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Positive Linear Maps Between Matrix Algebras Which Fix Diagonals*

Seung-Hyeok Kye

Department of Mathematics
Seoul National University
Seoul 151-742, Korea

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ABSTRACT

We consider a class of positive linear maps from the n-dimensional matrix algebra into itself which fix diagonal entries. We show that they are expressed by Hadamard products, and study their decompositions into the sums of completely positive linear maps and completely copositive linear maps. In the three-dimensional case, we show that every positive linear map in this type is decomposable, and give an intrinsic characterization for the positivity of these maps when the involving coefficients are real numbers.

1. INTRODUCTION

Let M_n be the C^*-algebra of all $n \times n$ matrices over the complex field. Because the structure of the positive cone $\mathcal{P}(M_n)$ of all positive linear maps between M_n is very complicated even in lower dimensions, it would be very useful to find extreme elements of this cone in various senses. In this vein, several authors [3, 8, 9] have constructed such extreme maps in the cases of $n = 2, 3$, or 4. In the case of $n = 3$, such maps were constructed by adjusting diagonal elements and attaching minus signs at offdiagonals (see also [1, 5, 6, 10]). In this note, we consider positive linear maps between matrix algebras which fix diagonal elements. It is easy to see that every positive linear map preserving the diagonals is of the form

$$\Phi_{A,B} : X \mapsto A \circ X + B \circ X^{\text{tr}} + I \circ X, \qquad X \in M_n,$$

for self-adjoint matrices A and B with zero diagonals, where $A \circ X$ (respectively X^{tr}) denotes the Hadamard product of A and X (respectivley the transpose of X), and I denotes the identity matrix.

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Recall that a positive linear map between matrix algebra is said to be *decomposable* if it is the sum of a completely positive linear map and a completely copositive linear map. For those linear maps in the above forms, they are completely positive (respectively completely copositive) if and only if they are 2-positive (respectively 2-copositive). Also, they are decomposable if and only if they are the sums of 2-positive and 2-copositive linear maps.

Restricting our attention to the three-dimensional case, we show that every positive linear map in this type is decomposable. We also get a necessary and sufficient condition for the positivity of the map $\Phi_{A,B}$ in terms of the entries of A and B, when the entries are real numbers.

Throughout this note, $A \geq 0$ means that A is positive semidefinite.

2. DECOMPOSABILITY

Let $\Phi : M_n \to M_n$ be a positive linear map which fixes diagonal elements, that is,

$$\Phi\left(E_{ii}\right) = E_{ii}, \qquad i = 1, 2, \ldots, n, \tag{2.1}$$

where $\{E_{ij}\}$ denotes the usual matrix units. For $j \neq k$, consider the positive semidefinite matrix

$$P_{x,y} = |X|^2 E_{jj} + |y|^2 E_{kk} + \bar{x} y E_{jk} + \bar{y} x E_{kj}$$

with arbitrary $x, y \in \mathbb{C}$. Then

$$\langle \Phi\left(P_{x,y}\right) e_i, e_i \rangle = |x|^2 \delta_{ij} + |y|^2 \delta_{ik} + 2 \operatorname{Re}\left[\bar{x} y \langle \Phi\left(E_{jk}\right) e_i, e_i \rangle\right]$$

becomes a nonnegative real number for each $x, y \in \mathbb{C}$, where $\{e_i\}$ is the usual orthonormal basis of $\mathbb{C}^n$. It follows that

$$\left\langle \Phi\left(E_{jk}\right) e_i, e_i \right\rangle = 0, \qquad j \neq k, \quad i = 1, 2, \ldots, n,$$

and so the matrix $\Phi(E_{jk})$ has zero diagonals for $j \neq k$. Therefore, $\Phi(P_{x,y})$ has zero diagonals except at the j-th and k-th positions. Because $\Phi(P_{x,y})$ is positive semidefinite, it follows that $\Phi(P_{x,y})$ is spanned by E_{jj}, E_{kk}, E_{jk}, and E_{kj}. From the relation

$$\Phi(E_{jk}) = \tfrac{1}{2}\left[\Phi(P_{1,1}) - i\Phi(P_{1,i})\right] - \frac{1-i}{2}\left[E_{jj} + E_{kk}\right],$$

we see that $\Phi(E_{jk})$ is spanned by E_{jk} and E_{kj}, and similarly for $\Phi(E_{kj})$. It follows that every positive linear map $\Phi : M_n \to M_n$ with the property (2.1) is of the form

$$\Phi([x_{ij}]) = \begin{pmatrix} x_{11} & a_{12}x_{12} + b_{12}x_{21} & \cdots & a_{1n}x_{1n} + b_{1n}x_{n1} \\ a_{21}x_{21} + b_{21}x_{12} & x_{22} & \cdots & a_{2n}x_{2n} + b_{2n}x_{n2} \\ \vdots & \vdots & & \vdots \\ a_{n1}x_{n1} + b_{n1}x_{1n} & a_{n2}x_{n2} + b_{n2}x_{2n} & \cdots & x_{nn} \end{pmatrix},$$

where $A = [a_{ij}]$ and $B = [b_{ij}]$ are self-adjoint matrices with zero diagonals. We will denote this linear map by $\Phi_{A,B}$, that is,

$$\Phi_{A,B}(X) = A \circ X + B \circ X^{\mathrm{tr}} + I \circ X, \qquad X \in M_n, \tag{2.2}$$

as in the introduction.

PROPOSITION 2.1. *Let A and B be self-adjoint matrices with zero diagonals. Then the following are equivalent*:

(i) $\Phi_{A,B}$ *is completely positive.*
(ii) $\Phi_{A,B}$ *is 2-positive.*
(iii) $B = 0$ *and* $A + I \geq 0$.

PROOF. If $\Phi_{A,B}$ is 2-positive, then the matrix

$$(\Phi_{A,B})_2 \begin{pmatrix} E_{ii} & E_{ij} \\ E_{ji} & E_{jj} \end{pmatrix} = \begin{pmatrix} E_{ii} & a_{ij}E_{ij} + b_{ij}E_{ji} \\ a_{ji}E_{ji} + b_{ji}E_{ij} & E_{jj} \end{pmatrix}$$

is positive. From this, we see that $b_{ij} = 0$ for $i \neq j$, and $B = 0$. It is well known that the map $X \mapsto (A + I) \circ X$ is positive if and only if $A + I \geq 0$. If $B = 0$ and $A + I \geq 0$, then we see that the $n^2 \times n^2$ matrix $[\Phi_{A,B}(E_{ij})]$ is positive semidefinite, and so $\Phi_{A,B}$ is completely positive by [2]. ■

By similar arguments to those for Proposition 2.1, we also have:

PROPOSITION 2.2. *Let A and B be self-adjoint matrices with zero diagonals. Then the following are equivalent*:

(i) $\Phi_{A,B}$ *is completely copositive.*
(ii) $\Phi_{A,B}$ *is 2-copositive.*
(iii) $A = 0$ *and* $B + I \geq 0$.

Now, we show that Φ is decomposable if and only if it is the sum of a 2-positive map and a 2-copositive map.

THEOREM 2.3. *Let A and B be self-adjoint matrices with zero diagonals. Then the following are equivalent*:

(i) $\Phi_{A,B}$ *is decomposable.*

(ii) $\Phi_{A,B}$ *is the sum of a 2-positive linear map and a 2-copositive linear map.*

(iii) *There exist diagonal matrices D_1 and D_2 in M_n such that*

$$D_1 + D_2 = I, \qquad A + D_1 \geq 0, \qquad B + D_2 \geq 0. \tag{2.3}$$

PROOF. Let $\Phi_{A,B} = \Phi_1 + \Phi_2$ with a 2-positive linear map Φ_1 and a 2-copositive linear map Φ_2. Considering the $n^2 \times n^2$ matrix $[\Phi_{A,B}(E_{ij})]$ again, we use the same arguments as in the proof of [5, Proposition 3.1] (see also [7]); then it is easy to see that Φ_1 and Φ_2 are of the forms

$$\Phi_1(X) = A_1 \circ X, \quad \Phi_2(X) = A_2 \circ X^{\mathrm{tr}}, \qquad X \in M_n,$$

for some positive semidefinite matrices A_1 and A_2. From this, we get condition (iii) with $A_1 = A + D_1$ and $A_2 = B + D_2$. The implication (iii) $\Rightarrow$ (i) follows from the facts that the above maps Φ_1 and Φ_2 are completely positive and completely copositive, respectively, as in the proof of Proposition 2.1 ■

Let A and B be $n \times n$ self-adjoint matrices with zero diagonals as before. We denote by Δ_A^+ [respectively Δ_B^-] the set of all diagonal matrices D such that $A + D \geq 0$ [respectively $B + (I - D) \geq 0$]. With the obvious identification $D = \mathrm{Diag}(d_1, d_2, \dots, d_n)$, Δ_A^+ and Δ_B^- are convex subsets of $\mathbb{R}^n$. We introduce the number

$$\delta_{A,B} := \sup\{r \geq 0 : \Phi_{rA,rB} \in \mathcal{D}\},$$

where $\mathcal{D}$ denotes the set of all decomposable linear maps.

PROPOSITION 2.4. *Let A and B be self-adjoint $n \times n$ matrices with zero diagonals. If $\Delta_A^+ \cap \Delta_B^-$ consists of one point, then we have $\delta_{A,B} = 1$. Conversely, if $\delta_{A,B} = 1$, then $\Delta_A^+ \cap \Delta_B^-$ has no interior point.*

PROOF. By definition, we see that $\Phi_{A,B} \in \mathcal{D}$. Suppose that there is $r > 1$ such that $\Phi_{rA,rB}$ is decomposable. By Theorem 2.3, there exist diagonal matrices D_1 and D_2 with $D_1 + D_2 = I$ such that $rA + D_1 \geq 0$ and $rB + D_2 \geq 0$. Then we see that $\Delta_A^+ \cap \Delta_B^-$ contains two points $I - (1/r)D_2$ and $(1/r)D_1$, by the following expressions:

$$\begin{aligned}\Phi_{A,B}(X) &= \left[\left(A + \frac{1}{r}D_1\right) \circ X + \left(I - \frac{1}{r}(D_1 + D_2)\right) \circ X\right] + \left(B + \frac{1}{r}D_2\right) \circ X^{\mathrm{tr}} \\ &= \left(A + \frac{1}{r}D_1\right) \circ X + \left[\left(B + \frac{1}{r}D_2\right) \circ X^{\mathrm{tr}} + \left(I - \frac{1}{r}(D_1 + D_2)\right) \circ X^{\mathrm{tr}}\right].\end{aligned}$$

For the second assertion, we assume that $\Delta_A^+ \cap \Delta_B^-$ has nonempty interior. Then we can find two points $(d_1, \dots, d_n)$ and $(d_1 - \epsilon, \dots, d_n - \epsilon)$ in $\Delta_A^+ \cap \Delta_B^-$ for some positive number ϵ. Define

$$\begin{aligned} \Phi_1(X) &= \left[A + \operatorname{Diag}(d_1 - \epsilon, \dots, d_n - \epsilon)\right] \circ X, \\ \Phi_2(X) &= \left[B + \operatorname{Diag}(1 - d_1, \dots, 1 - d_n)\right] \circ X^{\mathrm{tr}}. \end{aligned}$$

Then, we see that

$$\Phi_{\frac{1}{1-\epsilon}A, \frac{1}{1-\epsilon}B} = \frac{1}{1-\epsilon}(\Phi_1 + \Phi_2)$$

is decomposable, and so we have $\delta_{A,B} > 1$. ■

3. POSITIVITY IN THE THREE-DIMENSIONAL CASE

In the remainder of this note, we restrict ourselves to the case of three-dimensional matrix algebra, and put

$$A = \begin{pmatrix} 0 & a_1 & \overline{a}_2 \\ \overline{a}_1 & 0 & a_3 \\ a_2 & \overline{a}_3 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & b_1 & \overline{b}_2 \\ \overline{b}_1 & 0 & b_3 \\ b_2 & \overline{b}_3 & 0 \end{pmatrix}.$$

Then the linear map $\Phi_{A,B}$ is positive if and only if the matrix

$$\begin{pmatrix} |x|^2 & a_1\overline{x}y + b_1\overline{y}x & \overline{a}_2\overline{x}z + \overline{b}_2\overline{z}x \\ \overline{a}_1\overline{y}x + \overline{b}_1\overline{x}y & |y|^2 & a_3\overline{y}z + b_3\overline{z}y \\ a_2\overline{z}x + b_2\overline{x}z & \overline{a}_3\overline{z}y + \overline{b}_3\overline{y}z & |z|^2 \end{pmatrix} \tag{3.1}$$

is positive semidefinite for every $(x, y, z) \in \mathbb{C}^3$. Considering the 2×2 diagonal submatrices we have

$$|a_i| + |b_i| \le 1, \qquad i = 1, 2, 3. \tag{3.2}$$

If we consider the determinant of the matrix (3.1), then the inequality

$$\begin{aligned} &|z|^2|a_1\overline{x}y + b_1\overline{y}x|^2 + |y|^2|a_2\overline{z}x + b_2\overline{x}z|^2 + |x|^2|a_3\overline{y}z + b_3\overline{z}y|^2 \\ &\le |x|^2|y|^2|z|^2 + 2\operatorname{Re}(a_1\overline{x}y + b_1\overline{y}x)(a_2\overline{z}x + b_2\overline{x}z)(a_3\overline{y}z + b_3\overline{z}y) \end{aligned}$$

holds for every $x, y, z \in \mathbb{C}$. Because this inequality is trivial if one of x, y, z is zero, we divide by $|x|^2|y|^2|z|^2$, and may assume that $|x| = |y| = |z| = 1$. Put

$$F_{A,B}(\theta,\sigma,\tau) = \det \begin{pmatrix} 1 & a_1e^{i\theta} + b_1e^{-i\theta} & \overline{a}_2e^{-i\sigma} + \overline{b}_2e^{i\sigma} \\ \overline{a}_1e^{-i\theta} + \overline{b}_1e^{i\theta} & 1 & a_3e^{i\tau} + b_3e^{-i\tau} \\ a_2e^{i\sigma} + b_2e^{-i\sigma} & \overline{a}_3e^{-i\tau} + \overline{b}_3e^{i\tau} & 1 \end{pmatrix}. \tag{3.3}$$

We also put $\Omega = \{(\theta,\sigma,\tau) \in [-\pi,\pi]^3 : \theta + \sigma + \tau = 0\}$. Summing up, we have:

PROPOSITION 3.1. *The linear map $\Phi_{A,B}$ is positive if and only if the conditions* (3.2) *and the inequality $F_{A,B}(\theta,\sigma,\tau) \geq 0$ hold for each $(\theta,\sigma,\tau) \in \Omega$.*

Now, we introduce the number

$$\rho_{A,B} := \sup\{r \geq 0 : \Phi_{rA,rB} \in \mathcal{P}\},$$

where $\mathcal{P}$ denotes the set of all positive linear maps. Then it is clear that $\delta_{A,B} \leq \rho_{A,B}$.

PROPOSITION 3.2. *Assume that the linear map $\Phi_{A,B}$ is positive. Then $\rho_{A,B} = 1$ if and only if either an equality holds in* (3.2) *or $F_{A,B}(\theta,\sigma,\tau) = 0$ for some $(\theta,\sigma,\tau) \in \Omega$.*

PROOF. For $r \in [0,\infty)$ and $\Theta = (\theta,\sigma,\tau) \in \Omega$, we write

$$g(r,\Theta) = F_{rA,rB}(\theta,\sigma,\tau).$$

$\Rightarrow$: Assume that the strict inequalities hold in (3.2), and take a decreasing sequence $\{r_n\}$ in $(1,\infty)$ with $r_n|a_i| + r_n|b_i| \leq 1, i = 1,2,3$, which converges to 1. Becasue Φ_{r_nA,r_nB} is not positive, we can take $\Theta_n \in \Omega$ such that $g(r_n,\Theta_n) < 0$ for each $n = 1,2,\ldots$. From the compactness of Ω we may assume that the sequence $\{\Theta_n\}$ converges to a point $\Theta_0 = (\theta_0,\sigma_0,\tau_0) \in \Omega$. Since g is a continuous function on $[0,\infty) \times \Omega$, we have $F_{A,B}(\Theta_0) = g(1,\Theta_0) \leq 0$. The equality $F_{A,B}(\Theta_0) = 0$ follows from the positivity of the linear map $\Phi_{A,B}$.

$\Leftarrow$: If an equality holds in (3.2) then it is clear that $\rho_{A,B} = 1$. Assume that $g(1,\Theta_0) = 0$ for some $\Theta_0 = (\theta_0,\sigma_0,\tau_0)$. In order to prove $\rho_{A,B} = 1$, it suffices to show that the function $h : r \mapsto g(r,\Theta_0)$ is strictly decreasing at $r = 1$. Put

$$\alpha = a_1e^{i\theta_0} + b_1e^{-i\theta_0}, \qquad \beta = a_2e^{i\sigma_0} + b_2e^{-i\sigma_0}, \qquad \gamma = a_3e^{i\tau_0} + b_3e^{-i\tau_0}.$$

First, note that $|\alpha|^2 + |\beta|^2 + |\gamma|^2 > 0$, because $\alpha = \beta = \gamma = 0$ implies $g(1,\Theta_0) = 1$. If $\mathrm{Re}(\alpha\beta\gamma) \leq 0$, then h is strictly decreasing on the interval

$[0, \infty)$. If $\mathrm{Re}(\alpha\beta\gamma) > 0$, then h is strictly decreasing on the interval

$$\left[0, \frac{|\alpha|^2 + |\beta|^2 + |\gamma|^2}{3\,\mathrm{Re}(\alpha\beta\gamma)}\right).$$

From the inequalities in (3.2), we have $\mathrm{Re}(\alpha\beta\gamma) \le |\alpha\beta\gamma| < 1$. Therefore, it follows that

$$3\,\mathrm{Re}(\alpha\beta\gamma) < 2\,\mathrm{Re}(\alpha\beta\gamma) + 1 = |\alpha|^2 + |\beta|^2 + |\gamma|^2,$$

and so h is strictly decreasing at $r = 1$, as was desired. ■

PROPOSITION 3.3. *Assume that* $D \in \Delta_A^+ \cap \Delta_B^-$. *Then we have* $\rho_{A,B} = 1$ *if and only if one of the following two cases holds*:

(i) *An equality holds in* (3.2)

(ii) $A + D$ *and* $B + (I - D)$ *have null vector* (x_1, y_1, z_1) *and* (x_2, y_2, z_2), *respectively, with the condition*

$$|x_1| = |x_2|, \qquad |y_1| = |y_2|, \qquad |z_1| = |z_2|. \tag{3.4}$$

PROOF. Because $\Delta_A^+ \cap \Delta_B^-$ is nonempty, $\Phi_{A,B}$ is immediately positive. By Proposition 3.2, it suffices to show that $F_{A,B}(\theta, \sigma, \tau) = 0$ for some $(\theta, \sigma, \tau) \in \Omega$ if and only if case (ii) holds.

Assume that $F_{A,B}(\theta, \sigma, \tau) = 0$ for some $(\theta, \sigma, \tau) \in \Omega$. Note that the matrix in (3.3) is the sum of the following two positive semi-definite matrices:

$$\begin{pmatrix} d_1 & a_1 e^{i\theta} & \bar{a}_2 e^{-i\sigma} \\ \bar{a}_1 e^{-i\theta} & d_2 & a_3 e^{i\tau} \\ a_2 e^{i\sigma} & \bar{a}_3 e^{-i\tau} & d_3 \end{pmatrix}, \qquad \begin{pmatrix} 1 - d_1 & b_1 e^{-i\theta} & \bar{b}_2 e^{i\sigma} \\ \bar{b}_1 e^{i\theta} & 1 - d_2 & b_3 e^{-i\tau} \\ b_2 e^{-i\sigma} & \bar{b}_3 e^{-i\tau} & 1 - d_3 \end{pmatrix}. \tag{3.5}$$

Because the sum of these two matrices is singular, we see that they have a common null vector (x, y, z). Then,

$$(x, e^{i\theta} y, e^{-i\sigma} z) \quad \text{and} \quad (x, e^{-i\theta} y, e^{i\sigma} z)$$

are null vectors of $A + D$ and $B + (I - D)$, respectively, and satisfy the condition (3.4).

For the converse, we assume that (x, y, z) and $(xe^{2i\alpha}, ye^{2i\beta}, zr^{2i\gamma})$ are null vectors of $A + D$ and $B + (I - D)$, respectively. Then we see that $(x, ye^{i(\beta-\alpha)}, ze^{i(\gamma-\alpha)})$ is a common null vector for the matrices in (3.5), with $(\theta, \sigma, \tau) = (\alpha - \beta, \gamma - \alpha, \beta - \gamma)$. Therefore, we have $F_{A,B}(\theta, \sigma, \tau) = 0$. ■

In the remainder of this section, we give an intrinsic characterization of the positivity of the map $\Phi_{A,B}$ in terms of the entries of A and B, under the condition

$$a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}. \tag{3.6}$$

To do this, we need the following lemma.

LEMMA 3.4. *Let α, β, and γ be real numbers, and put*

$$\begin{aligned} F(\theta,\sigma,\tau) &= \alpha\cos\theta + \beta\cos\sigma + \gamma\cos\tau, \quad (3.7)\\ \theta+\sigma+\tau &= 0. \quad (3.8)\end{aligned}$$

We denote by Δ the set of all triplets (p, q, r) of nonnegative real numbers satisfying

$$p \le q + r, \qquad q \le r + p, \qquad r \le p + q.$$

(i) *If $\alpha\beta\gamma \ge 0$, then the maximum of* (3.7) *under the constraint* (3.8) *is*

$$|\alpha| + |\beta| + |\gamma|.$$

(ii) *If $\alpha\beta\gamma < 0$ and $(|\alpha\beta|, |\beta\gamma|, |\gamma\alpha|) \in \Delta$, then the maximum of* (3.7) *under the constraint* (3.8) *is*

$$-\frac{\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2}{2\alpha\beta\gamma}.$$

(iii) *If $\alpha\beta\gamma < 0$ and $(|\alpha\beta|, |\beta\gamma|, |\gamma\alpha|) \notin \Delta$, then the maximum of* (3.7) *under the constraint* (3.8) *is*

$$\max\{|\alpha| + |\beta| - |\gamma|, |\alpha| - |\beta| + |\gamma|, -|\alpha| + |\beta| + |\gamma|\}.$$

PROOF. There is nothing to prove for statement (i). For the remaining cases, we assume that $\alpha\beta\gamma < 0$. If (3.7) takes an extreme value at (θ, σ, τ) under the constraint (3.8), then we have

$$\alpha\sin\theta = \beta\sin\sigma = \gamma\sin\tau, \tag{3.9}$$

by the Lagrange multiplier method. First, we assume that $\sin\theta \ne 0$. From the relation

$$\alpha\sin\theta = \beta\sin\sigma = -\gamma\sin\theta\cos\sigma - \gamma\cos\theta\sin\sigma,$$

we get

$$\cos\theta = -\frac{\beta}{\gamma} - \frac{\beta}{\alpha}\cos\sigma.$$

Similarly, we have

$$\cos\sigma = -\frac{\gamma}{\alpha} - \frac{\gamma}{\beta}\cos\tau, \qquad \cos\tau = -\frac{\alpha}{\beta} - \frac{\alpha}{\gamma}\cos\theta.$$

Hence, it follows that

$$\cos\theta = P, \qquad \cos\sigma = Q, \qquad \cos\tau = R, \tag{3.10}$$

where

$$\begin{aligned} P &= \frac{-\alpha^2\beta^2 + \beta^2\gamma^2 - \gamma^2\alpha^2}{2\alpha^2\beta\gamma}, \\ Q &= \frac{-\alpha^2\beta^2 - \beta^2\gamma^2 + \gamma^2\alpha^2}{2\alpha\beta^2\gamma}, \\ R &= \frac{\alpha^2\beta^2 - \beta^2\gamma^2 - \gamma^2\alpha^2}{2\alpha\beta\gamma^2}. \end{aligned}$$

Therefore, an extreme value of (3.7) under the constraint (3.8) occurs only when

$$\sin\theta = \sin\sigma = \sin\tau = 0$$

or

$$\cos\theta = P, \quad \cos\sigma = Q, \quad \cos\tau = R,$$

for which (3.7) takes the values

$$|\alpha| + |\beta| - |\gamma|, \qquad |\alpha| - |\beta| + |\gamma|, \qquad -|\alpha| + |\beta| + |\gamma| \tag{3.11}$$

or

$$\frac{\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2}{2|\alpha\beta\gamma|}, \tag{3.12}$$

respectively. It is easy see that the value (3.12) is greater than or equal to the three values of (3.11). We note that there exist θ, σ, and τ satisfying (3.10) if and only if

$$-1 \le P \le 1, \qquad -1 \le Q \le 1, \qquad -1 \le R \le 1.$$

It is also easy to see that this is the case if and only if $(|\alpha\beta|, |\beta\gamma|, |\gamma\alpha|) \in \Delta$.

Now, if $(|\alpha\beta|, |\beta\gamma|, |\gamma\alpha|) \notin \Delta$, then the extreme values occur only when $\sin\theta = \sin\sigma = \sin\tau = 0$, and this proves (iii). If $(|\alpha\beta|, |\beta\gamma|, |\gamma\alpha|) \in \Delta$, then we see that there are θ, σ, and τ satisfying the equations (3.8) and (3.9), and this completes the proof of (ii). ■

Note that the inequality $F_{A,B}(\theta,\sigma,\tau)\ge 0$ for $(\theta,\sigma,\tau)\in\Omega$ is equivalent to

$$\begin{aligned}&|a_1+b_1e^{i\theta}|^2+|a_2+b_2e^{i\sigma}|^2+|a_3+b_3e^{i\tau}|^2\\&\qquad\le 1+2\,\mathrm{Re}\,(a_1+b_1e^{i\theta})(a_2+b_2e^{i\sigma})(a_3+b_3e^{i\tau}),\end{aligned}\tag{3.13}$$

for $\theta+\sigma+\tau=0$. Now, we impose the condition (3.6) and put

$$\begin{aligned}\alpha &= a_1b_1-a_1b_2b_3-b_1a_2a_3,\\ \beta &= a_2b_2-b_1a_2b_3-a_1b_2a_3,\\ \gamma &= a_3b_3-b_1b_2a_3-a_1a_2b_3,\\ \delta &= 1+2(a_1a_2a_3+b_1b_2b_3)-(a_1^2+a_2^2+a_3^2+b_1^2+b_2^2+b_3^2),\\ \Lambda &= \max\{|\alpha|+|\beta|-|\gamma|,|\alpha|-|\beta|+|\gamma|,-|\alpha|+|\beta|+|\gamma|\}.\end{aligned}\tag{3.14}$$

Then, by a direct calculation, we see that the condition (3.13) becomes

$$2(\alpha\cos\theta+\beta\cos\sigma+\gamma\cos\tau)\le\delta.$$

Now, we apply Lemma 3.4 to get

THEOREM 3.5. *Under the condition* (3.6), *the linear map* $\Phi_{A,B}$ *is positive if and only if the condition* (3.2) *together with the following conditions is satisfied*:

$$\begin{aligned}\alpha\beta\gamma\ge 0 &\;\Rightarrow\; 2(|\alpha|+|\beta|+|\gamma|)\le\delta,\\ \alpha\beta\gamma<0,\quad (|\alpha\beta|,|\beta\gamma|,|\gamma\alpha|)\in\Delta &\;\Rightarrow\; -\frac{\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2}{\alpha\beta\gamma}\le\delta,\\ \alpha\beta\gamma<0,\quad (|\alpha\beta|,|\beta\gamma|,|\gamma\alpha|)\notin\Delta &\;\Rightarrow\; 2\Lambda\le\delta,\end{aligned}$$

where the numbers $\alpha,\beta,\gamma,\delta$, *and* Λ *are given by* (3.14).

4. DECOMPOSABILITY IN THE THREE-DIMENSIONAL CASE

In this section, we show that the linear map $\Phi_{A,B}$ on M_3 is positive if and only if it is decomposable.

THEOREM 4.1. *Let* Φ *be a positive linear map on the three-dimensional matrix algebra which fixes diagonals. Then* Φ *is deocomposable.*

The following simple lemma will be useful. We omit an elementary proof.

LEMMA 4.2. *Assume that* $\alpha_0 < \alpha_1, \beta_0 < \beta_1, \gamma_0 \geq 0, \gamma_1 \geq 0$, *and one of* γ_0 *or* γ_1 *is nonzero. Then the equations*

$$(x-\alpha_0)(y-\beta_0)=\gamma_0^2, \qquad (x-\alpha_1)(y-\beta_1)=\gamma_1^2 \tag{4.1}$$

have a common solution (x, y) *with* $\alpha_0 \leq x \leq \alpha_1, \beta_0 \leq y \leq \beta_1$ *if and only if the inequality*

$$(\gamma_0+\gamma_1)^2 \leq (\alpha_1-\alpha_0)(\beta_1-\beta_0)$$

holds. Furthermore, the equality holds if and only if the solution is unique.

Also, assume that $\alpha_2 > 0, \beta_2 > 0$. *Then the first curve in* (4.1) *with* $x \geq \alpha_0, y \geq \beta_0$ *satisfies the condition*

$$\alpha_2 x + \beta_2 y \geq \gamma_2$$

if and only if we have the inequality

$$2\gamma_0\sqrt{\alpha_2\beta_2} + (\alpha_0\alpha_2 + \beta_0\beta_2 - \gamma_2) \geq 0.$$

For the proof of Theorem 4.1, we first consider the special case in which A or B has a zero row. For example, we assume that $b_2 = b_3 = 0$.

LEMMA 4.3. *Let* $b_2 = b_3 = 0$. *Then the map* $\Phi_{A,B}$ *is positive if and only if it is decomposable if and only if the condition*

$$(|b_1| + |\bar{a}_1| - a_2a_3|)^2 \leq (1-|a_2|^2)(1-|a_3|^2) \tag{4.2}$$

holds.

PROOF. Write $a_i = |a_i|e^{i\phi_i}, b_1 = |b_1|e^{i\psi_1}$. If $\Phi_{A,B}$ is positive, then it follows that

$$\begin{aligned}|a_1|^2 + |b_1|^2 + |a_2|^2 + |a_3|^2 + 2|a_1|\,|b_1|\cos(\theta+\psi_1-\phi_1)\\ \leq 1 + 2\,\mathrm{Re}(a_1a_2a_3) + 2|b_1a_2a_3|\cos(\theta+\psi_1+\phi_2+\phi_3),\end{aligned}$$

by (3.13), and so we have

$$\begin{aligned}2|b_1|\left[|a_1|\cos(\theta+\psi_1-\phi_1) - |a_2a_3|\cos(\theta+\psi_1+\phi_2+\phi_3)\right]\\ \leq 1 + 2\,\mathrm{Re}(a_1a_2a_3) - (|a_1|^2+|b_1|^2+|a_2|^2+|a_3|^2)\end{aligned}$$

for every θ. Because the maximum value of the left side with respect to θ is equal to $2|b_1|\,|\bar{a}_1 - a_2a_3|$, we have

$$2|b_1|\,|\bar{a}_1 - a_2a_3| \le (1-|a_2|^2)(1-|a_3|^2) - (|b_1|^2 + |\bar{a}_1 - a_2a_3|^2),$$

from which the condition (4.2) follows.

Assume that (4.2) holds. If $|a_2| = 1$ or $|a_3| = 1$ then $|b_1| = 0$. Also, note that if $b_1 = 0$ then $\Phi_{A,B}$ is already completely positive by Proposition 2.1. Hence, we may assume that $|a_2| < 1, |a_3| < 1$, and $b_1 \neq 0$. By Lemma 4.2, the equations

$$\left(x-|a_2|^2\right)\left(y-|a_3|^2\right) = |\bar{a}_1 - a_2a_3|^2 \quad \text{and} \quad (x-1)(y-1) = |b_1|^2$$

have a common solution (x, y) with $|a_2|^2 \le x \le 1$ and $|a_3|^2 \le y \le 1$. With these x and y, we see that

$$\begin{pmatrix} x & a_1 & \bar{a}_2 \\ \bar{a}_1 & y & a_3 \\ a_2 & \bar{a}_3 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1-x & b_1 & 0 \\ \bar{b}_1 & 1-y & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are positive semidefinite. Therefore, $\Phi_{A,B}$ is decomposable by Theorem 2.3. ∎

From now on, we assume

(I) Neither A nor B has a zero row.

In order to complete the proof of Theorem 4.1, it suffices to show

$$\delta_{A,B} = 1 \quad \Rightarrow \quad \rho_{A,B} = 1. \tag{4.3}$$

Therefore, we also assume

(II) $\delta_{A,B} = 1$.

This condition (II) implies that the set $\Delta_A^+ \cap \Delta_B^-$ contains a point $D = (d_1, d_2, d_3)$. From condition (I), we also have

$$\begin{aligned} (x,y,z) \in \Delta_A^+ \quad &(\text{respectively}\, \Delta_B^-) \\ \Rightarrow \quad x,y,z > 0 \quad &(\text{respectively} < 1). \end{aligned} \tag{4.4}$$

LEMMA 4.4. *Let S and T be convex sets in $\mathbb{R}^n$ with nonempty interiors. Assume that $S \cap T$ has no interior and contains a line segment L. Then L is contained in $\partial S \cap \partial T$, where ∂S denotes the boundary of S.*

PROOF. Assume that there is a point x of L which is an interior point of S. Let B be an open ball centered at x and contained in S. Take an interior point y of

T, and denote by M the line segment between x and y. Then every point of $M \cap B$ is an interior point of $S \cap T$ with the possible exception of x. ∎

We introduce the following equations:

$$\begin{aligned} s_A(x, y, z) &= \det[A + \mathrm{Diag}(x, y, z)] = 0 \\ s_B(x, y, z) &= \det[B + \mathrm{Diag}(1 - x, 1 - y, 1 - z)] = 0 \end{aligned} \tag{4.5}$$

LEMMA 4.5. *Assume condition* (I). *Then we have the following:*

(i) *A point* $(x, y, z) \in \Delta_A^+$ *lies on the boundary of* Δ_A^+ *if and only if* $s_A(x, y, z) = 0$.

(ii) *If a point* $D = (d_1, d_2, d_3) \in \partial\Delta_A^+$ *is a regular point of the surface* $s_A(x, y, z) = 0$, *then there is a neighborhood* U *of* D *such that*

$$U \cap \Delta_A^+ = U \cap \{(x, y, z) : s_A(x, y, z) \geq 0\}.$$

The analogous statements hold for Δ_B^- *and* s_B.

PROOF. Let $(x, y, z) \in \Delta_A^+$. Then we have that $x, y, z > 0$ by (4.4). Note that $(x, y, z) \in \partial\Delta_A^+$ if and only if one of the equalities

$$s_A(x, y, z) = 0, \qquad xy = |a_1|^2, \qquad yz = |a_3|^2, \qquad zx = |a_2|^2$$

holds. Therefore, the first statement is a direct consequence of the well-known fact: A positive semidefinite matrix with a singular principal submatrix is itself singular (see [4, Theorem 4.3.8] for example).

For the second statement, we assume that the gradient vector $\nabla_{s_A}(D)$ is nonzero, say

$$\frac{\partial s_A}{\partial z}(D) \neq 0.$$

Apply the inverse function theorem for the map $(x, y, z) \mapsto (x, y, s_A(x, y, z))$ to see that there is a neighborhood U of D such that $U \cap \{(x, y, z) : s_A(x, y, z) > 0\}$ is connected. On the other hand, statement (i) says that the interior of Δ_A^+ consists of one component of the region $\{(x, y, z) : s_A(x, y, z) > 0\}$, and this proves (ii). ∎

In order to prove (4.3), we first consider the case when $\Delta_A^+ \cap \Delta_B^-$ contains at least two points.

LEMMA 4.6. *Assume condition* (I). *Then we have the following:*

(i) *If a line segment L through a point $D = (d_1, d_2, d_3)$ lies in $\partial\Delta_A^+$ (or in $\partial\Delta_B^-$), then L is parallel to an axis.*

(ii) *If $\delta_{A,B} = 1$ and $\Delta_A^+ \cap \Delta_B^-$ contains at least two points, then $\Delta_A^+ \cap \Delta_B^-$ is a line segment which is parallel to an axis.*

PROOF. If $D = (d_1, d_2, d_3)$ and $E = (e_1, e_2, e_3)$ are distinct points on L, then we have the identity

$$\det[A + D + t(E - D)] = 0, \qquad t \in [0, 1],$$

by Lemma 4.5. Considering the coefficients, we see that two of $d_i - e_i$ become 0 by (4.4). For statement (ii), let D and E be distinct points of $\partial\Delta_A^+ \cap \partial\Delta_B^-$. Then the line segment L between D and E lies in $\partial\Delta_A^+ \cap \partial\Delta_B^-$ by Proposition 2.4 and Lemma 4.4. The above argument actually shows that $\Delta_A^+ \cap \Delta_B^-$ itself is a line segment containing L. ■

LEMMA 4.7. *Assume conditions* (I) *and* (II). *If $\partial\Delta_A^+ \cap \partial\Delta_B^-$ contains at least two points, then an equality holds in* (3.2).

PROOF. By Lemma 4.6, $\Delta_A^+ \cap \Delta_B^-$ is a line segment which is parallel to an axis, say the z-axis. If we denote $\Delta_A^+ \cap \Delta_B^- = \{(d_1, d_2, z) : z \in I\}$, then two equations in (4.5) with a fixed z have a unique common solution (d_1, d_2) for every $z \in I$. Note that (4.5) may be written as

$$\left(x - \frac{|a_2|^2}{z}\right)\left(y - \frac{|a_3|^2}{z}\right) = \left|\bar{a}_1 - \frac{a_2 a_3}{z}\right|^2, \tag{4.6}$$

$$\left[x - \left(1 - \frac{|b_2|^2}{1-z}\right)\right]\left[y - \left(1 - \frac{|b_3|^2}{1-z}\right)\right] = \left|\bar{b}_1 - \frac{b_2 b_3}{1-z}\right|^2. \tag{4.7}$$

It is easy to see that

$$I = \left[\max\left\{\frac{|a_2|^2}{d_1}, \frac{|a_3|^2}{d_2}\right\}, \min\left\{1 - \frac{|b_2|^2}{1-d_1}, 1 - \frac{|b_3|^2}{1-d_2}\right\}\right].$$

Therefore, for any interior point z of I, we have

$$\frac{|a_2|^2}{z} < d_1 < 1 - \frac{|b_2|^2}{1-z}, \qquad \frac{|a_3|^2}{z} < d_2 < 1 - \frac{|b_3|^2}{1-z},$$

and so we can take an open subinterval J of I such that Lemma 4.2 may be applied to (4.6) and (4.7) for each $z \in J$. Then it follows that the identity

$$\left(\left|\bar{a}_1 - \frac{a_2 a_3}{z}\right| + \left|\bar{b}_1 - \frac{b_2 b_3}{1-z}\right|\right)^2$$
$$= \left(1 - \frac{|a_2|^2}{z} - \frac{|b_2|^2}{1-z}\right)\left(1 - \frac{|a_3|^2}{z} - \frac{|b_3|^2}{1-z}\right)$$

holds for every $z \in J$. If we multiply by $z^2(1-z)^2$ on both sides and take squares in a suitable manner, then we can get an identity of real polynomials with respect to z. Comparing the coefficients of the highest degree, we get the relation $|a_1| + |b_1| = 1$. ■

If $\Delta_A^+ \cap \Delta_B^-$ contains at least two points, then the relation (4.3) follows by Lemma 4.7. Next, we assume that

(III) $\Delta_A^+ \cap \Delta_B^-$ consists of one point $D = (d_1, d_2, d_3)$.

Since $D \in \partial\Delta_A^+ \cap \partial\Delta_B^-$, both $A + D$ and $B + (I - D)$ are singular. If both $A + D$ and $B + (I - D)$ are of rank one, then they have a common null vector, and so we have $\rho_{A,B} = 1$ by Proposition 3.3.

Now, we consider the case when both $A + D$ and $B + (I - D)$ are of rank two. We denote by $[\alpha_{ij}]$ and $[\beta_{ij}]$ the classical adjoint matrices of $A + D$ and $B + (I - D)$, respectively. We note that the gradient vectors are given by

$$\nabla s_A(D) = (\alpha_{11}, \alpha_{22}, \alpha_{33}) \quad \text{and} \quad \nabla s_B(D) = (-\beta_{11}, -\beta_{22}, -\beta_{33}),$$

which are nonzero by the rank condition. By Lemma 4.5, we can take a neighborhood U of D such that

$$\begin{aligned} U \cap \Delta_A^+ &= \{(x,y,z) : (x,y,z) \in U, s_A(x,y,z) \geq 0\}, \\ U \cap \Delta_B^+ &= \{(x,y,z) : (x,y,z) \in U, s_B(x,y,z) \geq 0\}. \end{aligned}$$

By condition (III), we see that s_B takes a local maximum at D under the constraint $s_A(x,y,z) = 0$, and so $\nabla s_A(D)$ and $\nabla s_B(D)$ are linearly dependent. On the other hand, every column vector of $[\alpha_{ij}]$ (respectively $[\beta_{ij}]$) is a null vector of $A + D$ (respectively $B + (I - D)$), and $[\alpha_{ij}]$ (respectively $[\beta_{ij}]$) is of rank one. With this information in hand, it is easy to see that $A + D$ and $B + (I - D)$ have null vectors with the relation (3.4), and we conclude $\rho_{A,B} = 1$ by Proposition 3.3.

It remains to consider the case when one of $A + D$ or $B + (I - D)$ is of rank two and the other is of rank one. To do this, we need the following:

LEMMA 4.8. *Assume that* α, β, *and* γ *are positive numbers, and* $A + D$ *is of rank one. If every point* $X = (x, y, z)$ *in* Δ_A^+ *satisfies the relation*

$$\alpha(x - d_1) + \beta(y - d_2) + \gamma(z - d_3) \geq 0, \tag{4.8}$$

then we have

$$(|a_1a_2|\sqrt{\alpha}, |a_2a_3|\sqrt{\gamma}, |a_3a_1|\sqrt{\beta}) \in \Delta, \tag{4.9}$$

where Δ *is as was defined in Lemma* 3.4.

PROOF. From the relation (4.4), we see that $a_i \neq 0$ for $i = 1, 2, 3$. Therefore, we have

$$d_1 = \frac{|a_1a_2|}{|a_3|}, \qquad d_2 = \frac{|a_3a_1|}{|a_2|}, \qquad d_3 = \frac{|a_2a_3|}{|a_1|} = \frac{a_2a_3}{\bar{a}_1},$$

from the rank condition. We slice the set Δ_A^+ with the plane P_z through the point $(0, 0, z)$ which is parallel to the xy plane. Note that $P_{d_3} \cap \Delta_A^+ = \{(x, y, d_3) : x \geq d_1, y \geq d_2\}$ contains an interior point of Δ_A^+ by the first part of Lemma 4.6. Therefore, we see that the upper right component of the hyperbola (4.6) satisfies the relation (4.8) for each z in an open interval I containing d_3. From the second part of Lemma 4.2, it follows that

$$G(z) + H(z) \geq 0, \qquad z \in I, \tag{4.10}$$

where

$$\begin{aligned} G(z) &= \gamma z^2 - z(d_1\alpha + d_2\beta + d_3\gamma) + |a_2|^2\alpha + |a_3|^2\beta, \\ H(z) &= 2\sqrt{\alpha\beta}|z\bar{a}_1 - a_2a_3|. \end{aligned}$$

Note that $G(d_3) = H(d_3) = 0$ and $G'(d_3) = -d_1\alpha - d_2\beta + d_3\gamma$. We also have $H(z) = 2\sqrt{\alpha\beta}(z|a_1| - |a_2a_3|)$ because $a_1a_2a_3 = d_3|a_1|^2$ is a real number. By the relation (4.10), we have

$$|-d_1\alpha - d_2\beta + d_3\gamma| \leq 2|a_1|\sqrt{\alpha\beta},$$

from which we infer the desired condition (4.9). ∎

Now, we assume that $A + D$ is of rank one and $B + (I - D)$ is of rank two. Then the tangent plane of the surface $s_B(x, y, z) = 0$ at the point D is given by (4.8) with equality, where

$$\begin{aligned}\alpha &= \begin{vmatrix} 1-d_2 & b_3 \\ \bar{b}_3 & 1-d_3 \end{vmatrix}, \\ \beta &= \begin{vmatrix} 1-d_1 & \bar{b}_2 \\ b_2 & 1-d_3 \end{vmatrix}, \\ \gamma &= \begin{vmatrix} 1-d_1 & b_1 \\ \bar{b}_1 & 1-d_2 \end{vmatrix},\end{aligned} \tag{4.11}$$

and $|\cdot|$ denotes the determinant. It is easy to see that every point (x, y, z) of Δ_B^- satisfies

$$\alpha(x - d_1) + \beta(x - d_2) + \gamma(y - d_3) \leq 0, \tag{4.12}$$

by Lemma 4.2 together with the similar calculation as in the proof of Lemma 4.8. If a point $E \in \Delta_A^+$ satisfies (4.12) with the strict inequality, then the line segment between D and E has a nonempty intersection with $\Delta_B^- \setminus \{D\}$. Because this line segment lies in Δ_A^+ by the convexity, we get a contradiction with assumption (III). Therefore, we see that every point of Δ_A^+ satisfies (4.8), and so we have the relation (4.9) with α, β, and γ in (4.11).

In order to apply Proposition 3.3, it is more convenient to consider the quadratic Hermitian forms

$$|p_1x + p_2y + p_3z|^2 \quad \text{and} \quad |q_1x + q_2y + q_3z|^2 + |r_1x + r_2y + r_3z|^2,$$

which are associated with $A + D$ and $B + (I - D)$, respectively. Then the relation (4.9) says that

$$(|p_1x_0|, |p_2y_0|, |p_3z_0|) \in \Delta, \tag{4.13}$$

where (x_0, y_0, z_0) is the cross product of (q_1, q_2, q_3) and (r_1, r_2, r_3), and so it is a null vector of $B + (I - D)$. From the condition (4.13), we see that there are σ and τ with the relation

$$p_1x_0 + p_2y_0e^{i\sigma} + p_3z_0e^{i\tau} = 0.$$

Therefore, $(x_0, y_0e^{i\sigma}, z_0e^{i\tau})$ is a null vector of $A + D$. By Proposition 3.3, we have $\rho_{A,B} = 1$ and this completes the proof of Theorem 4.1

Note Added in the proof: In the paper (H.-J. Kim and S.-H. Kye, Indecomposable extreme positive linear maps in matrix algebras, Bull. London Math. Soc., to appear), the authors have shown that there is an indecomposable positive linear map between M_n which fixes diagonals, whenever $n \geq 4$.

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REFERENCES

1 S. J. Cho, S.-H. Kye, and S. G. Lee, Generalized Choi maps in 3-dimensional matrix algebras, *Linear Algebra Appl.* 171:213–224 (1992).
2 M.-D. Choi, Completely positive linear maps on complex matrices, *Linear Algebra Appl.* 10:285–290 (1985).
3 ——, Positive semidefinite biquadratic forms, *Linear Algebra Appl.* 12:95–100 (1975).
4 R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge U.P. 1990.
5 S.-H. Kye, A class of atomic positive linear maps in 3-dimensional matrix algebras, in *Elementary Operators and Applications* (M. Mathieu, Ed.), World-Scientific, 1992, pp. 205–209.
6 H. Osaka, A class of extremal positive maps in 3×3 matrix algebras, *Publ. Res. Inst. Math. Sci.* 28:747–756 (1992).
7 ——, A series of absolutely indecomposable positive maps in matrix algebras, *Linear Algebra Appl.*, to appear.
8 A. G. Robertson, Positive projections on C^*-algebras and extremal positive maps, *J. London Math. Soc. (2)* 32:133–140 (1985).
9 E. Størmer, Positive linear maps of operator algebras, *Acta Math.* 110:233–278 (1963).
10 K. Tanahashi and J. Tomiyama, Indecomposable positive maps in matrix algebras, *Canad. Math. Bull.* 31:308–317 (1988).

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