ON THE FACIAL STRUCTURES FOR POSITIVE LINEAR MAPS BETWEEN MATRIX ALGEBRAS

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Abstract. We give a lattice isomorphism between faces of the convex cone of all completely positive linear maps from M_m into M_n and subspaces of $m \times n$ matrices. Using this, we see that every face of the convex cone of all decomposable positive linear maps arises from a pair of subspaces of $m \times n$ matrices. Because every positive linear map from M_2 into M_2 is decomposable, we may determine completely the lattice structure for the faces of the convex cone of all positive linear maps between the 2×2 matrix algebras, in terms of pairs of subspaces in M_2.

1. Introduction

A linear map between C^*-algebras is said to be *positive* if it send every positive element to a positive element. The whole structures of the cone of all positive linear maps is extremely complicated even in low dimensional matrix algebras and far from being understood completely. After Stinespring's contribution [19], completely positive linear maps has been considered as the right order-preserving morphism for operator algebras. A linear map $\phi : A \to B$ is said to be *s-positive* if the map

$$\phi_s : M_s(A) \to M_s(B) : [a_{ij}] \mapsto [\phi(a_{ij})]$$

is positive, where $M_s(A)$ is the C^*-algebra of all $s \times s$ matrices over A. We say that ϕ is *completely positive* if ϕ is s-positive for every $s = 1, 2, \dots$.

Completely positive linear maps between matrix algebras are quite well understood [3], [10]. The transpose map

$$X \mapsto X^{\mathrm{t}}, \qquad X \in M_n$$

is an typical example of positive linear map which is not completely positive, where M_n denotes the C^*-algebra of all $n \times n$ matrices over the complex field.

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We say that a linear map $\phi : M_m \to M_n$ is said to be *s-copositive* if $X \mapsto \phi(X^{\mathrm{t}})$ is s-positive, and *completely copositive* if it is s-copositive for every $s = 1, 2, \dots$. A natural question is to ask whether every positive linear map is *decomposable*, that is, can be expressed as the sum of completely positive and completely copositive linear maps. It was shown by Choi [4] that positive linear maps between matrix algebras correspond to positive semi-definite biquadratic forms, and decomposability corresponds to the possibility of the expression as the sum of the squares of bilinear forms.

After Choi [4] found an example of non-decomposable positive linear maps between M_3, several authors [2], [5], [7], [9], [11], [17], [18], [21], [22], [23] produced and investigated nontrivial examples of positive linear maps between 2, 3 or 4 dimensional matrix algebras. We remark here that every extreme point of the convex set $\mathbb{P}_I[M_2, M_2]$ of all unital positive linear maps between M_2 was characterize by E. Størmer [20]. Recently, the theory of positive linear maps between matrix algebras plays an important rôle in the quantum information theory with relation to the notion of entanglement. See the survey article [8] and references there.

In this survey, we summarize the author's recent works [14], [6], [15], [1] to understand the facial structures of the convex cone $\mathbb{P}[M_m, M_n]$ of all positive linear maps from M_m into M_n. We first explain in Section 2 how a face of the cone $\mathbb{CP}[M_m, M_n]$ of all completely positive linear maps corresponds to a subspace of the vector space $M_{m\times n}$ of all $m \times n$ matrices over $\mathbb{C}$. It is also possible to explain the facial structures of $\mathbb{CP}[M_m, M_n]$ using the duality theory, arising from a bilinear map defined on $\mathcal{L}(M_m, M_n) \times M_{m\times n}$, where $\mathcal{L}(M_m, M_n)$ denotes the space of all linear maps from M_m into M_n. We proceed in Section 3 to characterize the facial structures for the cone $\mathbb{D}[M_m, M_n]$ of all decomposable positive linear maps. It turns out that that every face of $\mathbb{D}[M_m, M_n]$ is associated with a pair of subspaces of $M_{m\times n}$ in a unique way, and we will determine the faces of $\mathbb{D}[M_m, M_n]$ arising from the dual of $\mathbb{P}[M_m, M_n]$. With this machinery, we characterize all faces of the convex cone $\mathbb{P}[M_2, M_2]$ $(= \mathbb{D}[M_2, M_2])$ of all positive linear maps between M_2 in Section 4. In the forthcoming paper [16], we will characterize all faces of the convex set $\mathbb{P}_I[M_2, M_2]$ of all unital positive linear maps between M_2, which is a 12 affine dimensional convex body.

Throughout this note, we fix the natural numbers m and n and denote by $m \wedge n$ the minimum of m and n. We also denote by just $\mathbb{P}_s$, $\mathbb{P}^t$ and $\mathbb{D}$ for the convex cone mentioned above. Every vector x in a s-dimensional space will be considered as an $s \times 1$ matrix, and $\overline{x}$ denotes the vector whose entries are the conjugates of the corresponding entries of x. Hence, xy^* is an $m \times n$ matrix for each $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^n$.

2. Compeletely positive linear maps

For a subset $\mathcal{V} = \{V_1, V_2, \dots, V_\nu\}$ of $M_{m\times n}$, we define the linear maps $\phi_{\mathcal{V}}$ and $\phi^{\mathcal{V}}$ from M_m into M_n by

$$\phi_{\mathcal{V}} : X \mapsto \sum_{i=1}^{\nu} V_i^* X V_i, \qquad X \in M_m,$$
$$\phi^{\mathcal{V}} : X \mapsto \sum_{i=1}^{\nu} V_i^* X^{\mathrm{t}} V_i, \qquad X \in M_m,$$

where X^{t} denotes the transpose of the matrix X. We also denote by $\phi_V = \phi_{\{V\}}$ and $\phi^V = \phi^{\{V\}}$. It is well-known [3], [10] that every completely positive (respectively completely copositive) linear map from M_m into M_n is of the form $\phi_{\mathcal{V}}$ (respectively $\phi^{\mathcal{V}}$).

For a subspace E of $M_{m\times n}$, we define

$$\Phi_E = \{\phi_{\mathcal{V}} \in \mathbb{P}_{m\wedge n} : \operatorname{span} \mathcal{V} \subset E\}$$
$$\Phi^E = \{\phi_{\mathcal{V}} \in \mathbb{P}^{m\wedge n} : \operatorname{span} \mathcal{V} \subset E\},$$

where $\operatorname{span} \mathcal{V}$ denotes the span of the set $\mathcal{V}$. We have shown in [14] that the correspondences

$$E \mapsto \Phi_E : \mathcal{E}(M_{m\times n}) \longleftrightarrow \mathcal{F}(\mathbb{P}_{m\wedge n})$$
$$E \mapsto \Phi^E : \mathcal{E}(M_{m\times n}) \longleftrightarrow \mathcal{F}(\mathbb{P}^{m\wedge n})$$

give rise to lattice isomorphisms, where $\mathcal{E}(X)$ denotes the lattice of all subspaces of the vector space X, and $\mathcal{F}(C)$ denotes the lattice of all faces of the convex set C.

In order to describe the faces of $\mathbb{P}_{m\wedge n}$ in the other way, we identify a matrix $z \in M_{m\times n}$ and a vector $\widetilde{z} \in \mathbb{C}^n \otimes \mathbb{C}^m$ as follows: For $z = [z_{ik}] \in M_{m\times n}$, define

$$\begin{aligned} z_i &= \sum_{k=1}^{n} z_{ik} e_k \in \mathbb{C}^n, \qquad i = 1, 2, \dots, m, \\ \widetilde{z} &= \sum_{i=1}^{m} z_i \otimes e_i \in \mathbb{C}^n \otimes \mathbb{C}^m. \end{aligned}$$

Then $z \mapsto \widetilde{z}$ defines an inner product isomorphism from $M_{m\times n}$ onto $\mathbb{C}^n \otimes \mathbb{C}^m$. We also define

$$\begin{aligned} \mathbb{V}_s =& \operatorname{conv}\{\widetilde{z}\widetilde{z}^* \in M_n \otimes M_m : \text{ rank of } z \le s\} \\ \mathbb{V}^s =& \operatorname{conv}\{(\widetilde{z}\widetilde{z}^*)^\tau \in M_n \otimes M_m : \text{ rank of } z \le s\}, \end{aligned}$$

for $s = 1, 2, \dots, m \wedge n$, where $\operatorname{conv} X$ means the convex set generated by X, and A^τ denotes the *block-transpose* of A, that is,

$$\left(\sum_{i,j=1}^{m} a_{ij} \otimes e_{ij} \right)^\tau = \sum_{i,j=1}^{m} a_{ji} \otimes e_{ij}.$$

In [6], we have considered the bi-linear pairing between $M_n \otimes M_m$ and the space $\mathcal{L}(M_m, M_n)$ of all linear maps from M_m into M_n, given by

$$\langle A, \phi \rangle = \operatorname{Tr} \left[\left(\sum_{i,j=1}^{m} \phi(e_{ij}) \otimes e_{ij} \right) A^{\mathrm{t}} \right] = \sum_{i,j=1}^{m} \langle \phi(e_{ij}), a_{ij} \rangle, \tag{1}$$

for $A = \sum_{i,j=1}^{m} a_{ij} \otimes e_{ij} \in M_n \otimes M_m$ and $\phi \in \mathcal{L}(M_m, M_n)$, where the bi-linear form in the right-side is given by $\langle X, Y \rangle = \operatorname{Tr}(Y X^{\mathrm{t}})$ for $X, Y \in M_n$. This is equivalent to define

$$\langle y \otimes x, \phi \rangle = \operatorname{Tr}(\phi(x) y^{\mathrm{t}}), \qquad x \in M_m,\ y \in M_n.$$

In this duality, the pairs

$$(\mathbb{V}_{m\wedge n},\ \mathbb{P}_{m\wedge n}), \qquad (\mathbb{V}^{m\wedge n},\ \mathbb{P}^{m\wedge n}), \qquad (\mathbb{V}_{m\wedge n} \cap \mathbb{V}^{m\wedge n},\ \mathbb{D})$$

are dual each other, in the sense that

$$\begin{aligned} A \in \mathbb{V}_{m\wedge n} &\iff \langle A, \phi \rangle \ge 0 \text{ for each } \phi \in \mathbb{P}_{m\wedge n}, \\ \phi \in \mathbb{P}_{m\wedge n} &\iff \langle A, \phi \rangle \ge 0 \text{ for each } A \in \mathbb{V}_{m\wedge n}, \end{aligned}$$

and similarly for other pairs $(\mathbb{V}_s, \mathbb{P}_s)$ and $(\mathbb{V}^s, \mathbb{P}^s)$.

For a subspace E of $M_{m\times n}$, we also define

$$F_E = \{A \in \mathbb{V}_{m\wedge n} : \mathcal{R}(A) \subset \widetilde{E}\},$$

where $\mathcal{R}(A)$ denotes the range space of A, and $\widetilde{E} = \{\widetilde{z} : z \in E\} \subset \mathbb{C}^n \otimes \mathbb{C}^m$. It is well known that F_E is a face of the convex cone $\mathbb{V}_{m\wedge n} = (M_n \otimes M_m)^+ = M_{mn}^+$ of all positive semi-definite matrices, and every face of M_{mn}^+ arises in this form. By the duality between the faces of $\mathbb{V}_{m\wedge n}$ and $\mathbb{P}_{m\wedge n}$ discussed above, we also see that the set

$$\Psi_E = \{\phi \in \mathbb{P}_{m\wedge n} : \langle A, \phi \rangle = 0 \text{ for every } A \in F_E\}$$

is a face of $\mathbb{P}_{m\wedge n}$, and every face of $\mathbb{P}_{m\wedge n}$ is of the form Ψ_E for a subspace $E \subset M_{m\times n}$. Similarly, we also note that every face of $\mathbb{P}^{m\wedge n}$ is of the form

$$\Psi^E = \{\phi \in \mathbb{P}^{m\wedge n} : \langle A^\tau, \phi \rangle = 0 \text{ for every } A \in F_E\}.$$

Two faces Φ_E and Ψ_E are related as follows:

Theorem 2.1. *Let $\mathcal{V} = \{V_1, V_2, \dots, V_\nu\}$ be a finite set of $m \times n$ matrices, and E a subspace of $M_{m\times n}$. Then we have the following:*

(i) *$\phi_{\mathcal{V}} \in \Psi_E$ if and only if $E \perp \operatorname{span} \mathcal{V}$,*
(ii) *$\Psi_E = \Phi_{E^\perp}$, where $E^\perp$ denotes the orthogonal complement of E in $M_{m\times n}$.*

3. Decomposable positive linear maps

In this section, we consider the faces of the convex cone $\mathbb{D}$ consisting of all decomposable positive linear maps from M_m into M_n. Recall that $\mathbb{D}$ is the convex hull generated by $\mathbb{P}_{m\wedge n}$ and $\mathbb{P}^{m\wedge n}$. Let C be the convex hull of the cones C_1 and C_2. If F is a face of C then it is easy to see that $F \cap C_1$ and $F \cap C_2$ are faces of C_1 and C_2, respectively, and F is the convex hull of $F \cap C_1$ and $F \cap C_2$.

For a given face F of $\mathbb{D}$, we see that $F \cap \mathbb{P}_{m\wedge n}$ is a face of $\mathbb{P}_{m\wedge n}$, and so it is of the form Φ_D for a subspace $D \in \mathcal{E}(M_{m\times n})$. Similarly, $F \cap \mathbb{P}^{m\wedge n} = \Phi^E$ for a subspace $E \in \mathcal{E}(M_{m\times n})$. Therefore, we see that every face of $\mathbb{D}$ is of the form

$$\sigma(D, E) := \operatorname{conv} \{\Phi_D, \Phi^E\}, \tag{2}$$

for $D, E \in \mathcal{E}(M_{m\times n})$. Note that this expression is not unique, since different pairs of subspaces may give rise to the same face. But, if we assume the following condition

$$\sigma(D, E) \cap \mathbb{P}_{m\wedge n} = \Phi_D, \qquad \sigma(D, E) \cap \mathbb{P}^{m\wedge n} = \Phi^E \tag{3}$$

then it is clear that every face of $\mathbb{D}$ is uniquely expressed as in (2). Throughout in this note, we always assume the condition (3) whenever we mention $\sigma(D, E)$ for a pair (D, E) of subspaces of $M_{m\times n}$.

Recall that a face of $\mathbb{D}$ is said to be *exposed* if it is of the form

$$A^\circ = \{\phi \in \mathbb{D} : \langle A, \phi\rangle = 0\}$$

for an $A \in \mathbb{V}_{m\wedge n} \cap \mathbb{V}^{m\wedge n}$, since $(\mathbb{V}_{m\wedge n} \cap \mathbb{V}^{m\wedge n}, \mathbb{D})$ is a dual pair as was mentioned in Section 2. We say that a pair (D, E) of subspaces of $M_{m\times n}$ is a *decomposition pair* if $\sigma(D, E)$ is a face of $\mathbb{D}$ with the property (3). We also say that a decomposition pair (D, E) is *exposed* (respectively $\mathbb{V}_1$-*exposed*) if there is $A \in \mathbb{V}_{m\wedge n} \cap \mathbb{V}^{m\wedge n}$ (respectively $A \in \mathbb{V}_1$) such that $\sigma(D, E) = A^\circ$. It is easy to see that $\mathbb{V}_1 \subset \mathbb{V}_{m\wedge n} \cap \mathbb{V}^{m\wedge n}$.

It seems to be very hard to characterize decomposition pairs or exposed decomposition pairs. Assume that $n = 2$. Then it was shown by Woronowicz [23] that $\mathbb{P}_1 = \mathbb{D}$ if and only if $m \le 3$. In case of $m = n = 3$, an example of $\phi \in \mathbb{P}_1 \setminus \mathbb{D}$ has been constructed by Choi [4]. See also [5]. Therefore, we see that $\mathbb{P}_1 = \mathbb{D}$ holds, that is, every positive linear map from M_m into M_n is decomposable, if and only if $(m, n) = (2, 2), (2, 3)$ or $(3, 2)$. This is the case if and only if $\mathbb{V}_1 = \mathbb{V}_{m\wedge n} \cap \mathbb{V}^{m\wedge n}$ by the duality in Section 2. Therefore, we see that a pair of subspaces of $M_{m\times n}$ is an exposed decomposition pair if and only if it is a $\mathbb{V}_1$-exposed decomposition pair whenever $(m, n) = (2, 2), (2, 3)$ or $(3, 2)$, and the following is useful to find exposed faces of $\mathbb{P}_1 = \mathbb{D}$ in these cases.

Theorem 3.1. *A pair (D, E) of subspaces of $M_{m\times n}$ is a $\mathbb{V}_1$-exposed decomposition pair if and only if there exist $x_1, \dots, x_\alpha \in \mathbb{C}^m$ and $y_1, \dots, y_\alpha \in \mathbb{C}^n$ such that*

$$D = \{x_1 y_1^*, \dots, x_\alpha y_\alpha^*\}^\perp, \qquad E = \{\overline{x_1} y_1^*, \dots, \overline{x_\alpha} y_\alpha^*\}^\perp.$$

We note here that the dimensions of D and E in the Theorem may be different, as we will see in the next section. We say that a subspace E of $M_{m\times n}$ is a *decomposition conjugate* of a subspace D of $M_{m\times n}$ if (D, E) is a decomposition pair.

4. Positive linear maps between 2 by 2 matrix algebras

Now, we assume that $m = n = 2$, and characterize every decomposition pairs as was done in [1]. If D is a three dimension subspace of M_2 then it has a decomposition conjugate if and only if $D^\perp$ is spanned by a rank one matrix, say xy^*, and it has a unique decomposition pair whose orthogonal is spanned by $\overline{x}y^*$. This characterize maximal faces as was in [12], [13]. Note that every maximal face is exposed.

We proceed to characterize all exposed decomposition pairs. Note that if (D, E) is an exposed decomposition pair then $D^\perp$ is spanned by rank one matrices by Theorem 3.1. It is easy to see that a two dimensional subspace of M_2 is spanned by rank one matrices if and only if its orthogonal complement is spanned by rank one matrices. If $D = \mathrm{span}\,\{x_1y_1^*, x_2y_2^*\}$ is a two dimensional subspace of M_2 which is spanned by rank one matrices, then there is a unique decomposition conjugate $E = \mathrm{span}\,\{\overline{x}_1y_1^*, \overline{x}_2y_2^*\}$, and $\sigma(D, E)$ is an exposed face.

If D is a one dimensional subspace generated by a rank one matrix xy^* then there is a unique decomposition conjugate $E = \mathbb{C}\overline{x}y^*$ such that (D, E) is an exposed decomposition pair. This face $\sigma(D, E)$ is exposed and generates an extreme ray which belongs to $\mathbb{P}_2 \cap \mathbb{P}^2$, because $\phi_{xy^*} = \phi^{\overline{x}y^*}$.

Now, assume that D is a one dimensional subspace spanned by a rank two matrix V. Then every decomposition conjugate of D is zero or one dimensional. Note that every three dimensional subspace of M_2 is spanned by rank one matrices. It turns out that if E is a one dimensional subspace spanned by a rank two matrix W then (D, E) is a decomposition pair if and only if it is an exposed decomposition pair as follows:

Proposition 4.1. *Let V and W be rank two matrices in M_2, which span the one dimensional subspaces D and E, respectively. Then the following are equivalent:*

(i) *(D, E) is a decomposition pair,*
(ii) *(D, E) is an exposed decomposition pair,*
(iii) *There are nonzero $y_0, y_1, y_2 \in \mathbb{C}^2$ such that $Vy_i \parallel \overline{Wy_i}$ for each $i = 0, 1, 2$ and $y_i \nparallel y_j$ for $i \neq j$, where $x \parallel y$ means that x and y are parallel to each other.*
(iv) *$\overline{W}^{-1}V$ is of the form $\begin{pmatrix} r & se^{i\theta} \\ te^{i\theta} & re^{i(2\theta+\pi)} \end{pmatrix}$, where $r, s, t \in \mathbb{R}$ with $st > -r^2$, up to scalar multiplications.*

Finally, the pair $(\mathbb{C}V, \{0\})$ is an exposed decomposition pair for every rank two matrix V in M_2. To see this, we write $V = xy^* + \mu zw^*$ for nonzero

$\mu \in \mathbb{C}$ and unit vectors x, y, z and w with for $x \perp z$, $y \perp w$, by the polar decomposition, Then we see that

$$\mathbb{C}V = \{xw^*, zy^*, (\overline{\mu}x - z)(y+w)^*, (\overline{\mu}x - (1-i)z)((1+i)y+w)^*\}^\perp$$
$$\{0\} = \{\overline{x}w^*, \overline{z}y^*, \overline{(\overline{\mu}x - z)}(y+w)^*, \overline{(\overline{\mu}x - (1-i)z)}((1+i)y+w)^*\}^\perp,$$

and so $(\mathbb{C}V, \{0\})$ is an exposed decomposition pair by Theorem 3.1. Similarly, $(\{0\}, \mathbb{C}V)$ is also an exposed decomposition pair for each rank two matrix $V \in M_2$. This completes the classification of all exposed decomposition pairs of subspaces of M_2.

In order to characterize non-exposed faces, we proceed to consider two dimensional subspaces of M_2 which is not spanned by rank one matrices. Note that every two dimensional subspace of M_2 has a rank one matrix. Therefore, if a subspace $E \subset M_2$ is not spanned by rank two matrices then E has a unique rank one matrix up to scalar multiplication.

Proposition 4.2. *Let x, y, z and w be nonzero vectors in $\mathbb{C}^2$ with $x \perp z$ and $y \perp w$. Then the two dimensional subspace*

$$D = \operatorname{span}\{xy^*, xw^* + zy^*\} \subset M_2 \tag{4}$$

is not spanned by rank one matrices. Conversely, every two dimensional subspace D in M_2 which is not spanned by rank one matrices is of the above form for nonzero vectors x, y, z, w with $x \perp z$ and $y \perp w$.

If neither two dimensional subspaces $\operatorname{span}\{xy^*, V\}$ *nor* $\operatorname{span}\{zw^*, W\}$ *are spanned by rank one matrices, then these two spaces are orthogonal each other if and only if $x \perp z$ and $y \perp w$.*

The following proposition characterize all non-exposed faces, since every non-exposed decomposition pair has a two dimensional subspace which is not spanned rank one matrices.

Proposition 4.3. *Let D be a two dimensional subspace of M_2 which is not spanned by rank one matrices, and be the span of xy^* and $xw^* + zy^*$ with $x \perp z$ and $y \perp w$. Then we have the following:*

- (i) *There exists a unique one dimensional decomposition conjugate E which is spanned by $\overline{x}y^*$. In this case, we have $\sigma(D, E) = \Phi_D$.*
- (ii) *A two dimensional subspace E is a decomposition conjugate of D if and only if E is spanned by $\overline{x}y^*$ and $\overline{x}w^* + \mu\overline{z}y^*$ with $|\mu| = 1$.*

This finishes the classification of faces of the cone $\mathbb{P}[M_2, M_2] = \mathbb{D}[M_2, M_2]$, or equivalently all decomposition pairs (D, E). We summarize as follows, where the second column denotes $(\dim D, \dim E)$.

I	$(3,3)$	$D = (xy^*)^\perp,\ E = (\overline{x}y^*)^\perp$
II	$(2,2)$	$D = \{xy^*, zw^*\}^\perp,\ E = \{\overline{x}y^*, \overline{z}w^*\}^\perp$ ($x \nparallel z$ or $y \nparallel w$)
III	$(2,2)$	D, E are non type I
IV	$(2,1)$	D is non type I, E is spanned by a rank one rank
V	$(1,2)$	D is spanned by a rank one matrix, E is is non type I
VI	$(1,1)$	D, E are spanned by rank two matrices
VII	$(1,1)$	$D = \mathbb{C}xy^*, E = \mathbb{C}\overline{x}y^*$
VIII	$(1,0)$	D is spanned by a rank two matrix, $E = \{0\}$
IX	$(0,1)$	$D = \{0\}, E$ is spanned by a rank two matrix

For the possible combinations of two subspaces for the faces of types III, IV, V and VI, we refer to Proposition 4.3 and Proposition 4.1. We remark here that the faces of type I exhaust all maximal faces, and faces of type II (respectively VII) are the intersection of two (respectively three) maximal faces. The faces of types III, IV and V are non-exposed, and faces of types IV, VII and VIII (respectively V, VII and IX) consist of completely positive (respectively completely copositive) linear maps. The faces of types VII, VIII and IX are extreme rays of the cone $\mathbb{P}[M_2, M_2]$. Finally, faces of type II have different shapes according to whether D consists of rank one matrices or not. Note that D consists of rank one matrices if and only if $x \parallel z$ or $y \parallel w$. In this case, the face of type II is affine isomorphic to the cone M_2^+ of all positive semi-definite 2×2 matrices.

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