ENTANGLEMENT WITNESSES ARISING FROM EXPOSED POSITIVE LINEAR MAPS

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Abstract. We consider entanglement witnesses arising from positive linear maps which generate exposed extremal rays. We show that every entanglement can be detected by one of these witnesses, and this witness detects a unique set of entanglement among those. Therefore, they provide a minimal set of witnesses to detect all entanglement in a sense. Furthermore, if those maps are indecomposable then they detect large classes of entanglement with positive partial transposes which have nonempty relative interiors in the cone generated by all PPT states. We also provide a one parameter family of indecomposable positive linear maps which generate exposed extremal rays. This gives the first examples of such maps between three dimensional matrix algebra.

1. Introduction

The notion of entanglement is one of the key current research areas of quantum physics in the relation of possible applications to quantum computation and quantum information theories. Entanglement is a positive semi-definite $nm \times nm$ matrix in $M_{nm} = M_n \otimes M_m$ which is not separable, where M_n denotes the C^*-algebra of all $n \times n$ matrices over the complex field. Recall that a positive semi-definite matrix gives rise to a state on the matrix algebra through the Hadamard product if it is normalized. A positive semi-definite matrix in $M_n \otimes M_m$ is said to be separable if it is the convex sum of rank one positive semi-definite matrices onto product vectors of the form $x \otimes y \in \mathbb{C}^n \otimes \mathbb{C}^m$. We denote by $\mathbb{V}_1$ the cone of all separable ones. It is easy to see that the convex cone $\mathbb{V}_1$ coincides with the tensor product $M_n^+ \otimes M_m^+$ of the positive cones consisting of all positive semi-definite matrices. Therefore, entanglement consists of $(M_n \otimes M_m)^+ \setminus M_n^+ \otimes M_m^+$.

If $\mathcal{A}$ and $\mathcal{B}$ are commutative C^*-algebras then the two cones $(\mathcal{A} \otimes \mathcal{B})^+$ and $\mathcal{A}^+ \otimes \mathcal{B}^+$ coincide, and so the notion of entanglement reflects non-commutative order structures in nature.

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The most general method to determine if a given state is separable or not is to use the duality between the space $M_n \otimes M_m$ and the space $\mathcal{L}(M_m, M_n)$ of all linear maps from M_m into M_n. The bi-pairing between these spaces is given by

$$\langle y \otimes x, \phi \rangle = \mathrm{Tr}(\phi(x) y^{\mathrm{t}}) \tag{1}$$

for $x \in M_m, y \in M_n$ and $\phi \in \mathcal{L}(M_m, M_n)$. It is possible to define this pairing in more general context, and used [1] to define topologies in the space of linear maps between operator algebras, which are useful to prove Hahn-Banach type extension theorems for completely positive linear maps. This pairing had been also used by Woronowicz [41] to show that there exists an indecomposable positive linear map from M_2 into M_4. It had been also used in [37] to study extendibility of positive linear maps on C^*-algebras.

From now on, every vector in $\mathbb{C}^m$ may be considered as an $m \times 1$ matrix, and we denote by $\bar{x}$ and x^* the complex conjugate and the Hermitian conjugate of x. In this notation, rank one matrix $|x\rangle\langle y|$ is written by xy^*. A linear map between C^*-algebras is said to be positive if it sends positive elements into themselves. We denote by $\mathbb{P}_1$ the convex cone of all positive linear maps in $\mathcal{L}(M_m, M_n)$. It turns out [17] that the cones $\mathbb{V}_1$ and $\mathbb{P}_1$ are dual each other with respect to the pairing (1) in the following sense:

$$\begin{aligned} A \in \mathbb{V}_1 &\iff \langle A, \phi \rangle \ge 0 \text{ for each } \phi \in \mathbb{P}_1, \\ \phi \in \mathbb{P}_1 &\iff \langle A, \phi \rangle \ge 0 \text{ for each } A \in \mathbb{V}_1. \end{aligned} \tag{2}$$

Therefore, we see that a state A is entangled if and only if there exists a positive linear map such that $\langle A, \phi \rangle < 0$.

For a positive linear map $\phi \in \mathcal{L}(M_m, M_n)$, define the matrix $W_\phi \in M_m \otimes M_n$ by

$$W_\phi = (\mathrm{id} \otimes \phi) P^+ = \sum_{ij=1}^{m} e_{ij} \otimes \phi(e_{ij}),$$

where $\{e_{ij} : i, j = 1, 2, \dots, m\}$ denotes the usual matrix units in M_m and P^+ denotes the projector onto the maximally entangled state $\sum_{i=1}^m e_i \otimes e_i$ in $\mathbb{C}^m \otimes \mathbb{C}^m$. Since we have the relation

$$\langle x \otimes y | W_\phi | x \otimes y \rangle = \mathrm{Tr}(W_\phi | x \otimes y \rangle \langle x \otimes y |) = y^* \phi(\bar{x}\bar{x}^*) y = \langle \bar{y}\bar{y}^* \otimes \bar{x}\bar{x}^*, \phi \rangle, \tag{3}$$

and $\mathbb{V}_1 = M_n^+ \otimes M_m^+$ is spanned by $\bar{y}\bar{y}^* \otimes \bar{x}\bar{x}^*$, we see that A_0 is an entangled state if and only if there is a hermitian matrix W with the property:

$$\mathrm{Tr}\,(W A_0) < 0, \qquad \mathrm{Tr}\,(W A) \ge 0 \text{ for each } A \in \mathbb{V}_1. \tag{4}$$

In this sense, the duality (2) is equivalent to the separability criterion given in [21] under the Jamiołkowski-Choi isomorphism [8] $\phi \mapsto W_\phi$. Note that the inverse $W \mapsto \phi$ is given by

$$\phi(X) = \sum_{i,j=1}^{m} x_{ij} W_{ij},$$

for $X = [x_{ij}] \in M_m$ and $W = \sum_{i,j} e_{ij} \otimes W_{ij} \in M_m \otimes M_n$.

A Hermitian matrix W is said to be an entanglement witness if there is an entanglement A_0 with the property (4). Therefore, any entanglement witness is of the form W_ϕ for a positive map ϕ which is not completely positive. In the language of the duality (2), we see that every positive map ϕ detects an entangled state A in the sense of $\langle A, \phi \rangle < 0$ whenever ϕ is not completely positive. Since our presentation heavily depends on the duality (2), we will call sometimes a positive map itself an entanglement witness if it detect entanglement. After Terhal [40] introduced the terminology of entanglement witness, Lewenstein, Kraus, Cirac and Horodecki [28] studied the notion of optimal entanglement witness which detects a maximal set of entanglement, and addressed [29] a fundamental question to find a minimal set of witnesses to detect all entanglement.

In this note, we consider the entanglement witnesses arising from positive linear maps which generate exposed extremal rays of the cone $\mathbb{P}_1$. A positive linear map will be said to be just *exposed* if it generates an exposed ray of the cone $\mathbb{P}_1$. Note that an exposed positive map automatically generates an extremal ray. We denote by $\mathbb{W}$ the set of all those witnesses arising from exposed positive maps. We show that any entanglement can be detected by a witness in the family $\mathbb{W}$. Furthermore, different witnesses in $\mathbb{W}$ detect different sets of entanglement. Therefore, the family $\mathbb{W}$ provides a minimal set of witnesses in a sense. It should be noted that any dense subset of $\mathbb{W}$ also detects all entanglement. Recently, there are researches [18], [22] on the witnesses detecting the common set of entangled states.

In 1980, Choi [10] observed that if $A \in (M_n \otimes M_m)^+$ belongs to $M_n^+ \otimes M_m^+$ then the partial transpose A^τ of A is again positive semi-definite. An equivalent formulation was given in [32]. This gives us a simple necessary condition for separability, and called the PPT criterion for separability. We denote by $\mathbb{T}$ the convex cone of all positive semi-definite matrices in $(M_n \otimes M_m)^+$ whose partial transposes are also positive semi-definite. The PPT criterion tells us the relation $\mathbb{V}_1 \subset \mathbb{T}$. When $m = 2$, Woronowicz [41] show that $\mathbb{V}_1 = \mathbb{T}$ if and only if $n \le 3$, and exhibited an explicit example in $\mathbb{T} \setminus \mathbb{V}_1$ for the case of $m = 2$ and $n = 4$. This kind of example is called a

PPT entangled state if it is normalized. The first example of PPT entangled state in the case of $m = n = 3$ was given in [10].

The most important examples of positive linear maps come from elementary operators of the forms

$$\phi_V : X \mapsto V^* X V,$$

for $m \times n$ matrices V. The convex sum of these maps are said to be a completely positive linear map. After the notion of completely positive linear maps arises [34] in the context of the representation theory of C^*-algebras in the fifties, they played key roles in several research areas of the theory of operator algebras, notably in the characterization of nuclear C^*-algebras during the seventies.

The transpose map which assigns the transpose X^{t} to a given matrix X is a typical example of a positive map which is not completely positive. The convex sum of the following maps

$$\phi^V : X \mapsto V^* X^{\mathrm{t}} V$$

with $m \times n$ matrices V is said to be completely copositive, and the convex sum of a completely positive map and a completely copositive map is called a decomposable positive linear map. We denote by $\mathbb{D}$ the convex cone of all decomposable positive linear maps. The first example of an indecomposable positive linear map was given by Choi [9], in the case of $m = n = 3$. It was also shown by Woronowicz [41] that every positive linear map from M_2 into M_n is decomposable if and only if $n \le 3$ by the duality. The first explicit example of an indecomposable positive linear map from M_2 into M_4 was given in terms of non-jordanian type in [42]. See also [39] for more such examples.

It was also shown in [17] that the cones $\mathbb{T}$ and $\mathbb{D}$ are dual each other in the same sense as in (2). Therefore, in order to detect PPT entangled states lying in $\mathbb{T} \setminus \mathbb{V}_1$, we need indecomposable positive linear maps in $\mathbb{P}_1 \setminus \mathbb{D}$. This explains the importance of indecomposable positive linear maps as detectors of PPT entangled states. We show that witness arising from an exposed indecomposable positive map detects a set of PPT entangled states which has a nonempty relative interior in the cone $\mathbb{T}$.

It was shown in [44] and [30] that the map ϕ_V is exposed for each matrix V. It is now clear that the maps ϕ_V and ϕ^V exhaust all decomposable positive linear maps which generate exposed rays of $\mathbb{P}_1$. In spite of the importance of exposed indecomposable positive maps, very few such maps seem to be known to the specialists. For example, Woronowicz [43] kindly showed the authors that the example of a non-extendible positive map from M_2 into M_4 in [42] is actually exposed. To the best knowledge of the authors, there is no known example of such a map between M_3. We provide a one parameter family of such maps from M_3 into M_3, by showing that some maps in [5] are exposed indecomposable positive linear maps.

In the next section, we consider the notion of duality for convex cones in a general framework, and apply that to show the above mentioned result for witnesses arising from exposed positive linear maps. After we consider the witnesses arising from exposed indecomposable positive linear maps in Section 3, we provide the above mentioned examples in the last section.

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2. Convex cones and their duals

Let X and Y be finite dimensional normed spaces, which are dual each other with respect to a bilinear pairing $\langle\ ,\ \rangle$. For a subset C of X, we define the *dual cone* C° by

$$C^\circ = \{y \in Y : \langle x, y\rangle \geq 0 \text{ for each } x \in C\},$$

and the dual cone $D^\circ \subset X$ similarly for a subset D of Y. We assume that the pairing is non-degenerate on the convex cone C of X in the following sense:

$$x \in C,\ \langle x, y\rangle = 0 \text{ for each } y \in C^\circ \implies x = 0.$$

For a face F of a closed convex cone C of X, we define the subset F' of C° by

$$F' = \{y \in C^\circ : \langle x, y\rangle = 0 \text{ for each } x \in F\}.$$

It is then clear that F' is a closed face of C°. It is also clear that $F \subset F''$ for any face F of C. Note that a closed face F of a closed convex cone C is exposed with respect to the pairing $\langle\ ,\ \rangle$ if and only if the equality $F = F''$ holds. We say that a point x in a convex cone C is *exposed* if it generates an exposed ray of C. It is clear that an exposed point of a cone generates an extremal ray.

It was shown in [17] that if F is a maximal face of C° then there is an extremal ray L of C such that $F = L'$. Especially, every maximal face is exposed. Conversely, if L is an exposed face of C which is minimal among nonzero exposed faces, then L' is a maximal face of C°. Especially, if L is an exposed ray of C then L' is a maximal face of C°.

Let C be a closed convex cone of X. If $y \in Y \setminus C^\circ$ then there exists $x \in C$ such that $\langle x, y\rangle < 0$ by the definition of the dual cone. In this case, we say that $x \in C$ *detects* the element $y \in Y \setminus C^\circ$. Since every element of the cone C is the convex sum of points of C generating extremal rays, it is apparent to see that $y \in C^\circ$ if and only if $\langle x, y\rangle \geq 0$ for every $x \in C$ generating an extremal ray. Since every extremal ray is the limit of exposed rays by Straszewicz's Theorem (see Theorem 18.6 of [33]), we also see that $y \in C^\circ$ if and only if $\langle x, y\rangle \geq 0$ for every exposed point $x \in C$. In other words, every element of $y \in Y \setminus C^\circ$ is detected by an exposed point of C. We state this as follows:

Proposition 2.1. *Let C be a closed convex cone of X. For $y \in Y$, the following are equivalent:*

(i) $y \notin C^\circ$.
(ii) *There exists an exposed point $x \in C$ such that $\langle x, y\rangle < 0$.*

By Proposition 2.1, we see that a positive semi-definite matrix $A \in M_n \otimes M_m$ is entangled if and only if there exists an exposed positive linear map such that $\langle A, \phi\rangle < 0$. In this sense, we see that every entanglement is detected by an exposed positive linear map.

We denote by E_ϕ the set of all entanglement detected by $\phi \in \mathbb{P}_1$. Let $\phi, \psi \in \mathbb{W}$. If E_ϕ and E_ψ coincide then the maximal faces ${L_\phi}'$ and ${L_\psi}'$ also coincide, where L_ϕ denotes the ray generated by ϕ. Note that L_ϕ itself is an exposed face by definition. Therefore, we have

$$L_\phi = {L_\phi}'' = {L_\psi}'' = L_\psi.$$

This means that ϕ is a scalar multiple of ψ. Therefore, we conclude that if $\phi, \psi \in \mathbb{W}$ are different with the exception of scalar multiplication then they detect different sets of entanglement.

We know all extremal rays of the cone $\mathbb{V}_1$, by the definition of separability. They consist of all rank one projectors onto product vectors in $\mathbb{C}^n \otimes \mathbb{C}^m$. This enables us to find all maximal faces of the cone $\mathbb{P}_1$, as was done in [25]. We will explain that for the later use.

For a projector $(\bar{y} \otimes x)(\bar{y} \otimes x)^* \in \mathbb{V}_1$ onto the product vector $\bar{y} \otimes x \in \mathbb{C}^n \otimes \mathbb{C}^m$, we have

$$\begin{aligned}\langle(\bar{y} \otimes x)(\bar{y} \otimes x)^*, \phi\rangle &= \langle \bar{y}\bar{y}^* \otimes xx^*, \phi\rangle \\ &= \operatorname{Tr}(\phi(xx^*)yy^*) = (\phi(xx^*)y \mid y)_{\mathbb{C}_n},\end{aligned}$$

where the last expression denotes the inner product which is linear in the first variable and conjugate linear in the second variable. Therefore, if L is an extremal ray of the cone $\mathbb{V}_1$ determined by a product vector $\bar{y} \otimes x$ then the dual face L' is the set of all positive linear maps satisfying the condition

$$(\phi(xx^*)y \,|\, y) = 0.$$

We denote by P_ϕ the set of all product vectors $\bar{y} \otimes x$ satisfying the above relation. Therefore P_ϕ determines the dual face L'_ϕ of $\mathbb{V}_1$ for the ray L_ϕ generated by the positive linear map ϕ.

3. PPT entanglement detected by exposed indecomposable maps

In this section, we compare the boundary structures of the two cones $\mathbb{V}_1$ and $\mathbb{T}$, to distinguish the role of entanglement witnesses arising from exposed indecomposable positive maps among all those from exposed positive maps. To begin with, we recall that every face of the cone $\mathbb{D}$ is determined [27] by a pair (D, E) of subspaces of the space $M_{m\times n}$ of all $m \times n$ matrices. More precisely, every face of the cone $\mathbb{D}$ is of the form

$$\sigma(D, E) = \left\{ \textstyle\sum_i \phi_{V_i} + \sum_j \phi^{W_j} : V_i \in D,\ W_j \in E \right\}$$

for a pair (D, E). It should be noted that not every pair gives rise to a face of the cone $\mathbb{D}$. Using the duality between two cones $\mathbb{D}$ and $\mathbb{T}$, the authors [19] showed that every face of $\mathbb{T}$ is of the form

$$\tau(D, E) = \{A \in \mathbb{T} : \mathcal{R}A \subset D,\ \mathcal{R}A^\tau \subset E\},$$

which is nothing but the dual face of the face $\sigma(D^\perp, E^\perp)$ of the cone $\mathbb{D}$, where $\mathcal{R}A$ denotes the range space of A with the identification of the space $M_{m\times n}$ with $\mathbb{C}^n \otimes \mathbb{C}^m$, under which a product vector $\bar{y} \otimes x$ corresponds to a rank one matrix xy^*. Especially, every face of $\mathbb{T}$ is exposed. Note [4] that there is an unexposed face of $\mathbb{D}$, even in the simplest case $m = n = 2$.

Because the cone $\mathbb{V}_1$ is a subcone of $\mathbb{T}$, the faces of $\mathbb{V}_1$ are naturally divided in two categories as was studied in [6]. A face of F of $\mathbb{V}_1$ is said to be induced by the face $\tau(D, E)$ of $\mathbb{T}$ if it is of the form

$$F = \mathbb{V}_1 \cap \tau(D, E)$$

with the additional condition $\mathbb{V}_1 \cap \operatorname{int} \tau(D, E) \neq \emptyset$, where $\operatorname{int} C$ denotes the relative interior of the convex set C with respect to the affine manifold generated by C. It was also shown in [6] that a pair (D, E) gives rise to the face $\tau(D, E)$ of $\mathbb{T}$ which induces a face of $\mathbb{V}_1$ if and only if the pair (D, E) satisfies the range criterion, that is, there is a family of product vector $\{y_\iota \otimes x_\iota\}$ satisfying the relation

$$D = \operatorname{span}\{y_\iota \otimes x_\iota\}, \qquad E = \operatorname{span}\{y_\iota \otimes \bar{x}_\iota\}.$$

Now, we restrict our attention to maximal faces of the cone $\mathbb{V}_1$ which determine the whole boundary. We begin with the maximal face dual to a completely copositive map ϕ^V, which may be an optimal entanglement witness under the Jamiołkowski-Choi isomorphism. In this case, we have

$$
\begin{aligned}
\langle (\bar{y}\otimes x)(\bar{y}\otimes x)^*, \phi^V\rangle &= y^*\phi^V(xx^*)y \\
&= y^*V^*\bar{x}\bar{x}^*Vy \\
&= |\bar{x}^*Vy|^2 \\
&= |(V|\bar{x}y^*)|^2 = |(V|\bar{y}\otimes\bar{x})|^2
\end{aligned}
\tag{5}
$$

from the equation (3) and the identification between $\bar{y}\otimes x$ and xy^*. Therefore, we see that the set P_{ϕ^V} consists of the partial conjugates of all product vectors orthogonal to V. This relation (5) also tells us generally that for $\phi\in\mathbb{P}_1$ the partial conjugates of the product vectors in P_ϕ has a nontrivial orthogonal complement if and only if the double dual $L_\phi{}''$ contains a completely copositive map. By the same calculation, we also have

$$\langle (\bar{y}\otimes x)(\bar{y}\otimes x)^*, \phi_V\rangle = |(V\,|\,\bar{y}\otimes x)|^2,$$

and so the set P_{ϕ_V} consist of all product vectors orthogonal to V.

If nonzero V is not of rank one, then Lemma 2.3 of [30] tells us that the pair $(V^\perp, \{0\}^\perp)$ satisfies the range criterion. This is equivalent to say that both ϕ^V and ϕ_V generate rays in the cone of $\mathbb{D}$ which are exposed by separable states by [27]. If $V=xy^*$ is of rank one, then we see that $\phi_{\bar{x}y^*}=\phi^{xy^*}$ is both completely positive and completely copositive. In any cases, we conclude that if ϕ is an exposed decomposable positive map then both P_ϕ and the partial conjugates of P_ϕ do not span the whole space. Conversely, if both P_ϕ and the partial conjugates of P_ϕ do not span the whole space, then the double dual $L_\phi{}''$ contains a completely positive or a completely copositive map, and so ϕ is not exposed indecomposable. Therefore, we have shown that an exposed positive map ϕ is decomposable if and only if both P_ϕ and the partial conjugates of P_ϕ do not span the whole space.

We recall that the interior of the face $\tau(D,E)$ is given by

$$\operatorname{int}\tau(D,E)=\{A\in\mathbb{T}:\mathcal{R}A=D,\ \mathcal{R}A^\tau=E\}.$$

Especially, the interior of the $\mathbb{T}$ itself consists of all $A\in\mathbb{T}$ such that both A and A^τ have the full ranges. Therefore, for a positive map ϕ, we see that the dual face $L_\phi{}'$ lies in the interior of $\mathbb{T}$ if and only if both P_ϕ and the partial conjugates of P_ϕ span the whole space. We summarize as follows:

Proposition 3.1. *Let ϕ be a positive linear map. Then the dual face $L_\phi{}'$ lies in the interior of the cone $\mathbb{T}$ if and only if both P_ϕ and the partial conjugates of P_ϕ span the whole space. If ϕ is exposed then this is the case if and only if ϕ is indecomposable.*

For $\phi \in \mathbb{P}_1$, consider the set $E^{\mathbb{T}}_\phi$ of all entangled states with positive partial transposes which are detected by the positive map ϕ:

$$E^{\mathbb{T}}_\phi = \{A \in \mathbb{T} : \langle A, \phi \rangle < 0\}.$$

Note that $E^{\mathbb{T}}_\phi$ is nonempty if and only if ϕ is indecomposable. Let A be an interior point of the cone $\mathbb{T}$, which contains the identity matrix I as a typical interior point. If we take a line segment from I to the boundary point of $\mathbb{T}$ through A, then any point in the interior of this line segment is an interior point of $\mathbb{T}$. This shows the implication (i) $\Longrightarrow$ (ii) of the followings:

Theorem 3.2. *For a positive linear map ϕ, the followings are equivalent:*

(i) *Both product vectors in P_ϕ and their partial conjugates span the whole space.*
(ii) *the set $E^{\mathbb{T}}_\phi$ has contains an entangled state A such that both A and A^τ have trivial kernels.*
(iii) *The set $E^{\mathbb{T}}_\phi$ has a nonempty relative interior in $\mathbb{T}$.*

If ϕ is an exposed indecomposable positive map then the above these conditions hold.

Proof. It remains to prove the implication (ii) $\Longrightarrow$ (i). Suppose that both $A \in E^{\mathbb{T}}_\phi$ and A^τ have the full ranges, and consider the line segment between A and the identity matrix I in $M_n \otimes M_m$. Since $\langle A, \phi \rangle < 0$ and $\langle I, \phi \rangle > 0$, there is A_0 on the line segment such that $\langle A_0, \phi \rangle = 0$. Denote by D and E the orthogonal complements of the product vectors in P_ϕ and the partial conjugates of product vectors in P_ϕ, respectively. Then we see that A_0 belongs to the face $\sigma(D, E)'$ of $\mathbb{T}$. Since A_0 is an interior point of $\mathbb{T}$, we conclude that both D and E are zeroes. □

Recall the generalized Choi maps between M_3 defined by

$$\Phi[a,b,c](X) = \begin{pmatrix} ax_{11}+bx_{22}+cx_{33} & -x_{12} & -x_{13} \\ -x_{21} & cx_{11}+ax_{22}+bx_{33} & -x_{23} \\ -x_{31} & -x_{32} & bx_{11}+cx_{22}+ax_{33} \end{pmatrix} \tag{6}$$

for $X = [x_{ij}] \in M_3$, as was introduced in [5], where a, b and c are nonnegative numbers. Note that the set $P_{\Phi[1,0,1]}$ for the Choi map $\Phi[1,0,1]$ spans 7-dimensional subspace and their partial conjugates span the whole space $\mathbb{C}^3 \otimes \mathbb{C}^3$, as was shown in [6]. This means that the set $E^{\mathbb{T}}_{\Phi[1,0,1]}$ of PPT entangled states detected by the Choi map $\Phi[1,0,1]$ lies on the boundary of the cone $\mathbb{T}$. Even if it happens that the dual face F of the Choi map in $\mathbb{V}_1$ is on the boundary, it should be noted that the smallest face F_1 of $\mathbb{T}$ containing F is much bigger than F itself, as was shown in [6]. The face F is a face of the convex cone $\mathbb{V}_1 \cap F_1$, which is a part of the boundary of $\mathbb{V}_1 \cap F_1$.

Especially, we see that the Choi map is not exposed as was already noticed in [17] and [25], even though it generates an extremal ray [11]. In the next section, we will show that $\Phi[a, b, c]$ is an exposed positive linear map whenever the following conditions

$$0 < a < 1, \qquad a + b + c = 2, \qquad bc = (1 - a)^2 \tag{7}$$

hold, which gives us a one parameter family of exposed positive maps.

4. EXPOSED INDECOMPOSABLE POSITIVE MAPS

In spite of their usefulness, the whole structures of the convex cone $\mathbb{P}_1$ is far from being completely understood even for the cases when m and n are small numbers. For the case of $m = m = 2$, all extreme points of the convex set consisting of unital positive linear maps were found by Størmer [35] in 1963. Furthermore, all faces of the cone $\mathbb{P}_1$ were characterized in [4] in terms of certain pairs of subspaces of M_2. See also [26] for the faces of the convex set of all unital positive linear maps in M_2.

The first example of a map of the type (6) was given by Choi [7], who showed that the map $\Phi[1, 2, 2]$ is a 2-positive linear map which is not completely positive. This is the first example to distinguish n-positivity for different n's. The first examples of indecomposable positive linear maps by Choi mentioned in Introduction are $\Phi[1, 0, \mu]$ for $\mu \geq 1$. See also [36] for another method to prove that. The map $\Phi[1, 0, 1]$, which is usually called the Choi map, was shown [11] to generate an extremal ray of the cone $\mathbb{P}_1$. Furthermore, it turns out [38] that this map $\Phi[1, 0, 1]$ is an atom, that is, it is not the sum of a 2-positive map and a 2-copositive map.

There had been very few examples of indecomposable positive linear maps in the literature until it was shown in [5] that $\Phi[a, b, c]$ is positive if and only if

$$a + b + c \geq 2, \qquad 0 \leq a \leq 1 \implies bc \geq (1 - a)^2,$$

and decomposable if and only if

$$0 \leq a \leq 2 \implies bc \geq \left(\frac{2 - a}{2}\right)^2.$$

It was also shown that $\Phi[a, b, c]$ is completely positive if and only if $a \geq 2$ and it is completely copositive if and only if $bc \geq 1$. For more extensive examples of indecomposable positive linear maps, we refer to [12]. We note that there are another variant of the Choi map as was considered in [24]. Some of them, parameterized by three real variables, were shown [31] to be extremal. See also [3] and [13] for another variations of the Choi map.

The most interesting cases arise when

$$0 \le a \le 1, \qquad a+b+c=2, \qquad bc=(1-a)^2,$$

as was analyzed in [15]. See also [23] and [14] for the different approach and generalization of these maps. Note that the maps $\Phi[1,0,1]$ and $\Phi[1,1,0]$ reproduce the Choi map and its dual, respectively, in the case of $a=1$. On the other hand, if $a=0$ then the map $\Phi[0,1,1]$ is completely copositive. Answering a question raised in [15], the authors [20] have shown that the map $\Phi[a,b,c]$ with the condition (7) gives rise to an indecomposable optimal witness. After the authors circulated and posted the paper [20], Chruściński [16] kindly sent the authors their proof which is independent from the authors.

In this section, we show that the map $\Phi[a,b,c]$ generates an exposed ray of the cone $\mathbb{P}_1$ under the condition (7). For those maps, the authors [20] could find product vectors in $P_{\Phi[a,b,c]}$ both of whom and whose partial conjugates span the whole space $\mathbb{C}^3 \otimes \mathbb{C}^3$. The point was to parameterize them by

$$a(t)=\frac{(1-t)^2}{1-t+t^2}, \quad b(t)=\frac{t^2}{1-t+t^2}, \quad c(t)=\frac{1}{1-t+t^2}.$$

Note that

$$0 \le a(t) \le 1, \quad a(t)+b(t)+c(t)=2, \quad b(t)c(t)=(1-a(t))^2.$$

We denote by $\Phi(t):=\Phi[a(t),b(t),c(t)]$. Note that $t=1$ corresponds the completely copositive reduction map $\Phi(1)=\Phi[0,1,1]$, and $t=0$ corresponds to the Choi map $\Phi(0)=\Phi[1,0,1]$. With this parameterization, we found product vectors $\bar{y}_i \otimes x_i$ in $P_{\Phi(t)}=L'_{\Phi(t)}$ as follows:

$$\begin{aligned} x_1&=(1,1,1)^{\rm t}, \quad x_2=(1,-1,1)^{\rm t}, \quad x_3=(1,i,-i)^{\rm t},\\ \bar{y}_1&=(1,1,1)^{\rm t}, \quad \bar{y}_2=(1,-1,1)^{\rm t}, \quad \bar{y}_3=(1,-i,i)^{\rm t},\end{aligned}$$

$$\begin{aligned} x_4&=(0,\sqrt{t},1)^{\rm t}, \quad x_5=(0,\sqrt{t},i)^{\rm t}, \quad \bar{y}_4=(0,\sqrt{t},t)^{\rm t}, \quad \bar{y}_5=(0,\sqrt{t},-ti)^{\rm t},\\ x_6&=(1,0,\sqrt{t})^{\rm t}, \quad x_7=(i,0,\sqrt{t})^{\rm t}, \quad \bar{y}_6=(t,0,\sqrt{t})^{\rm t}, \quad \bar{y}_7=(-ti,0,\sqrt{t})^{\rm t},\\ x_8&=(\sqrt{t},1,0)^{\rm t}, \quad x_9=(\sqrt{t},i,0)^{\rm t}, \quad \bar{y}_8=(\sqrt{t},t,0)^{\rm t}, \quad \bar{y}_9=(\sqrt{t},-ti,0)^{\rm t}.\end{aligned}$$

Note [20] that both these product vectors $\bar{y}_i \otimes x_i$ and their partial complex conjugates $\bar{y}_i \otimes \bar{x}_i$ span the whole space $\mathbb{C}^3 \otimes \mathbb{C}^3$ for any positive number t except for $t=1$.

Actually, all (unnormalized) product vectors $\bar{y} \otimes x$ in $P_{\Phi(t)}$ are of the form

$$(\bar{a}_1, \bar{a}_2, \bar{a}_3)^{\rm t} \otimes (a_1, a_2, a_3)^{\rm t} \text{ with } |a_1| = |a_2| = |a_3|, \tag{8}$$

$$(0, \bar{a}_2, t\bar{a}_3)^{\rm t} \otimes (0, a_2, a_3)^{\rm t} \text{ with } |a_2|^2 = t|a_3|^2, \tag{9}$$

$$(t\bar{a}_1, 0, \bar{a}_3)^{\rm t} \otimes (a_1, 0, a_3)^{\rm t} \text{ with } |a_3|^2 = t|a_1|^2, \tag{10}$$

$$(\bar{a}_1, t\bar{a}_2, 0)^{\rm t} \otimes (a_1, a_2, 0)^{\rm t} \text{ with } |a_1|^2 = t|a_2|^2. \tag{11}$$

Now, we consider the 9×9 matrices which represent the positive linear maps in the double dual of the map $\Phi(t)$ under the Jamiołkowski-Choi isomorphism. We note that the vectors in (9) determine the 4×4 submatrices taking the entries from the $5, 6, 8$ and 9-th rows and columns, which represent a positive linear maps from M_2 into M_2. We also recall that every positive maps between M_2 is decomposable, and the orthogonal matrices to product vectors in (9) and their partial conjugates are

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ -t & 0 \end{pmatrix},$$

respectively. Note that the map $\alpha\phi_A + q\phi^B$ is represented by the matrix

$$\alpha\phi_A + q\phi^B = \begin{pmatrix} \alpha & 0 & 0 & -\alpha - tq \\ 0 & q & 0 & 0 \\ 0 & 0 & t^2 q & 0 \\ -\alpha - tq & 0 & 0 & \alpha \end{pmatrix}.$$

Considering the other cases of (10) and (11) similarly, we see that the matrices representing positive maps in the double dual face ${L_{\Phi(t)}}''$ are of the form

$$\begin{pmatrix}
\alpha & \cdot & \cdot & \cdot & -\alpha - tp & \cdot & \cdot & \cdot & -\alpha - tr \\
\cdot & p & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & t^2 r & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & t^2 p & \cdot & \cdot & \cdot & \cdot & \cdot \\
-\alpha - tp & \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & -\alpha - tq \\
\cdot & \cdot & \cdot & \cdot & \cdot & q & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & r & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t^2 q & \cdot \\
-\alpha - tr & \cdot & \cdot & \cdot & -\alpha - tq & \cdot & \cdot & \cdot & \alpha
\end{pmatrix},$$

for nonnegative numbers α, p, q, r. Considering the product vectors (8), we have also the condition

$$3\alpha = (1 - t)^2 (p + q + r). \tag{12}$$

Note that the corresponding maps send $[x_{ij}]$ to

$$\begin{pmatrix} \alpha x_{11}+t^2px_{22}+rx_{33} & (-\alpha-tp)x_{12} & (-\alpha-tr)x_{13} \\ (-\alpha-tp)x_{21} & \alpha x_{22}+t^2qx_{33}+px_{11} & (-\alpha-tq)x_{23} \\ (-\alpha-tr)x_{31} & (-\alpha-tq)x_{32} & \alpha x_{33}+t^2rx_{11}+qx_{22} \end{pmatrix}$$

Therefore, this map is positive if and only if

$$\begin{pmatrix} \alpha x^2+4py^2+rz^2 & (-\alpha-2p)xy & (-\alpha-2r)xz \\ (-\alpha-2p)yx & \alpha y^2+4qz^2+px^2 & (-\alpha-2q)yz \\ (-\alpha-2r)zx & (-\alpha-2q)zy & \alpha z^2+4rx^2+qy^2 \end{pmatrix} \tag{13}$$

is positive semi-definite for every real numbers x, y, z.

Define S_1, S_2, S_3 by

$$S_1 = p+q+r, \quad S_2 = pq+qr+rp, \quad S_3 = pqr.$$

By considering the determinant of the matrix (13) with $(x, y, z) = (1, 1, 1)$, we have the following necessary condition for the positivity of the maps

$$D[\alpha,p,q,r] = S_3t^6+\alpha S_2t^4-2(S_3+\alpha S_2)t^3-\alpha S_2t^2-2\alpha(S_2+2\alpha S_1)t+S_3+\alpha S_2-4\alpha^3 \geq 0.$$

On the other hand, since $\alpha = \frac{1}{3}(1-t)^2S_1$ from (12), we can write

$$\begin{aligned} D[\alpha,p,q,r] =&(t^6-2t^3+1)T_1-(t^5-t^4-t^2+t)T_2 \\ =&(t-1)^2(t^2+t+1)(t^2+1)T_1+(t-1)^2(t^2+t+1)t(T_1-T_2), \end{aligned}$$

where

$$\begin{aligned} T_1 =& -\frac{4}{27}S_1^3+\frac{1}{3}S_1S_2+S_3, \\ T_2 =& -\frac{4}{9}S_1^3+\frac{4}{3}S_1S_2. \end{aligned}$$

Now, by the inequality of arithmetic and geometric means, we have

$$\begin{aligned} T_1 =&\frac{10}{9}pqr-\frac{4}{27}(p^3+q^3+r^3)-\frac{1}{9}(p^2q+p^2r+pq^2+pr^2+q^2r+qr^2) \\ \leq&\frac{4}{9}pqr-\frac{4}{27}(p^3+q^3+r^3). \end{aligned}$$

Denote T_3 by the right hand side of the above inequality, then

$$\begin{aligned} T_1-T_2 =&\frac{8}{27}(p^3+q^3+r^3)-\frac{1}{9}(p^2q+p^2r+pq^2+pr^2+q^2r+qr^2)-\frac{2}{9}pqr \\ \leq&\frac{8}{27}(p^3+q^3+r^3)-\frac{8}{9}pqr = -2T_3, \end{aligned}$$

Therefore, we have

$$\begin{aligned} D[\alpha,p,q,r] &\leq (t-1)^2(t^2+t+1)(t^2+1)T_3+(t-1)^2(t^2+t+1)t(-2T_3) \\ &= (t-1)^4(t^2+t+1)T_3 \leq 0, \end{aligned}$$

by the inequality of arithmetic and geometric means again.

By combining this inequality with the necessary condition for positivity, we conclude that $D[\alpha, p, q, r] = 0$. This is the case only when $p = q = r$ and $\alpha = (1-t)^2 p$ by the equality conditions for the inequalities between arithmetic and geometric means.

Consequently, the corresponding positive maps are the scalar multiples of the generalized Choi map

$$(1 - t + t^2)p\, \Phi[a(t), b(t), c(t)].$$

This means that ${L_{\Phi(t)}}'' = L_{\Phi(t)}$, that is, every $\Phi(t)$ generates an exposed extremal ray for any positive number t except for $t = 1$.

Note that an indecomposable positive linear map $\Phi[a, b, c]$ with $a + b + c = 2$ can be written as the following convex combination

$$(1 - \alpha)\Phi[2, 0, 0] + \alpha\Phi(t)$$

where $t = \sqrt{\frac{b}{c}}$ and $\alpha = c(1 - t + t^2)$. Since $\Phi[2, 0, 0]$ is completely positive linear map, the only optimal entanglement witnesses arising from these maps are witnesses arising from the exposed indecomposable positive maps $\Phi(t)$.

There are bunch of examples of optimal decomposable entanglement witness which is not extremal. See [2] for examples. However, all the known examples of indecomposable optimal entanglement witnesses arise from positive linear maps which generate extremal rays. Therefore, it would be interesting to know if there exists an example of an optimal indecomposable entanglement witness arising from a positive linear map which is not extremal. Finally, it would be also interesting to determine if every positive map satisfying the conditions in Theorem 3.2 is exposed or not.

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