Classification of bi-qutrit PPT entangled edge states by their ranks

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- Y.-H. Kiem, S.-H. Kye and J. Lee, Existence of product vectors and their partial conjugates in a pair of spaces, J. Math. Phys. **52** (2011), 122201 arXiv:1107.1023.
- S.-H. Kye and H. Osaka, Classification of bi-qutrit PPT entangled edge states by their ranks, preprint.

1. Separable states and PPT states

A positive semi-definite matrix in $M_m \otimes M_n = M_m(M_n)$ is said to be *separable* if it is the sum of rank one projectors onto product vectors in $\mathbb{C}^m \otimes \mathbb{C}^n$. A *product vector* is a simple tensor $\xi \otimes \eta \in \mathbb{C}^m \otimes \mathbb{C}^n$.

Therefore, if A is separable then it is of the form

$$A = \sum_i z_i z_i^* = \sum_i |z_i\rangle\langle z_i| \in M_m \otimes M_n$$

with product vectors $z_i = \xi_i \otimes \eta_i \in \mathbb{C}^m \otimes \mathbb{C}^n$.
We denote by $\mathbb{V}_1$ the convex cone consisting of all separable ones.

A positive semi-definite matrix in $M_m \otimes M_n$ is said to be *entangled* if it is not separable.

By the relation

$$(\xi \otimes \eta)(\xi \otimes \eta)^* = \xi\xi^* \otimes \eta\eta^*$$

we see that

$$\mathbb{V}_1 = M_n^+ \otimes M_m^+,$$

and so, entanglement consists of

$$(M_n \otimes M_m)^+ \setminus M_n^+ \otimes M_m^+.$$

Note that $(A \otimes B)^+ = A^+ \otimes B^+$ for *commutative* C^*-algebras A and B.

For $A \in M_m \otimes M_n$, define the *partial transpose* $A^\tau \in M_m \otimes M_n$ by

$$(X \otimes Y)^\tau = X^{\mathrm{t}} \otimes Y,$$

for $X \in M_n$ and $Y \in M_m$. Then

$$\left(\sum_{ij=1}^{m} e_{ij} \otimes x_{ij} \right)^\tau = \sum_{ij=1}^{m} e_{ji} \otimes x_{ij} = \sum_{ij=1}^{m} e_{ij} \otimes x_{ji}$$

So, partial transpose is nothing but the block-wise transpose.

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^\tau = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}^\tau = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For $\xi \in \mathbb{C}^m$ and $\eta \in \mathbb{C}^n$, we have

$$\begin{aligned}
[(\xi \otimes \eta)(\xi \otimes \eta)^*]^\tau &= [\xi\xi^* \otimes \eta\eta^*]^\tau \\
&= (\xi\xi^*)^{\mathrm{t}} \otimes \eta\eta^* \\
&= \bar{\xi}\bar{\xi}^* \otimes \eta\eta^* \\
&= (\bar{\xi} \otimes \eta)(\bar{\xi} \otimes \eta)^*
\end{aligned}$$

The partial transpose of a rank one projection onto a *product vector* is again a rank one projection, especially positive semi-definite.

Therefore, if $A \in M_n \otimes M_m$ is separable then its partial transpose A^τ is also positive semi-definite.

The product vector $\bar{\xi} \otimes \eta \in \mathbb{C}^m \otimes \mathbb{C}^n$ is called the *partial conjugate* of $\xi \otimes \eta$.

This gives us a simple necessary condition, called the PPT (positive partial transpose) criterion for separability, as was observed by Choi (1982) and Peres (1996). Denote by

$$\mathbb{T} = \{A \in (M_n \otimes M_m)^+ : A^\tau \in (M_n \otimes M_m)^+\}.$$

With this notation, the PPT criterion says that

$$\mathbb{V}_1 \subseteq \mathbb{T}.$$

The equality holds if and only if $(m, n) = (2, 2), (2, 3)$ or $(3, 2)$, by Woronowicz (1976) and Choi (1982).

Woronowicz (1976): When $m = 2$, $\mathbb{V}_1 = \mathbb{T}$ if and only if $n \leq 3$, and gave an explicit example in $\mathbb{T} \setminus \mathbb{V}_1$ for $(m, n) = (2, 4)$.
Choi (1982): gave an explicit example in $\mathbb{T} \setminus \mathbb{V}_1$ for $(m, n) = (3, 3)$.

These examples in $\mathbb{T} \setminus \mathbb{V}_1$ are said to be PPTES (positive partial transpose entangled state) if it is normalized.

Identify the vector space $\mathbb{C}^m \otimes \mathbb{C}^n$ with the space $M_{m\times n}$ of all $m \times n$ matrices. Every vector $z \in \mathbb{C}^m \otimes \mathbb{C}^n$ is uniquely expressed by

$$z = \sum_{i=1}^{m} e_i \otimes z_i \in \mathbb{C}^m \otimes \mathbb{C}^n, \qquad z_i = \sum_{k=1}^{n} z_{ik} e_k \in \mathbb{C}^n, \qquad i = 1, 2, \dots, m.$$

Then we get $z = [z_{ik}] \in M_{m\times n}$. This identification

$$\sum_{i=1}^{m} e_i \otimes \left(\sum_{k=1}^{n} z_{ik} e_k \right) \longleftrightarrow [z_{ik}] \tag{1}$$

is an inner product isomorphism from $\mathbb{C}^m \otimes \mathbb{C}^n$ onto $M_{m\times n}$.

$$\xi \otimes \bar{\eta} \leftrightarrow \xi\eta^* \in M_{m\times n},$$
$$e_i \otimes e_j \leftrightarrow e_{ij} \in M_{m\times n},$$
$$\sum_i e_i \otimes e_i \leftrightarrow \mathrm{Identity}.$$

A convex subset F of a convex set C is said to be a face of C if the following condition

$$x, y \in C,\ (1-t)x + ty \in F \text{ for some } t \in (0,1) \implies x, y \in F$$

holds.

Every face of the convex cone $\mathbb{T}$ is determined (Ha, K, 2005) by a pair (D, E) of subspaces of $M_{m\times n} = \mathbb{C}^m \otimes \mathbb{C}^n$:

$$\tau(D,E) = \{A \in \mathbb{T} : \mathcal{R}A \subset D,\ \mathcal{R}A^\tau \subset E\}.$$

Note also that

$$\operatorname{int}\tau(D,E) = \{A \in \mathbb{T} : \mathcal{R}A = D,\ \mathcal{R}A^\tau = E\}.$$

Finally, we note that a pair (D, E) gives rise to a face of $\mathbb{T}$ if and only if there is a nonzero $A \in \mathbb{T}$ such that $\mathcal{R}A = D$ and $\mathcal{R}A^\tau = E$.

$2 \otimes 2$ case: We can list up all faces of the cone $\mathbb{V}_1 = \mathbb{T}$:

	D	E	$D^\perp$	$E^\perp$		
$(1,1)$	SP	SP	SP	SP	$\xi\eta^*$	$\bar{\xi}\eta^*$
$(2,2)$	SP	SP	SP	SP	$\langle \xi\eta^*, \zeta\omega^* \rangle$	$\langle \bar{\xi}\eta^*, \bar{\zeta}\omega^* \rangle$
$(3,3)$	SP	SP	SP	SP	$(\xi\eta^*)^\perp$	$(\bar{\xi}\eta^*)^\perp$
$(3,3)$	SP	SP	CE	CE	$V^\perp$	$W^\perp$
$(3,4)$	SP	SP	CE	$\{0\}$	$V^\perp$	M_2
$(4,3)$	SP	SP	$\{0\}$	CE	M_2	$W^\perp$

with some rank two matrices V and W.
SP: spanned by product vectors
CE: completely entangled, that is, has no product vector.

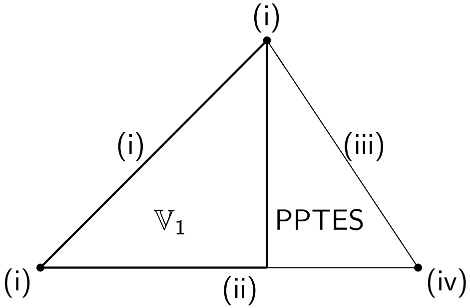
Comparing boundaries of the two cones $\mathbb{V}_1 \subset \mathbb{T}$, we have the following four possibilities:

(i) $\tau(D, E) \subset \mathbb{V}_1$

(ii) $\tau(D, E) \not\subset \mathbb{V}_1$ but $\mathrm{int}\,\tau(D, E) \cap \mathbb{V}_1 \neq \emptyset$

(iii) $\mathrm{int}\,\tau(D, E) \cap \mathbb{V}_1 = \emptyset$ but $\partial\tau(D, E) \cap \mathbb{V}_1 \neq \emptyset$

(iv) $\tau(D, E) \cap \mathbb{V}_1 = \emptyset$



Note that every point x of a convex set determines a unique face in which x is an interior point. This is the smallest face containing x. A PPT state A determines the smallest face by $\mathcal{R}A$ and $\mathcal{R}A^\tau$.

Recall that the *range criterion* (P. Horodecki, 1997) tells us that if a PPT state A is separable then there exists product vector $\xi_i \otimes \eta_i$ such that $\mathcal{R}A = \mathrm{span}\,\{\xi_i \otimes \eta_i\}$ and $\mathcal{R}A^\tau = \mathrm{span}\,\{\bar{\xi}_i \otimes \eta_i\}$. This condition holds if and only if either the case (i) or the case (ii) occurs.

A PPT state A is said to be an *edge state* (P. Horodecki, Lewenstein, Vidal and Cirac, 2000) if the case (iv) occurs; there exists no product vectors $\xi_i \otimes \eta_i$ such that $\mathcal{R}A = \mathrm{span}\,\{\xi_i \otimes \eta_i\}$ and $\mathcal{R}A^\tau = \mathrm{span}\,\{\bar{\xi}_i \otimes \eta_i\}$.

Choi's example (1982):

$$A = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & 2 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \frac{1}{2} & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \\ \cdot & 1 & \cdot & \frac{1}{2} & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & 1 & \cdot \\ \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \frac{1}{2} & \cdot \\ 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \end{pmatrix} \in \mathbb{T}$$

Note that the range is a 4-dimensional space spanned by

$$e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3$$

and

$$\sqrt{2} e_1 \otimes e_2 + \frac{1}{\sqrt{2}} e_2 \otimes e_1, \qquad \sqrt{2} e_2 \otimes e_3 + \frac{1}{\sqrt{2}} e_3 \otimes e_2, \qquad \sqrt{2} e_3 \otimes e_1 + \frac{1}{\sqrt{2}} e_1 \otimes e_3.$$

The corresponding ($e_i \otimes e_j \leftrightarrow e_{ij}$) 4-dimensional subspace D of $M_{3\times 3}$ is spanned by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \end{pmatrix}$$

which has no rank one matrices. So, A is entangled.

Note that

$$\begin{pmatrix} a & \sqrt{2}b & \frac{d}{\sqrt{2}} \\ \frac{b}{\sqrt{2}} & a & \sqrt{2}c \\ \sqrt{2}d & \frac{c}{\sqrt{2}} & a \end{pmatrix}$$

is never of rank one.

Størmer's example (1982)

$$A = \begin{pmatrix}
2\mu & \cdot & \cdot & \cdot & 2\mu & \cdot & \cdot & \cdot & 2\mu \\
\cdot & 4\mu^2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & & \cdot & \cdot \\
\\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
2\mu & \cdot & \cdot & \cdot & 2\mu & \cdot & \cdot & \cdot & 2\mu \\
\cdot & \cdot & \cdot & \cdot & \cdot & 4\mu^2 & \cdot & \cdot & \cdot \\
\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4\mu^2 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
2\mu & \cdot & \cdot & \cdot & 2\mu & \cdot & \cdot & \cdot & 2\mu
\end{pmatrix}$$

If we identity $\mathbb{C}^3 \otimes \mathbb{C}^3$ and $M_{3\times 3}$ in the usual way, then we see that

$$\mathcal{R}A = \{e_{11} - e_{22},\ e_{22} - e_{33}\}^{\perp}.$$

$$A^\tau = \begin{pmatrix} 2\mu & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 4\mu^2 & \cdot & 2\mu & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 2\mu & \cdot & \cdot \\ \\ \cdot & 2\mu & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 2\mu & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 4\mu^2 & \cdot & 2\mu & \cdot \\ \\ \cdot & \cdot & 2\mu & \cdot & \cdot & \cdot & 4\mu^2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2\mu & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2\mu \end{pmatrix},$$

and so, we see that

$$\mathcal{R}A = \{e_{11} - e_{22},\ e_{22} - e_{33}\}^\perp,$$
$$\mathcal{R}A^\tau = \{2\mu e_{12} - e_{21},\ 2\mu e_{23} - e_{32},\ 2\mu e_{31} - e_{13}\}^\perp.$$

The space $\mathcal{R}A$ has product vectors, but there is no product vector $\xi \otimes \eta$ such that

$$\xi \otimes \eta \in \mathcal{R}A, \qquad \bar{\xi} \otimes \eta \in \mathcal{R}A^{\tau}.$$

except for $\mu \neq \frac{1}{2}$. Therefore, A for $\mu \neq \frac{1}{2}$ violates the range criterion in an extreme way.

Note that $\dim \mathcal{R}A = 7, \ \dim \mathcal{R}A^{\tau} = 6$

2. Edge PPTES

An edge state A is said to be of *type* (p, q) if $\dim \mathcal{R}(A) = p$ and $\dim \mathcal{R}(A^\tau) = q$.

Question: Classify edge states by their types (p, q).

Lower bounds for p and q were given by P. Horodecki, Lewenstein, Vidal and Cirac (2000):

$$A \in \mathbb{T}, \ \dim \mathcal{R}(A) \le \max\{m, n\} \implies A \in \mathbb{V}_1,$$

So, we have

$$p, q > \max\{m, n\}.$$

What about upper bounds ?

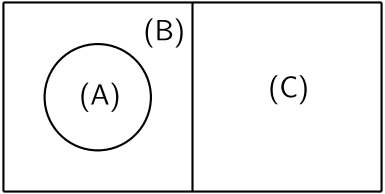
We consider the following three conditions for a pair (D, E) of subspaces of $\mathbb{C}^m \otimes \mathbb{C}^n$:

(A) There exists a PPT entangled edge state A such that $\mathcal{R}A = D$ and $\mathcal{R}A^\tau = E$.

(B) There exists NO nonzero product vector $\xi \otimes \eta \in \mathbb{C}^m \otimes \mathbb{C}^n$ with

$$\xi \otimes \eta \in D, \qquad \bar{\xi} \otimes \eta \in E.$$

(C) Negation of (B).



Find a condition for a quadruplet (k, ℓ, m, n) of natural numbers to satisfy the following:

(C) For any pair (D, E) of subspaces of $\mathbb{C}^m \otimes \mathbb{C}^n$ with $\dim D^\perp = k$, $\dim E^\perp = \ell$, there exists a nonzero product vector

$$\xi \otimes \eta = (\xi_1\eta_1, \xi_1\eta_2, \dots, \xi_1\eta_n\,;\, \xi_2\eta_1, \dots, \xi_2\eta_n,\,;\, \dots\,;\, \xi_1\eta_n, \dots \xi_m\eta_n)$$

in $\mathbb{C}^m \otimes \mathbb{C}^n$ with

$$\xi \otimes \eta \in D, \qquad \bar{\xi} \otimes \eta \in E.$$

- Every system of equations consisting of k linear equations w.r.t. $\xi_i\eta_j$ and ℓ linear equations w.r.t. $\bar{\xi}_i\eta_j$ has a nonzero solution.

If this condition (C) holds then there is no edge state of type $(mn - k, mn - \ell)$, which gives us upper bounds for range dimensions of an edge state A and its partial transpose A^τ.

Results with Y.-H. Kiem and J. Lee
(J. Math. Phys. **52** (2011), 122201 arXiv:1107.1023.):

Theorem

Let (k, ℓ, m, n) be a quadruplet of natural numbers with the relation $k, \ell \le m \times n$. If

$$(-\alpha + \beta)^k (\alpha + \beta)^\ell \neq 0 \qquad \textit{modulo} \quad \alpha^m, \beta^n,$$

in the polynomial ring $\mathbb{Z}[\alpha, \beta]$, then the above condition (C) holds.

Proof involves techniques from algebraic geometry.

- If $k + \ell < m + n - 2$ then the condition (C) holds:
  If $(mn - p) + (mn - q) < m + n - 2$ then there is no edge state of type (p, q):
  If there is an edge state of type (p, q) then we have

$$p + q \leq 2mn - m - n + 2.$$

- If $k + \ell > m + n - 2$ then the condition (C) does not hold: no information for the existence of edge states.
- What happen when $k + \ell = m + n - 2$, that is, $p + q = 2mn - m - n + 2$??

We expand the polynomial to write

$$(-\alpha+\beta)^k(\alpha+\beta)^\ell = \sum_{t=0}^{k+\ell} C_t^{k,\ell}\alpha^t\beta^{k+\ell-t}$$

with the coefficients

$$C_t^{k,\ell} = \sum_{r+s=t}(-1)^r\binom{k}{r}\binom{\ell}{s}.$$

If $k+\ell = m+n-2$ then we have

$$(-\alpha+\beta)^k(\alpha+\beta)^\ell = \cdots + C_{m-2}^{k,\ell}\alpha^{m-2}\beta^n + C_{m-1}^{k,\ell}\alpha^{m-1}\beta^{n-1} + C_m^{k,\ell}\alpha^m\beta^{n-2} + \cdots .$$

We see that the polynomial is zero modulo α^m and β^n if and only if

$$C_{m-1}^{k,\ell} = 0.$$

Therefore, if

$$\begin{cases} k + \ell = m + n - 2 \\ C^{k,\ell}_{m-1} \neq 0 \end{cases}$$

and $\dim D^\perp = k$, $\dim E^\perp = \ell$ then there exist product vector $x \otimes y \in D$ with $\bar{x} \otimes y \in E$,
which implies that there is no $m \otimes n$ edge state of type $(mn - k, mn - \ell)$.

Converse ??

If

$$\begin{cases} k+\ell = m+n-2 \\ C^{k,\ell}_{m-1} = 0 \end{cases}$$

then there exists (D, E) such that

• $\dim D^\perp = k$, $\dim E^\perp = \ell$

• there exist no nonzero product vector $x \otimes y \in D$ with $\bar{x} \otimes y \in E$.

Yes, for $m = 2$ or $(m, n) = (3, 3)$.

We do not know for general cases.

The above equation is known to the Krawtchouk polynomial, which plays an important role in coding theory.

If this equation is satisfied then there may exist an $m \otimes n$ edge state of type $(mn - k, mn - \ell)$.

List of solutions for some easy cases:

(i) When $m = 2$: $n = 2k$ and $\ell = k$ for $k = 1, 2, \dots$.

(ii) When $m = 3$: $n = r(r+2)$, $\quad k = \binom{r+1}{2} \quad$ and $\quad \ell = \binom{r+2}{2}$ for $r = 1, 2, \dots$.

(iii) When $m = n$: k and ℓ are odd.

(iv) When $k = \ell$: m and n are even.

For small numbers with $mn \le 10$, we have the solutions

$$(1,1) \text{ in } 2 \otimes 2, \qquad (2,2) \text{ in } 2 \otimes 4, \qquad (1,3) \text{ in } 3 \otimes 3,$$

$$\left\{ \begin{array}{l} \bar{x}_1 y_1 + \bar{x}_2 y_2 = 0 \\ x_1 y_2 - x_2 y_1 = 0 \end{array} \right. \qquad \left\{ \begin{array}{l} \bar{x}_1 y_1 + \bar{x}_2 y_2 + \bar{x}_3 y_3 = 0 \\ x_1 y_2 - x_2 y_1 = 0 \\ x_2 y_3 - x_3 y_2 = 0 \\ x_3 y_1 - x_1 y_3 = 0 \end{array} \right.$$

These examples do not give rise to examples of PPT edge states.

$2 \otimes 4$ case:

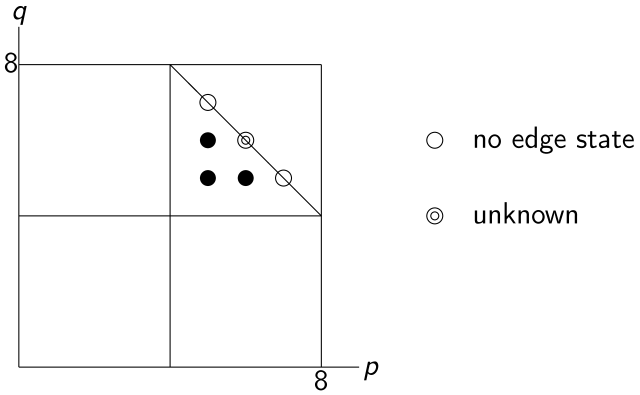
(i) $k + \ell < 2 + 4 - 2 = 4$; $p + q \leq 12$ gives an upper bound

(ii) When $k + \ell = 4$ $(p + q = 12)$, we have $\sum_{r+s=2-1} (-1)^r \binom{k}{r}\binom{\ell}{s} = 0$ if and only if $(k, \ell) = (2, 2)$. The case $(k, \ell) = (3, 1)$ is not the root of the equation. This means that there is no edge state of type $(5, 7)$. This special case was shown by Samsonowicz, Kuś and Lewenstein (2007).

Actually, all possible types are

$$(5,5), \qquad (5,6), \qquad (6,5), \qquad (6,6)$$

- Woronowicz (1976): edge state of $(5,5)$ type
- P. Horodecki (1997): parameterized examles of $(5,5)$ type
- Augusiak, J. Grabowski, M. Kuś and M. Lewenstein (2010): edge state of type $(5,6)$.
- It is not known that if there is a $2 \otimes 4$ edge state of type $(6,6)$.



$3 \otimes 3$ case:

(i) $k + \ell < 3 + 3 - 2 = 4$. $p + q \leq 14$ gives an upper bound

(ii) When $k + \ell = 4$ $(p + q = 14)$, we have $\sum_{r+s=3-1} (-1)^r \binom{k}{r} \binom{\ell}{s} = 0$ if and only if $(k, \ell) = (1, 3)$. The case $(k, \ell) = (2, 2)$ is not the root of the equation. This means that there is no edge state of type $(7, 7)$.

Actually, all possible types are

$$(4,4), \quad (5,5), \quad (5,6), \quad (5,7), \quad (6,6), \quad (5,8), \quad (6,7), \quad (6,8),$$

here we list up the cases $s \leq t$ by the symmetry.

- Choi (1982): The first example of PPTES is turned out to be an edge state of type $(4,4)$.
- Other examples of edge states of this type $(4,4)$ were constructed using orthogonal unextendible product bases (Bennett, . . . ,1999) and indecomposable positive linear maps (Ha, K, Park, 2003). The latter is generic, in the sense that generic 5-dim subspaces of $\mathbb{C}^3 \otimes \mathbb{C}^3$ have six product vectors. Choi's example is of second kind.
- Størmer (1982): an edge state of type $(6,7)$.
- Ha, K (2004): types $(5,6)$, $(5,7)$ and $(5,8)$
- Clasrisse (2006), Ha (2007): types $(5,5)$ and $(6,6)$
- type $(6,8)$??

3. Construction of edge states of type $(8, 6)$

(with H. Osaka)
We begin with the following 3×3 matrix

$$P[\theta] := \begin{pmatrix} e^{i\theta} + e^{-i\theta} & -e^{i\theta} & -e^{-i\theta} \\ -e^{-i\theta} & e^{i\theta} + e^{-i\theta} & -e^{i\theta} \\ -e^{i\theta} & -e^{-i\theta} & e^{i\theta} + e^{-i\theta} \end{pmatrix}$$

which has a kernel vector $(1, 1, 1)^{\mathrm{t}}$.
Note that

- $P[\theta]$ is positive semi-definite if and only if $\cos\theta \geq 0$ and $2\cos 2\theta \geq -1$ if and only if $-\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}$.
- If $-\frac{\pi}{3} < \theta < \frac{\pi}{3}$ then $P[\theta]$ is of rank two.
- If $\theta = -\frac{\pi}{3}$ or $\theta = \frac{\pi}{3}$ then $P[\theta]$ is of rank one.

$$A = \begin{pmatrix} e^{i\theta}+e^{-i\theta} & \cdot & \cdot & \cdot & -e^{i\theta} & \cdot & \cdot & \cdot & -e^{-i\theta} \\ \cdot & \frac{1}{b} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ -e^{-i\theta} & \cdot & \cdot & \cdot & e^{i\theta}+e^{-i\theta} & \cdot & \cdot & \cdot & -e^{i\theta} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{b} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{b} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot \\ -e^{i\theta} & \cdot & \cdot & \cdot & -e^{-i\theta} & \cdot & \cdot & \cdot & e^{i\theta}+e^{-i\theta} \end{pmatrix}$$

in $M_3 \otimes M_3$ with the conditions

$$b > 1, \qquad -\frac{\pi}{3} < \theta < \frac{\pi}{3}, \qquad \theta \neq 0,$$

where $\cdot$ denotes the zero.

The partial transpose A^τ of A is given by

$$\begin{pmatrix} a_\theta & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \frac{1}{b} & \cdot & -e^{-i\theta} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b & \cdot & \cdot & \cdot & -e^{i\theta} & \cdot & \cdot \\ \cdot & -e^{i\theta} & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & a_\theta & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{b} & \cdot & -e^{-i\theta} & \cdot \\ \cdot & \cdot & -e^{-i\theta} & \cdot & \cdot & \cdot & \frac{1}{b} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -e^{i\theta} & \cdot & b & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_\theta \end{pmatrix},$$

with $a_\theta = e^{i\theta} + e^{-i\theta} = 2\cos\theta$.
It is clear that A is of PPT, and we have

$$\mathrm{rank}\, A = 8, \qquad \mathrm{rank}\, A^\tau = 6.$$

It is straightforward to show that A is an edge state.

Consider

$$X = \begin{pmatrix} e^{i\theta}+e^{-i\theta} & \cdot & \cdot & \cdot & -e^{i\theta} & \cdot & \cdot & \cdot & -e^{-i\theta} \\ \cdot & \frac{1}{b} & \cdot & \alpha^* & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b & \cdot & \cdot & \cdot & \gamma & \cdot & \cdot \\ \cdot & \alpha & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ -e^{-i\theta} & \cdot & \cdot & \cdot & e^{i\theta}+e^{-i\theta} & \cdot & \cdot & \cdot & -e^{i\theta} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{b} & \cdot & \beta^* & \cdot \\ \cdot & \cdot & \gamma^* & \cdot & \cdot & \cdot & \frac{1}{b} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \beta & \cdot & b & \cdot \\ -e^{i\theta} & \cdot & \cdot & \cdot & -e^{-i\theta} & \cdot & \cdot & \cdot & e^{i\theta}+e^{-i\theta} \end{pmatrix},$$

whose partial transpose X^τ is given by

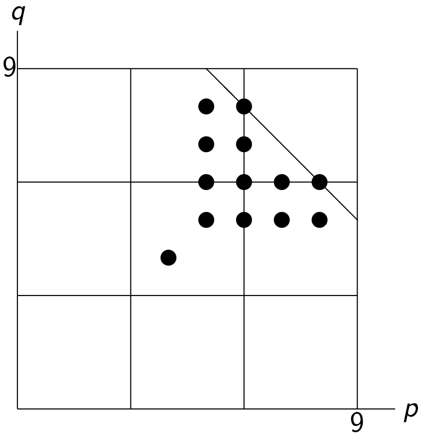
$$\begin{pmatrix}
a_\theta & \cdot & \cdot & \cdot & \alpha & \cdot & \cdot & \cdot & \gamma^* \\
\cdot & \frac{1}{b} & \cdot & -e^{-i\theta} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & b & \cdot & \cdot & \cdot & -e^{i\theta} & \cdot & \cdot \\
\cdot & -e^{i\theta} & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot \\
\alpha^* & \cdot & \cdot & \cdot & a_\theta & \cdot & \cdot & \cdot & \beta \\
\cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{b} & \cdot & -e^{-i\theta} & \cdot \\
\cdot & \cdot & -e^{-i\theta} & \cdot & \cdot & \cdot & \frac{1}{b} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -e^{i\theta} & \cdot & b & \cdot \\
\gamma & \cdot & \cdot & \cdot & \beta^* & \cdot & \cdot & \cdot & a_\theta
\end{pmatrix},$$

With suitable choice of α, β, γ, we may get edge states of types

$$(8,6), \qquad (7,6), \qquad (6,6), \qquad (5,6)$$

and

$$(8,5), \qquad (7,5), \qquad (6,5), \qquad (5,5)$$



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