{C}^m$ and $\eta \in \mathbb{C}^n$ the following are equivalent:

(i) $\phi_v \in F[I][\xi, \eta]$.

(ii) $\langle \xi, V_k \eta \rangle = 0$ for each $k = 1, \ldots, s$.

**Proof.** We have

$$\phi_v \in F[I][\xi, \eta] \Leftrightarrow \left\langle \sum_{k=1}^s V_k^* \xi \xi^* V_k \eta, \eta \right\rangle = 0 \Leftrightarrow \sum_{k=1}^s |\xi^* V_k \eta|^2 = 0.$$

**Lemma 4.2.** Let $V = \xi \eta^*$ be a rank one matrix with unit vectors $\xi \in \mathbb{C}^m$ and $\eta \in \mathbb{C}^n$. Then we have the identity

$$F[I][V] = F[I][\xi, \eta] \cap \partial(CP_I).$$

**Proof.** If $W = \xi_1 \eta_1^*$ is a rank one matrix then we have

$$\langle V, W \rangle = \text{Tr}(\eta_1 \xi_1^* \xi_1 \eta_1^*) = \langle \xi, \xi_1 \rangle \langle \eta, \eta_1 \rangle = \langle \xi, \xi_1 \eta_1^* \eta \rangle = \langle \xi, W \eta \rangle,$$

and so the same relation holds for each matrix $W \in M_{m,n}$. Therefore, for each $V = \{V_k: k = 1, \ldots, s\}$ we have

$$\phi_v \in F[I][V] \Leftrightarrow \langle V, V_k \rangle = 0 \quad \text{for each} \quad k = 1, \ldots, s$$

$$\Leftrightarrow \langle \xi, V_k \eta \rangle = 0 \quad \text{for each} \quad k = 1, \ldots, s$$

$$\Leftrightarrow \phi_v \in F[I][\xi, \eta].$$

The conclusion follows since $F[I][V] \subseteq \partial(CP_I)$.

We denote by $P$ (respectively $CP$) the convex cone of all positive (respectively completely positive) linear maps from $M_{m,n}$ into $M_n$. We also define

$$F[I][V] = \{\phi_v \in CP: \text{span } \mathcal{F} \subseteq V^\perp\}, \quad F[I][\xi, \eta] = \{\phi \in P: \langle \phi(\xi \xi^* \eta^*), \eta \rangle = 0\}.$$

Then every argument in Lemmas 4.1 and 4.2 is still valid if $P_I$, $CP_I$, $F[I][V]$ and $F[I][\xi, \eta]$ are replaced by $P$, $CP$, $F[V]$ and $F[\xi, \eta]$, respectively.

**Corollary 4.3.** Let $\mathcal{F}^* = \{V_k: k = 1, \ldots, s\}$ be a family of $m \times n$ matrices. Then the following are equivalent:

(i) $\phi_v$ is an interior point of $P$.

(ii) The set $\{V_k \eta: k = 1, \ldots, s\}$ spans $\mathbb{C}^m$ for each non-zero $\eta \in \mathbb{C}^n$.

(iii) $\mathcal{F}^\perp$ does not contain a rank one matrix.

**Proof.** The equivalence of (i) and (ii) follows from Lemma 4.1, because the boundary of a convex set in a Euclidean space is the union of all maximal faces (see [Rf, theorem 18.2]). We prove (ii) $\Rightarrow$ (iii). If $\mathcal{F}^\perp$ contains a rank one matrix $V = \xi \eta^*$ then span $\mathcal{F} \subseteq V^\perp$. Hence, $\phi_v \in F[V] \subseteq F[\xi, \eta]$. This means that $E_v := \text{span } \{V_k \eta: k = 1, \ldots, s\}$ is contained in $\xi^\perp$ by Lemma 4.1. Conversely, if $E_v$ is a proper subspace of $\mathbb{C}^m$ and $\xi \in E_v^\perp$ then $\phi_v \in F[\xi, \eta]$. Hence, $\phi_v \in F[V]$ with $V = \xi \eta^*$ by Lemma 4.2, and so it follows that span $\mathcal{F} \subseteq V^\perp$ and $V \in \mathcal{F}^\perp$.

**Theorem 4.4.** The maximal face $F[I][V]$ of $CP_I$ lies on the boundary of $P_I$ if and only if $V$ is of rank one. If this is the case with $V = \xi \eta^*$ then $F[I][V] \subseteq F[I][\xi, \eta]$. 
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Proof. If $V$ is of rank one then $F[V]$ lies in a maximal face of $P$, by Lemma 4.2. For the converse, we assume that $V = \sum_{j=1}^{r} y_j^*$ is a matrix of rank $r \geq 2$ where $\langle y_j: j = 1, \ldots, r \rangle$ is orthonormal, and $F[V]$ lies on the boundary of $P$. Then $F[V]$ lies in a maximal face of $P$, say $F[\xi, \eta]$. We show that $\xi = 0$ or $\eta = 0$, to get a contradiction. Assume that $\eta \neq 0$.

We retain every notation in the proof of Theorem 3, and write $\xi = \sum_{i=1}^{m} a_i \xi_i$ and $\eta = \sum_{j=1}^{n} b_j \eta_j$. Apply Lemma 4, to get

$$0 = \langle \xi, U_{kl} \eta \rangle = \langle \xi, b_l \xi_k \rangle = a_k \overline{b_l}$$

for each $k = r+1, \ldots, m$ and $l = 1, \ldots, n$. Therefore, we have $(a_{r+1}, \ldots, a_m) = 0$ or $(b_1, \ldots, b_n) = 0$. But, since $\eta \neq 0$, we have $(a_{r+1}, \ldots, a_m) = 0$ and $\xi \in \text{span} \{\xi_1, \ldots, \xi_r\}$. For each $l = r+1, \ldots, n$ and $k = 1, \ldots, r$, we also have

$$0 = \langle \xi, U_{kl} \eta \rangle = \langle \xi, b_l \xi_k \rangle = \sum_{i=1}^{r} a_i \overline{b_l} \langle \xi_i, \xi_k \rangle.$$ 

Since $\{\xi_i: i = 1, \ldots, r\}$ is linearly independent, the $r \times r$ matrix $[\langle \xi_i, \xi_k \rangle]_{i,k=1}^{r}$ is non-singular and so $a_i \overline{b_l} = 0$ for each $i = 1, \ldots, r$ and $l = r+1, \ldots, n$. If $(b_1, \ldots, b_n) = 0$ then $(a_1, \ldots, a_r) = 0$ and so we have $\xi = 0$. We proceed to consider the case $(b_1, \ldots, b_n) = 0$ and $y \in \text{span} \{\eta_1, \ldots, \eta_r\}$.

Fix $\lambda = 1, 2, \ldots, r-1$. In the proof of Theorem 3, the family $\{\xi_{ij}\}$ may be chosen so that the relations

$$\xi_{ij} \in \{\xi_i: l = 1, \ldots, r \}, \quad l \neq j (\mod r), \quad j = 1, \ldots, r$$

hold. Then it follows that $\langle \xi_{ij}, \xi_{i+\lambda (\mod r)} \rangle \neq 0$, and

$$0 = \langle \xi, V_{ij} \eta \rangle = \sum_{i=1}^{r} a_i \lambda_j \langle \xi_i, \xi_{ij} \rangle = a_{j+i \lambda (\mod r)} \overline{b_j} \langle \xi_{ij}, \xi_{ij} \rangle.$$ 

Therefore, we have $a_{j+i \lambda (\mod r)} \overline{b_j} = 0$ for each $j = 1, \ldots, r$. Since this is true for each $\lambda = 1, \ldots, r-1$, we see that

$$a_i \overline{b_j} = 0, \quad \text{whenever } i \neq j.$$ 

For each $h = 2, \ldots, r$, we have

$$0 = \langle \xi, W_{hh} \eta \rangle = \langle \xi, b_1 x_h + b_h y_h \rangle = a_1 \overline{b_1} \langle \xi_1, x_h \rangle + a_h \overline{b_h} \langle \xi_h, y_h \rangle.$$ 

If $b_i \neq 0$ then $a_i \overline{b_1} = 0$ implies that $a_i = 0$. Since $\langle \xi_1, x_h \rangle \neq 0$ we have $a_1 \overline{b_1} = 0$. Therefore, we have $a_i \overline{b_h} = 0$ in any cases, and the relation $\langle \xi_h, y_h \rangle \neq 0$ gives us $a_h \overline{b_h} = 0$ for each $h = 2, \ldots, r$. Hence, we see that $a_i \overline{b_j} = 0$ for each $i, j = 1, 2, \ldots, r$. Because $(b_1, b_2) = 0$, we have $(a_1, \ldots, a_r) = 0$ and conclude that $\xi = 0$.

By Theorem 4.4, we see that whenever $V$ is a matrix with rank $\geq 2$ the interior of the maximal face $F[V]$ of $CP$ lies in the interior of $P$. With this information, one may suspect there might be an extreme point of $CP$ which lies in the interior of $P$. But this is not the case, whenever $m \geq n$.

**Proposition 4.5.** If $m \geq n$ then every extreme point of $CP$ lies on the boundary of $P$.

**Proof.** If $\phi_\nu$ is an extreme point of $CP$ then the cardinality of $\nu$ is less than or
equal to $n$ by [C, remark 6]. Applying Corollary 4.3, it suffices to show that for any $m \times n$ matrices $V_1', \ldots, V_n'$ there is $\eta \in \mathbb{C}^n$ such that $\{V_k'\eta : k = 1, \ldots, n\}$ does not span $\mathbb{C}^m$. If $m > n$ then this is clear, and so it remains to consider the case $m = n$. If one of $V_k'$ is singular then this is also clear by taking a non-zero null vector. If each $V_k'$ is non-singular, we take an eigenvector $\eta$ of $V_1'^{-1}V_2$. Then $\{V_1'\eta, V_2'\eta\}$ is linearly dependent.

REFERENCES


