

Construction of entangled states with positive partial transposes based on indecomposable positive linear maps

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Abstract

We show that every entanglement with positive partial transpose may be constructed from an indecomposable positive linear map between matrix algebra.

Key words: entanglement, positive partial transpose, indecomposable positive map

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1 Introduction

The theory of entanglement in quantum physics is now playing a key rôle in the quantum information theory and quantum communication theory. In order to characterize various kinds of entangled states, the notion of positive linear maps is turned out to be very useful as was developed by quantum physicists [5], [6] and [13]. On the other hand, some operator algebraists [3], [12], [15] studied various kinds of positive linear maps and showed their non-decomposability using several kinds of block matrices, which are nothing but entanglement.

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The duality between entangled states and positive linear maps is also used to construct them. The notion of entanglement was used [14] to construct a new kind of indecomposable positive linear maps. This construction was generalized in [10], [11] to give a method of creating entanglement witnesses and corresponding positive linear maps starting from the edge states. Thus they presented a canonical form of non-decomposable entanglement witnesses and the corresponding indecomposable positive linear maps.

On the other hand, the authors [4] have constructed entangled states with positive partial transposes from decomposable positive linear maps. This methods provided us an explicit example of one parameter family of $3 \otimes 3$ edge states which are not arising from unextendible product basis.

In the present Letter, we show that every edge state with positive partial transpose arises from an indecomposable positive linear map in the way described in the previous paper [4]. Therefore, our result complements the opposite direction of [10] and [11], where it was shown that, given an edge state, its entanglement witness (and so the corresponding linear map) can be determined. For the interplay between the notions of entanglement and positive linear maps, we refer to the paper [4] together with the references there.

As in the paper [4], we denote by \mathbb{D} the convex cone of all decomposable positive linear maps from the C^* -algebra M_m of all $m \times m$ matrices over the complex field into M_n , and by \mathbb{T} the cone of all positive semi-definite $m \times m$ matrices over M_n whose partial transposes are also positive semi-definite. The key step is to show that every face of the cone \mathbb{T} is exposed. This is quite surprising, since not every face of the dual cone \mathbb{D} of \mathbb{T} is exposed as was shown in [1] and [9].

We first develop in Section 2 some generalities about duality theory between the cones \mathbb{D} and \mathbb{T} , and characterize faces of the cone \mathbb{T} . In Section 3, we show that every face of the cone \mathbb{T} is exposed, and find all faces of \mathbb{T} in the simplest nontrivial case of $m = n = 2$. In the final section, we show that edge state with positive partial transpose arises from an indecomposable positive linear map in the way described in [4]. Using our terminologies, it is easy to check the following statement: An entangled state ρ with positive partial transpose is an edge state if and only if the proper face of \mathbb{T} containing ρ as an interior point does not contain a separable state.

Throughout this Letter, we will not use bra-ket notation as in the previous paper [4]. Every vector will be considered as a column vector. If $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^n$ then x will be considered as an $m \times 1$ matrix, and y^* will be considered as a $1 \times n$ matrix, and so xy^* is an $m \times n$ rank one matrix whose range is generated by x and whose kernel is orthogonal to y . For a vector x , the notation \bar{x} will be used for the vector whose entries are conjugate of the corresponding entries. The notation $\langle \cdot, \cdot \rangle$ will be used for bi-linear pairing. On the other hand, $(\cdot | \cdot)$ will be used for the inner product, which is sesqui-linear, that is, linear in the first variable and conjugate-linear in the second variable.

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2 Duality between two cones

In this section, we first consider the faces of the cones $\text{conv}(C_1, C_2)$ and $C_1 \cap C_2$ of the cones C_1 and C_2 , where $\text{conv}(C_1, C_2)$ denotes the convex hull generated by C_1 and C_2 . We have already seen in [9] that every face F of the cone $C = \text{conv}(C_1, C_2)$ associates with a unique pair (F_1, F_2) of faces of C_1 and C_2 , respectively, with the properties

$$F = \text{conv}(F_1, F_2), \quad F_1 = F \cap C_1, \quad F_2 = F \cap C_2. \quad (1)$$

Actually, it is easy to see that $F_i = F \cap C_i$ is a face of C_i for $i = 1, 2$ and the identity $F = \text{conv}(F_1, F_2)$ holds. It should be noted [1] that $\text{conv}(F_1, F_2) = \text{conv}(E_1, F_2)$ may be a face of C even though $E_1 \neq F_1$. If F_1 and F_2 satisfy the conditions in (1) then we will write

$$\sigma(F_1, F_2) := \text{conv}(F_1, F_2).$$

We always assume that

$$\sigma(F_1, F_2) \cap C_i = F_i, \quad i = 1, 2, \quad (2)$$

whenever we use the notation $\sigma(F_1, F_2)$.

On the other hand, if F_i is a face of the cone C_i for $i = 1, 2$ then $F_1 \cap F_2$ is a face of $C_1 \cap C_2$. Conversely, every face F of the cone $C = C_1 \cap C_2$ associates with

a unique pair (F_1, F_2) of faces of C_1 and C_2 , respectively, with the properties

$$F = F_1 \cap F_2, \quad \text{int } F \subset \text{int } F_1, \quad \text{int } F \subset \text{int } F_2,$$

where $\text{int } F$ denotes the set of all interior point of the convex set F , which is the topological interior of F relative to the hyperplane generated by F . To see that, take an interior point x of F in $C_1 \cap C_2$. If we take the face F_i of C_i with $x \in \text{int } F_i$ for $i = 1, 2$ then we have

$$x \in \text{int } F_1 \cap \text{int } F_2 \subset \text{int } (F_1 \cap F_2).$$

Since $F_1 \cap F_2$ is a face of C , we conclude that $F = F_1 \cap F_2$. The uniqueness is clear, because every convex set is decomposed into the interiors of faces. We will use the notation

$$\tau(F_1, F_2) := F_1 \cap F_2$$

only when

$$\text{int } \tau(F_1, F_2) \subset \text{int } F_i, \quad i = 1, 2. \quad (3)$$

Now, we proceed to consider the duality. Let X and Y be finite dimensional normed spaces, which are dual each other with respect to a bilinear pairing $\langle \cdot, \cdot \rangle$. For a subset C of X (respectively D of Y), we define the *dual cone* C° (respectively D°) by the set of all $y \in Y$ (respectively $x \in X$) such that $\langle x, y \rangle \geq 0$ for each $x \in C$ (respectively $y \in D$). It is clear that $C^{\circ\circ}$ is the closed convex cone of X generated by C . We have

$$[\text{conv}(C_1, C_2)]^\circ = C_1^\circ \cap C_2^\circ, \quad (C_1 \cap C_2)^\circ = \text{conv}(C_1^\circ, C_2^\circ),$$

whenever C_1 and C_2 are closed convex cones of X , as was seen in [3]. For a subset S of a closed convex cone C of X , we define the subset S' of C° by

$$S' = \{y \in C^\circ : \langle x, y \rangle = 0 \text{ for each } x \in S\}.$$

It is then clear that S' is a closed face of C° . If we take an interior point x_0 of a face F then we see that $F' = \{x_0\}'$. We will write x'_0 for $\{x_0\}'$.

Let F_i be a face of the convex cone C_i , for $i = 1, 2$, satisfying the conditions in (1) then $\sigma(F_1, F_2) = \text{conv}(F_1, F_2)$ is a face of $C = \text{conv}(C_1, C_2)$. It is easy to see that

$$\sigma(F_1, F_2)' = F_1' \cap F_2', \quad (4)$$

where it should be noted that the dual faces should be taken in the corresponding duality. For example, $\sigma(F_1, F_2)'$ is the set of all $y \in C^\circ = C_1^\circ \cap C_2^\circ$ such that $\langle x, y \rangle = 0$ for each $x \in \text{conv}(F_1, F_2)$. On the other hand, F_i' is the set of all $y \in C_i^\circ$ such that $\langle x, y \rangle = 0$ for each $x \in F_i$. Analogously, we also have

$$\tau(F_1, F_2)' = \text{conv}(F_1', F_2'). \quad (5)$$

From the easy inclusion $F_i' \subset \tau(F_1, F_2)'$, one direction comes out. For the reverse inclusion, let $y \in \tau(F_1, F_2)'$. Since $y \in (C_1 \cap C_2)^\circ = \text{conv}(C_1^\circ, C_2^\circ)$, we may write $y = y_1 + y_2$ with $y_i \in C_i^\circ$ for $i = 1, 2$. We also take an interior point x of $\tau(F_1, F_2)$. Then we have $x \in \text{int } F_i \subset C_i$ by (3), and so $\langle x, y_i \rangle \geq 0$ for $i = 1, 2$. From the relation

$$0 = \langle x, y \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle,$$

we conclude that $\langle x, y_i \rangle = 0$. Since x is an interior point of F_i , we see that $y_i \in F_i'$ for $i = 1, 2$, and $y \in \text{conv}(F_1', F_2')$.

Now, we consider the cone $C_1 = \mathbb{P}_{m \wedge n}$ of all completely positive linear maps from M_m into M_n , and the cone $C_2 = \mathbb{P}^{m \wedge n}$ of all completely copositive linear maps from M_m into M_n . Both cones are sitting in the linear space $\mathcal{L}(M_m, M_n)$ of all linear maps from M_m into M_n . It is well-known that every map in the cone $\mathbb{P}_{m \wedge n}$ (respectively $\mathbb{P}^{m \wedge n}$) is of the form $\phi_{\mathcal{V}}$ (respectively $\phi^{\mathcal{V}}$) for a subset $\mathcal{V} = \{V_1, V_2, \dots, V_\nu\}$ of $M_{m \times n}$:

$$\begin{aligned} \phi_{\mathcal{V}} : X &\mapsto \sum_{i=1}^{\nu} V_i^* X V_i, & X \in M_m, \\ \phi^{\mathcal{V}} : X &\mapsto \sum_{i=1}^{\nu} V_i^* X^t V_i, & X \in M_m, \end{aligned}$$

where X^t denote the transpose of X . We denote by $\phi_{\mathcal{V}} = \phi_{\{V\}}$ and $\phi^{\mathcal{V}} = \phi^{\{V\}}$. For a subspace E of $M_{m \times n}$, we define

$$\begin{aligned} \Phi_E &= \{\phi_{\mathcal{V}} \in \mathbb{P}_{m \wedge n} : \text{span } \mathcal{V} \subset E\}, \\ \Phi^E &= \{\phi^{\mathcal{V}} \in \mathbb{P}^{m \wedge n} : \text{span } \mathcal{V} \subset E\}, \end{aligned}$$

where $\text{span } \mathcal{V}$ denotes the span of the set \mathcal{V} . We have shown in [7] that the correspondence

$$E \mapsto \Phi_E \quad (\text{respectively } E \mapsto \Phi^E)$$

gives rise to a lattice isomorphism between the lattice of all subspaces of the vector space $M_{m \times n}$ and the lattice of all faces of the convex set $\mathbb{P}_{m \wedge n}$

(respectively $\mathbb{P}^{m \wedge n}$). It is also known that

$$\text{int } \Phi_E = \{\phi_{\mathcal{V}} \in \mathbb{P}_{m \wedge n} : \text{span } \mathcal{V} = E\}$$

and similarly for $\text{int } \Phi^E$.

A linear map in the cone

$$\mathbb{D} := \text{conv}(\mathbb{P}_{m \wedge n}, \mathbb{P}^{m \wedge n}) \subset \mathcal{L}(M_m, M_n)$$

is said to be *decomposable*. Every decomposable map is positive, that is, sends positive semi-definite matrices into themselves, but the converse is not true. There are bunch of examples of indecomposable positive linear maps in the literature as was referred in [4]. We see that every face of the cone \mathbb{D} is of the form $\text{conv}(\Phi_D, \Phi^E)$ for a pair (D, E) of subspaces of $M_{m \times n}$. We say that a pair (D, E) is a *decomposition pair* if the set $\text{conv}(\Phi_D, \Phi^E)$ is a face of \mathbb{D} together with the corresponding condition (2). By an abuse of notation, we will write

$$\sigma(D, E) := \sigma(\Phi_D, \Phi^E)$$

for a decomposition pair (D, E) . The condition (2) is then written by

$$\sigma(D, E) \cap \mathbb{P}_{m \wedge n} = \Phi_D, \quad \sigma(D, E) \cap \mathbb{P}^{m \wedge n} = \Phi^E.$$

In this way, every decomposition pair gives rise to a face of \mathbb{D} , and every face of \mathbb{D} corresponds to a unique decomposition pair.

In [3], we have considered the bi-linear pairing between the spaces $\mathcal{L}(M_m, M_n)$ and $M_n \otimes M_m$, given by

$$\langle A, \phi \rangle = \text{Tr} \left[\left(\sum_{i,j=1}^m \phi(e_{ij}) \otimes e_{ij} \right) A^t \right] = \sum_{i,j=1}^m \langle \phi(e_{ij}), a_{ij} \rangle, \quad (6)$$

for $A = \sum_{i,j=1}^m a_{ij} \otimes e_{ij} \in M_n \otimes M_m$ and $\phi \in \mathcal{L}(M_m, M_n)$, where the bi-linear form in the right-side is given by $\langle X, Y \rangle = \text{Tr}(Y X^t)$ for $X, Y \in M_n$. It is well-known [2] (see also [3]) that the dual cone $(\mathbb{P}_{m \wedge n})^\circ = (M_n \otimes M_m)^+$ consists of all positive semi-definite block matrices in $M_n \otimes M_m$. Recall that every element in the tensor product $M_n \otimes M_m$ may be considered as an $m \times m$ matrix over M_n . We define the *partial transpose* or *block transpose* A^τ by

$$\left(\sum_{i,j=1}^m a_{ij} \otimes e_{ij} \right)^\tau = \sum_{i,j=1}^m a_{ji} \otimes e_{ij}.$$

Then we also see that the dual cone $(\mathbb{P}^{m \wedge n})^\circ$ consists of all block matrices in $M_n \otimes M_m$ whose block transposes are positive semi-definite.

We identify a matrix $z \in M_{m \times n}$ and a vector $\tilde{z} \in \mathbb{C}^n \otimes \mathbb{C}^m$ as follows: For $z = [z_{ik}] \in M_{m \times n}$, define

$$\begin{aligned} z_i &= \sum_{k=1}^n z_{ik} e_k \in \mathbb{C}^n, & i = 1, 2, \dots, m, \\ \tilde{z} &= \sum_{i=1}^m z_i \otimes e_i \in \mathbb{C}^n \otimes \mathbb{C}^m. \end{aligned}$$

Then $z \mapsto \tilde{z}$ defines an inner product isomorphism from $M_{m \times n}$ onto $\mathbb{C}^n \otimes \mathbb{C}^m$. We also note that $\tilde{z} \tilde{z}^*$ is a positive semi-definite matrix in $M_n \otimes M_m$ of rank one, and the cone $(\mathbb{P}^{m \wedge n})^\circ$ is spanned by $\{\tilde{z} \tilde{z}^* : z \in M_{m \times n}\}$. We have shown in [9] that

$$\langle \tilde{z} \tilde{z}^*, \phi_V \rangle = \langle (\tilde{z} \tilde{z}^*)^\tau, \phi^V \rangle = |z|V|^2, \quad (7)$$

for $m \times n$ matrices $z, V \in M_{m \times n}$. Therefore, we see that $\tilde{z} \tilde{z}^*$ belongs to $(\Phi_D)'$ if and only if $z \in D^\perp$, and so we have

$$(\Phi_D)' = \text{conv} \{ \tilde{z} \tilde{z}^* : z \in D^\perp \} = \{ A \in (M_n \otimes M_m)^+ : \mathcal{R}A \subset \widetilde{D}^\perp \},$$

where $\mathcal{R}A$ is the range space of A and $\widetilde{D} = \{ \tilde{z} \in \mathbb{C}^n \otimes \mathbb{C}^m : z \in D \}$. If we denote by

$$\begin{aligned} \Psi_D &= \{ A \in (M_n \otimes M_m)^+ : \mathcal{R}A \subset \widetilde{D} \}, \\ \Psi^E &= \{ A \in M_n \otimes M_m : A^\tau \in \Psi^E \} \end{aligned}$$

then we have

$$\text{int } \Psi_D = \{ A \in \Psi_D : \mathcal{R}A = \widetilde{D} \}, \quad \text{int } \Psi^E = \{ A \in \Psi^E : \mathcal{R}A^\tau = \widetilde{E} \}. \quad (8)$$

In short, we have

$$(\Phi_D)' = \Psi_{D^\perp}, \quad (\Psi_D)' = \Phi_{D^\perp}, \quad (\Phi^E)' = \Psi^{E^\perp}, \quad (\Psi^E)' = \Phi^{E^\perp}.$$

It is well known that every face of the cone $(M_n \otimes M_m)^+$ is of the form Ψ_D for a subspace D of $M_{m \times n}$.

We say that a pair (D, E) of subspaces of $M_{m \times n}$ is an *intersection pair* if the faces Ψ_D and Ψ^E satisfy the condition (3). More precisely, a pair (D, E) is an intersection pair if and only if

$$\text{int}(\Psi_D \cap \Psi^E) \subset \text{int } \Psi_D \cap \text{int } \Psi^E.$$

Then, every face of the cone

$$\mathbb{T} := (\mathbb{P}_{m \wedge n})^\circ \cap (\mathbb{P}^{m \wedge n})^\circ = \{A \in (M_n \otimes M_m)^+ : A^\tau \in (M_n \otimes M_m)^+\}$$

is of the form $\tau(\Psi_D, \Psi^E)$ for a unique intersection pair (D, E) . We also write

$$\tau(D, E) := \tau(\Psi_D, \Psi^E)$$

if (D, E) is an intersection pair, by another abuse of notation. By (4) and (5), we have

$$\sigma(D, E)' = \Psi_{D^\perp} \cap \Psi^{E^\perp}, \quad \tau(D, E)' = \text{conv}(\Phi_{D^\perp}, \Phi^{E^\perp}). \quad (9)$$

It should be noted that (D^\perp, E^\perp) need not to be an intersection pair even if (D, E) is a decomposition pair.

We denote by \mathbb{P}_1 the cone of all positive linear maps from M_m into M_n . Then the dual cone $\mathbb{V}_1 := (\mathbb{P}_1)^\circ$ of the cone \mathbb{P}_1 with respect to the pairing (6) is given by

$$\mathbb{V}_1 = \text{conv} \{ \tilde{z} \tilde{z}^* : \text{rank } z = 1 \}.$$

By the relation

$$\widetilde{xy^*} \widetilde{xy^*}^* = (\bar{y} \otimes x)(\bar{y} \otimes x)^* = \bar{y} \bar{y}^* \otimes xx^*,$$

we have $\mathbb{V}_1 = M_n^+ \otimes M_m^+$. It is easy to see that $\mathbb{V}_1 \subset \mathbb{T}$. A density matrix in $\mathbb{T} \setminus \mathbb{V}_1$ is said to be an *entangled state with positive partial transpose*.

3 Exposed faces

A face F of a convex set C is said to be *exposed* if there is $x \in C^\circ$ such that $F = x'$. It is easy to see that a face F is exposed if and only if $F = F''$. A decomposition pair (respectively an intersection pair) (D, E) is said to be *exposed* if the face $\sigma(D, E)$ (respectively $\tau(D, E)$) is exposed.

Lemma 1 *Let $A \in \mathbb{T}$. If $A' = \sigma(D, E)$ then $\mathcal{R}A = \widetilde{D}^\perp$ and $\mathcal{R}A^\tau = \widetilde{E}^\perp$.*

PROOF. First of all, the relation

$$A \in A'' = \sigma(D, E)' = \Psi_{D^\perp} \cap \Psi^{E^\perp}$$

implies that $\mathcal{R}A \subset \widetilde{D}^\perp$ and $\mathcal{R}A^\tau \subset \widetilde{E}^\perp$. For the reverse inclusion, let $V \in M_{m \times n}$ with $\widetilde{V} \in (\mathcal{R}A)^\perp$, and write $A = \sum_i \widetilde{z}_i \widetilde{z}_i^*$ with $z_i \in M_{m \times n}$. Then we have

$$\langle A, \phi_V \rangle = \sum |(z_i | V)|^2 = 0,$$

and $\phi_V \in A'$. Since $A' = \sigma(D, E)$ by the assumption, we have

$$\phi_V \in A' \cap \mathbb{P}_{m \wedge n} = \sigma(D, E) \cap \mathbb{P}_{m \wedge n} = \Phi_D.$$

This implies $V \in D$, and so we have $\mathcal{R}A = \widetilde{D}^\perp$. For the second relation $\mathcal{R}A^\tau = \widetilde{E}^\perp$, we note the following easy identities

$$\langle A^\tau, \phi_V \rangle = \langle A, \phi^V \rangle, \quad \langle A^\tau, \phi^W \rangle = \langle A, \phi_W \rangle,$$

by (7). This implies that $A' = \sigma(D, E)$ if and only if $(A^\tau)' = \sigma(E, D)$. Therefore, the second relation $\mathcal{R}A^\tau = \widetilde{E}^\perp$ follows from the first.

Proposition 2 *Let (D, E) be a pair of subspaces of $m \times n$ matrices. Then (D, E) is an exposed decomposition pair if and only if (D^\perp, E^\perp) is an intersection pair.*

PROOF. Assume that (D, E) is an exposed decomposition pair, and take an element $A \in \text{int } \sigma(D, E)'$. Then we have

$$A' = \sigma(D, E)'' = \sigma(D, E)$$

by the assumption. This implies that $\mathcal{R}A = \widetilde{D}^\perp$ and $\mathcal{R}A^\tau = \widetilde{E}^\perp$ by Lemma 1, and so we see that $A \in \text{int } \Psi_{D^\perp} \cap \text{int } \Psi^{E^\perp}$ by (8). This proves the required relation

$$\text{int } (\Psi_{D^\perp} \cap \Psi^{E^\perp}) = \text{int } \sigma(D, E)' \subset \text{int } \Psi_{D^\perp} \cap \text{int } \Psi^{E^\perp}$$

by the relation (9). Therefore, we see that (D^\perp, E^\perp) is an intersection pair.

For the converse, assume that (D^\perp, E^\perp) is an intersection pair. First of all, we see that $\text{conv } (\Phi_D, \Phi^E) = \tau(D^\perp, E^\perp)'$ is an exposed face of \mathbb{D} . We may take an exposed decomposition pair (D_1, E_1) such that $\text{conv } (\Phi_D, \Phi^E) = \sigma(D_1, E_1)$. It suffices to show that $D = D_1$ and $E = E_1$. To do this, take $A \in \text{int } \tau(D^\perp, E^\perp)$. Then we have $A \in \text{int } \Psi_{D^\perp} \cap \text{int } \Psi^{E^\perp}$ since (D^\perp, E^\perp) is an intersection pair, and so

$$\widetilde{D}^\perp = \mathcal{R}A, \quad \widetilde{E}^\perp = \mathcal{R}A^\tau,$$

by (8). On the other hand, we also have $A' = \tau(D^\perp, E^\perp)' = \sigma(D_1, E_1)$, and

$$\widetilde{D}_1^\perp = \mathcal{R}A, \quad \widetilde{E}_1^\perp = \mathcal{R}A^\tau,$$

by Lemma 1. Therefore, we have $D = D_1$ and $E = E_1$.

Theorem 3 *Every face of the convex cone \mathbb{T} consisting of all positive semi-definite block matrices with positive semi-definite block transpose is exposed.*

PROOF. Every face of \mathbb{T} is of the form $\tau(D, E)$ for an intersection pair (D, E) of spaces of matrices. Then (D^\perp, E^\perp) is an exposed decomposition pair by Proposition 2, and so $\tau(D, E)' = \sigma(D^\perp, E^\perp)$. Therefore, we have

$$\tau(D, E)'' = \sigma(D^\perp, E^\perp)' = \tau(D, E)$$

by (9).

For the simplest case of $m = n = 2$, every decomposition pair has been characterized in [1]. See also [8]. For every $x, y \in \mathbb{C}^2$ the pair

$$([xy^*], [\bar{x}y^*]) \tag{10}$$

is an intersection pair, where $[z_i]$ denotes the span of $\{z_i\}$. These give us all extremal rays of the cone \mathbb{T} . We note that $\widetilde{xy^*} = \bar{y} \otimes x$, and so the face $\tau([xy^*], [\bar{x}y^*])$ consists of nonnegative scalar multiples of $(\bar{y} \otimes x)(\bar{y} \otimes x)^* = \bar{y}y^* \otimes xx^*$. For every vectors x, y, z, w , the pair

$$([xy^*, zw^*], [\bar{x}y^*, \bar{z}w^*]) \tag{11}$$

is also an exposition pair, where $x \not\parallel z$ or $y \not\parallel w$. Here, $x \not\parallel z$ means that x and z are not parallel each other.

Next, we also have intersection pairs whose subspaces spanned by three rank one matrices. They are

$$([V]^\perp, [W]^\perp) \tag{12}$$

with rank two matrices V and W satisfying the properties in Proposition 3.6 of [1], or

$$([xy^*]^\perp, [\bar{x}y^*]^\perp) \tag{13}$$

for arbitrary vectors $x, y \in \mathbb{C}^2$. For the case (12), note that if $([V], [W])$ is an exposed decomposition pair then we may take vectors $x_i, y_i, i = 1, 2, 3$, such that

$$Vy_i \perp x_i, \quad i = 1, 2, 3,$$

and we have

$$[V]^\perp = [x_1y_1^*, x_2y_2^*, x_3y_3^*], \quad [W]^\perp = [\bar{x}_1y_1^*, \bar{x}_2y_2^*, \bar{x}_3y_3^*]$$

as is in the proof of [1] Proposition 3.6. In the case (13), we may take vector z, w with $z \perp x$ and $w \perp y$ so that

$$[xy^*]^\perp = [zy^*, xw^*, zw^*], \quad [\bar{x}y^*]^\perp = [\bar{z}y^*, \bar{x}w^*, \bar{z}w^*].$$

The case (13) gives rise to a maximal face of the cone \mathbb{T} .

Finally, we also have intersection pairs

$$([V]^\perp, M_2) \tag{14}$$

and

$$(M_2, [W]^\perp) \tag{15}$$

with rank two matrices V and W . These also give us maximal faces. We may take four rank one matrices $x_iy_i^*, i = 1, 2, 3, 4$, so that

$$[V]^\perp = [x_iy_i^* : i = 1, 2, 3, 4], \quad M_2 = [\bar{x}_iy_i^* : i = 1, 2, 3, 4]$$

by Proposition 3.7 of [1]. For example, consider the pair $([I]^\perp, M_2)$, where I is the identity matrix. If we take

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} -i \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

for $x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4$, then we have

$$\sum_{i=1}^4 \widetilde{x_iy_i^*} \widetilde{x_iy_i^*}^* = \begin{pmatrix} 2 & 1-i & -1-i & -2 \\ 1+i & 3 & 0 & -1-i \\ -1+i & 0 & 3 & 1-i \\ -2 & -1+i & 1+i & 2 \end{pmatrix}$$

whose range space is \widetilde{I}^\perp , which is 3-dimensional. On the other hand, $\sum_i \widetilde{\bar{x}_iy_i^*} \widetilde{\bar{x}_iy_i^*}^*$ is the block transpose of the above matrix, and the range space is 4-dimensional.

These six cases (10)-(15) list up all faces of the convex cone \mathbb{T} for the case of $m = n = 2$.

4 Entangled states with positive partial transposes

In order to construct entanglement with positive partial transpose, we have shown in [4] that if (D, E) is a decomposition pair so that $\sigma(D, E)$ is a proper face of \mathbb{D} such that $\text{int } \sigma(D, E) \subset \text{int } \mathbb{P}_1$ then every non-zero element A of the face $\sigma(D, E)'$ belongs to $\mathbb{T} \setminus \mathbb{V}_1$. To begin with, we show the converse. Note that we have the following two cases

$$\text{int } \sigma(D, E) \subset \text{int } \mathbb{P}_1 \quad \text{or} \quad \sigma(D, E) \subset \partial \mathbb{P}_1,$$

since $\sigma(D, E)$ is a convex subset of the cone \mathbb{P}_1 , where $\partial C := C \setminus \text{int } C$ denotes the boundary of the convex set C .

Theorem 4 *Let (D, E) be a decomposition pair which gives rise to a proper face of \mathbb{D} . Then the following are equivalent:*

- (i) $\text{int } \sigma(D, E) \subset \text{int } \mathbb{P}_1$,
- (ii) $\sigma(D, E)' \setminus \{0\} \subset \mathbb{T} \setminus \mathbb{V}_1$.

PROOF. The direction (i) \implies (ii) was proved in Theorem 1 of [4]. For the reverse direction, it suffices to show that

$$\sigma(D, E) \subset \partial \mathbb{P}_1 \implies \sigma(D, E)' \cap \mathbb{V}_1 \not\subseteq \{0\}.$$

To do this, assume that $\sigma(D, E) \subset \partial \mathbb{P}_1$. Take $\phi \in \text{int } \sigma(D, E)$, and take the face F of \mathbb{P}_1 such that $\phi \in \text{int } F$. We note that F is a proper face of \mathbb{P}_1 since $\phi \in \partial \mathbb{P}_1$ by the assumption. We also note that F is a face of $\mathbb{P}_1 = (\mathbb{V}_1)^\circ$ and $\sigma(D, E)$ is a face of $\mathbb{D} = \mathbb{T}^\circ$, and so we have

$$\begin{aligned} F' &= \{A \in \mathbb{V}_1 : \langle A, \phi \rangle = 0\} \\ \sigma(D, E)' &= \{A \in \mathbb{T} : \langle A, \phi \rangle = 0\}. \end{aligned}$$

This shows that $\sigma(D, E)' \cap \mathbb{V}_1 = F'$, which has a non-zero element since F is a proper face of \mathbb{P}_1 .

Now, we show that every edge state A with positive partial transpose arise in the way described in [4]. First of all, we take the face F of \mathbb{T} such that $A \in \text{int } F$. Since A is an edge state, we see that F has no separable state. We also see that $F = \sigma(D, E)'$ for an exposed decomposition pair (D, E) by Theorem 3, and so we have

$$\sigma(D, E)' \setminus \{0\} = F \setminus \{0\} \subset \mathbb{T} \setminus \mathbb{V}_1.$$

It follows from Theorem 4 that

$$\text{int } \sigma(D, E) \subset \text{int } \mathbb{P}_1. \quad (16)$$

Of course, a face $\sigma(D, E)$ of \mathbb{D} with the property (16) arises from an indecomposable positive linear map in $\mathbb{P}_1 \setminus \mathbb{D}$, as was explained in [4].

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