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The structural physical approximations and optimal entanglement witnesses

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We introduce the notions of positive and copositive types for entanglement witnesses, depending on the distance to the positive part and copositive part. An entanglement witness W is of positive type if and only if its partial transpose W^Γ is of copositive type. We show that if the structural physical approximation (SPA) of W is separable, then W should be of copositive type, and the SPA of W^Γ is never separable unless W is of both positive and copositive types. This shows that the SPA conjecture is meaningful only for those of copositive type. We provide examples to show that the SPA conjecture fails even for the case of copositive types. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4754279]

I. INTRODUCTION

Quantum entanglement is the key resource for applications to quantum computation and quantum information theory (see Ref. 20). One of the most important topics in the theory of entanglement is how to distinguish entangled states from separable states. In the early 1980s, Choi[5] observed that if a positive ("positive" means "positive semi-definite") matrix in the tensor product belongs to tensor product of positive parts, then its partial transpose must be positive. Since the tensor product of positive parts is just the convex cone generated by separable states, this is equivalent to the positive partial transposed (PPT) criterion which was rediscovered later by Peres.[29] This criterion is already implicit in the Woronowicz' earlier work[40] who initiated the duality theory between positivity of linear maps and the separability of states. He constructed an example of PPT entangled state in order to show the existence of indecomposable positive linear maps.

It was Horodecki[17] who used the relation between positive maps and separable states to get the complete criterion for separability. The Woronowicz idea has been developed for general cases in Ref. 10 to get the duality theory for s-positive linear maps. For the case of $s = 1$, this is equivalent to the Horodecki criterion, through the Jamiołkowski-Choi (JC) isomorphism.[4,21] We note that the general cases give rise to the notion of Schmidt numbers and the relation to s-positivity of linear maps, which was obtained independently in Refs. 31 and 36. See also Refs. 32 and 33 for approaches to the duality theory which works for an arbitrary mapping cone. Horodecki's criterion tells us that we need positive maps in order to detect entanglement, which has been reformulated in terms of entanglement witnesses.[35] Under the JC isomorphism, an entanglement witness is just a positive linear map which is not completely positive. An entanglement witness which detects a maximal set of entanglement is said to be optimal, as was introduced in Ref. 26.

In spite of its importance, the whole structure of the convex cone of all positive linear maps is far from being completely understood, and there had even been very few known nontrivial examples of positive linear maps until the 1980s. To overcome this difficulty, the idea to consider the line segment from the trace map to a given positive map had been used in mathematical literature to distinguish various notions of positivity (see Refs. 37, 38, and 39). From the point of view of physics, positive maps which are not completely positive, such as the transpose maps, are non-physical operations. The main idea of structural physical approximation (SPA)[11,18,19] is to approximate positive maps by the nearest completely positive maps, which are physical operations, and so we can implement them in the real world. It was theoretically shown[22] and later demonstrated in practice[27,28] that the SPAs of the transpose map and the partial transpose map are experimentally feasible. On the mathematics side, SPAs correspond to the points at which a segment between the trace map and a given positive map crosses the border of the set of completely positive maps.

We recall that positive maps (respectively, completely positive maps) correspond to block-positive (respectively, positive) matrices in $M_m \otimes M_n$ under the JC isomorphism, where M_n denotes the $*$-algebra of all $n \times n$ matrices over the complex field, with the identity $\mathbb{1}_n$. We also note that the trace map corresponds to $\mathbb{1}_m \otimes \mathbb{1}_n$. For an entanglement witness $W \in M_m \otimes M_n$, we consider the line segment L_W from $\mathbb{1}_m \otimes \mathbb{1}_n$ to W. The SPA of W is the positive matrix on L_W nearest to W. It was conjectured in Ref. 22 that if W is an optimal entanglement witness, then its SPA is separable. Several authors[1,6–9,30] considered various classes of optimal entanglement witnesses to support the conjecture. Here, we provide a counterexample. Since non-decomposable optimal witnesses need not be optimal in the sense of Sec. IV of Ref. 26, we use the term PPTES witnesses.[15] Here, PPTES refers to PPT entangled states. PPTES optimality of a witness W means that W detects a maximal set (in the sense of inclusion) of PPT entangled states. This is equivalent to say that W is indecomposable and both W and W^Γ is optimal,[26] where W^Γ denotes the partial transpose of W. Let us recall that a special role is played by the product vectors $\phi \otimes \psi$ that satisfy $\langle \phi \otimes \psi | W | \phi \otimes \psi \rangle = 0$. If these vectors span $\mathbb{C}^m \otimes \mathbb{C}^n$, we say that W has the *spanning property*. We say that W is co-optimal (respectively, co-spanning) if W^Γ is optimal (respectively, spanning), bi-optimal if it is both optimal and co-optimal, and bi-spanning similarly.

In order to deal with the SPA conjecture in a systematic way, we introduce the notions of positive type and copositive type for entanglement witnesses in Sec. II, and show that if the SPA of an entanglement witness is separable, then the witness must be of copositive type. We also see that W is of positive type if and only if W^Γ is of copositive type. Because the optimality of PPTES witnesses is invariant under the operation of partial transpose, we conclude that only approximately half of optimal PPTES witnesses can satisfy the SPA conjecture. In Sec. III, we exhibit examples to show that the SPA conjecture fails even for the case of copositive type. We consider entanglement witnesses given in Ref. 16 to find examples of optimal PPTES witnesses of copositive type whose SPAs are not separable. The SPAs of our examples are exactly $3 \otimes 3$ PPT edge states[25] of type (6, 8). Very recently, after the authors posted this paper, Størmer[34] also examined the same class of entanglement witnesses[16] to find optimal indecomposable entanglement witnesses whose SPAs are not separable.

II. POSITIVE AND COPOSITIVE TYPE

From now on, we say that a block matrix is *copositive* if its partial transpose is positive. Copositive matrices correspond to completely copositive maps under the JC isomorphism. For a given block-positive matrix W, we consider the line segment L_W from $\mathbb{1}_n \otimes \mathbb{1}_m$ to W, and compare the distances to the nearest positive matrix and the nearest copositive matrix on L_W. We say that W is *of positive* (respectively, *copositive*) *type* if the distance to the positive (respectively, copositive) part is shorter than or equal to the other. We also say that a block-positive matrix is *of PPT type* if two distances coincide.

To be precise, we consider self-adjoint matrix $W_t = (1-t)/(mn)\mathbb{1}_m \otimes \mathbb{1}_n + tW$. When t_0 is the largest number in the interval [0, 1] for which W_{t_0} is positive, W_{t_0} is said to be the SPA of W. It is clear that the SPA of W is separable if and only if the following condition:

$$0 \le t \le 1,\ \ W_t \text{ is positive} \implies W_t \text{ is separable} \tag{1}$$

holds. It is also clear that the validity of the condition (1) is not changed when we replace $1/(mn)$ $\mathbb{1}_m \otimes \mathbb{1}_n$ in the definition of W_t by $\mathbb{1}_m \otimes \mathbb{1}_n$ alone due to the convexity. Therefore, we may redefine W_t as

$$W_t := (1-t)\mathbb{1}_m \otimes \mathbb{1}_n + tW$$

for every block-positive W, so far as we are concerned with the SPA conjecture.

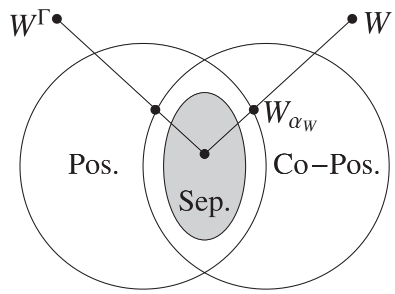

FIG. 1. W has the mirror image W^Γ. W is nearer to the copositive part if and only if W^Γ is nearer to the positive part.

Since separable states are of PPT, we see that if the SPA conjecture is true, then every optimal entanglement witness W must satisfy the following condition:

$$0 \le t \le 1,\ \ W_t \text{ is positive } \Longrightarrow\ W_t \text{ is of PPT},$$

which is weaker than (1), and so satisfies the following equivalent condition:

$$0 \le t \le 1,\ \ W_t \text{ is positive } \Longrightarrow\ W_t \text{ is copositive.} \tag{2}$$

We define two real numbers α_W and β_W in the interval [0, 1] by

$$\alpha_W := \sup\{t \in [0, 1] : W_t \text{ is positive}\},$$
$$\beta_W := \sup\{t \in [0, 1] : W_t \text{ is copositive}\},$$

for an arbitrary block-positive W. We see that both α_W and β_W are nonzero, since $\mathbb{I}_m \otimes \mathbb{I}_n$ is an interior point of the convex cone generated by all separable states. We also note that the number $1 - \alpha_W$ (respectively, $1 - \beta_W$) plays a role of the distance from W to the positive (respectively, copositive) part through the line segment L_W. It is clear that W satisfies the condition (2) if and only if the inequality $\alpha_W \le \beta_W$ holds. Since W_t is positive if and only if W_t^Γ is copositive, we also have $\alpha_W = \beta_{W^\Gamma}$ and $\alpha_{W^\Gamma} = \beta_W$. Therefore, we see that $\alpha_W < \beta_W$ if and only if $\alpha_{W^\Gamma} > \beta_{W^\Gamma}$. In other words, W satisfies the condition (2) if and only if W^Γ does not satisfy (2), whenever $\alpha_W \ne \beta_W$.

We see that a block-positive matrix W is of positive type if and only if $\alpha_W \ge \beta_W$, and of copositive type if and only if $\alpha_W \le \beta_W$ if and only if the condition (2) holds. We also see that W is of PPT type if and only if $\alpha_W = \beta_W$. The above discussion tells us that W is of positive type if and only if W^Γ is of copositive type, and the SPA conjecture can only hold for witnesses of copositive type. Especially, if the SPA of W is separable, then W is of copositive type and the SPA of W^Γ is never separable unless W is of PPT type (see Fig. 1).

III. EXAMPLES OF PPT TYPE

In this section, we exhibit examples of indecomposable entanglement witnesses of PPT type with the bi-spanning property in the sense of Ref. 15, whose SPAs are not separable.

For non-negative real numbers a, b, c, and $-\pi \le \theta \le \pi$, we consider the following self-adjoint block matrix in $M_3 \otimes M_3$:

$$W[a,b,c;\theta] = \begin{pmatrix} a & \cdot & \cdot & \cdot & -e^{i\theta} & \cdot & \cdot & \cdot & -e^{-i\theta} \\ \cdot & c & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ -e^{-i\theta} & \cdot & \cdot & \cdot & a & \cdot & \cdot & \cdot & -e^{i\theta} \\ \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & b & \cdot \\ -e^{i\theta} & \cdot & \cdot & \cdot & -e^{-i\theta} & \cdot & \cdot & \cdot & a \end{pmatrix}.$$

We also put $p_\theta = \max\{q_{(\theta-\frac{2}{3}\pi)}, q_\theta, q_{(\theta+\frac{2}{3}\pi)}\}$, where $q_\theta = e^{i\theta} + e^{-i\theta}$. We note that $1 \le p_\theta \le 2$. We also see that $p_\theta = 2$ if and only if $\theta = 0, \pm 2\pi/3$, and $p_\theta = 1$ if and only if $\theta = \pm\pi/3, \pm\pi$. It is easy to see that $W[a, b, c; \theta]$ is of PPT if and only if

$$a \ge p_\theta, \qquad bc \ge 1. \tag{3}$$

If $1 < p_\theta < 2$, the cases of $a = p_\theta$ and $bc = 1$ give rise to new kinds of examples[25] of PPT edge states of type (8, 6). If $\theta = 0$, then the notion of PPT coincides[25] with separability. On the other hand, the authors[14] explored the boundary structures between the separability and the inseparability for PPT states when $\theta = \pi$. We recall that the case $\theta = 0$ had been considered in Ref. 3.

It turns out[16] that $W[a, b, c; \theta]$ is block-positive if and only if the condition

$$a + b + c \ge p_\theta, \qquad a \le 1 \Longrightarrow bc \ge (1-a)^2 \tag{4}$$

holds. We refer to Ref. 16 for the pictures of the three-dimensional convex bodies determined by (3) and (4) for fixed θ. From now on, we concentrate on the case $a < p_\theta$ and $bc < 1$, for which $W[a, b, c; \theta]$ is neither positive nor copositive. We have

$$W_t[a, b, c; \theta] = t\, W\left[\frac{a_t}{t}, \frac{b_t}{t}, \frac{c_t}{t}; \theta\right], \qquad 0 < t \le 1,$$

with $a_t = 1 - t + ta$, $b_t = 1 - t + tb$, and $c_t = 1 - t + tc$. We see that $W_t[a, b, c; \theta]$ is positive if and only if $a_t \ge tp_\theta$ if and only if $t \le \alpha_W = 1/(p_\theta + 1 - a)$, and $W_t[a, b, c; \theta]$ is copositive if and only if $b_t c_t \ge t^2$ if and only if

$$F(t) := (b + c - bc)t^2 - (b + c - 2)t - 1 \le 0.$$

Therefore, we see that $W[a, b, c; \theta]$ is of copositive type if and only if $0 \le t \le \alpha_W$ implies $F(t) \le 0$. Since $F(0) = -1 < 0$ and $F(1) = 1 - bc > 0$, we see that this happens if and only if $F(\alpha_W) \le 0$ if and only if

$$(p_\theta - a + b)(p_\theta - a + c) \ge 1.$$

Analogously, $W[a, b, c; \theta]$ is of positive type if and only if the reverse inequality holds, and of PPT type if and only if the equality holds.

On the other hand, we note that the SPA of $W[a, b, c; \theta]$ is given by

$$W_{\alpha_W}[a, b, c; \theta] = \frac{1}{p_\theta + 1 - a} W[p_\theta, p_\theta - a + b, p_\theta - a + c; \theta].$$

We note[25] that $W_{\alpha_W}[a, b, c; \theta]$ is not separable when

$$(p_\theta - a + b)(p_\theta - a + c) = 1, \qquad 1 < p_\theta < 2, \tag{5}$$

that is when $W[a, b, c; \theta]$ is of PPT type. We concentrate on the following two subcases:

$$2 - p_\theta \le a < 1, \quad a + b + c = p_\theta, \quad bc = (1-a)^2, \tag{6}$$

$$1 \le a < p_\theta, \quad a + b + c = p_\theta, \quad bc = 0. \tag{7}$$

We note[16] that $W[a, b, c; \theta]$ is indecomposable in the above cases; has the bi-spanning property in the case (6) and has the co-spanning property in the case (7). See Refs. 12 and 13 for the case of $\theta = 0$. We also note that condition (5) together with (6) is translated into

$$3a^2 - 2(2p_\theta + 1)a + 2p_\theta^2 = 0, \quad 2 - p_\theta \le a < 1,$$

and it is easy to see that

(i) there exists a, b, and c satisfying (5) and (6) if and only if $4/3 \le p_\theta < 1 + 1/\sqrt{2}$.

In this case, there are two solutions as is seen in Fig. 2 when $2 - p_\theta < a < 1$. If $a = 2 - p_\theta$, then we have only one solution with $b = c = p_\theta - 1$. For the case (7), we note that the condition

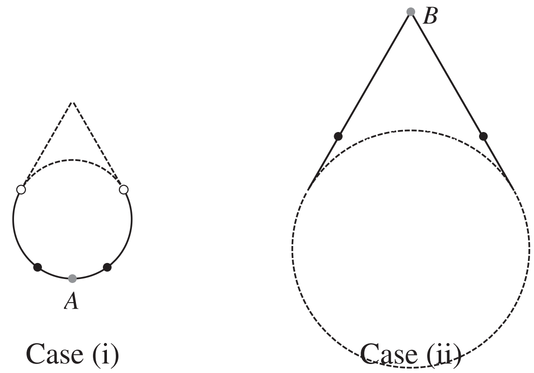

FIG. 2. Figure of the boundary of the convex body determined by (3) and (4), which is lying on the plane $a + b + c = p_\theta$ for fixed θ. Continuous line in the case (i) (respectively (ii)) indicates a, b, c satisfying the condition (6) (respectively (7)). In both cases, black dots represent PPT types which violate the SPA conjecture. We refer to Fig. 3 of Ref. 16 for the pictures of the three-dimensional convex bodies. The point A represents the case $a = 2 - p_\theta$, $b = c = p_\theta - 1$, and the point B represents the case $a = p_\theta$, $b = c = 0$. We refer to the figure in Ref. 23, Sec. V for the case of $\theta = 0$.

(5) is equivalent to

$$2(p_\theta - a)^2 = 1, \quad 1 \le a < p_\theta < 2,$$

and thus we have the following:

(ii) There exists a, b, and c satisfying (5) and (7) if and only if $1 + 1/\sqrt{2} \le p_\theta < 2$.

In the first case (i), we have indecomposable entanglement witnesses with the bi-spanning properties whose SPAs are not separable. These entanglement witnesses are optimal PPTES witnesses in the sense of Ref. 15 which detect maximal sets of PPTES. In the case (ii), we see that the partial transposes W^Γ give rise to examples of optimal indecomposable entanglement witnesses whose SPAs are not separable. It should be noted that they are not "nd-OWEs" in the sense of Ref. 26, or PPTES optimal witnesses using our terminology, when $1 + 1/\sqrt{2} < p_\theta < 2$, because they are not co-optimal; the smallest face determined by W^Γ contains $W[p_\theta, 0, 0; \theta]^\Gamma$ which is copositive. If $p_\theta = 1 + 1/\sqrt{2}$, then we have the solution $a = 1, b = p_\theta - 1, c = 0$, or $a = 1, b = 0, c = p_\theta - 1$, for which $W[a, b, c; \theta]$ are bi-optimal entanglement witnesses without the spanning property.[16]

IV. CONCLUSION

In conclusion, we propose the notions of positive type and copositive type for block-positive matrices W. We have shown that if the SPA of W is separable, then W must be of copositive type, and W is copositive type if and only if its partial transpose W^Γ is of positive type. Furthermore, a PPTES witness W is optimal if and only if W^Γ is optimal. Therefore, using the above facts about positive and copositive types, we conclude that the separability of the SPA of an optimal PPTES witness W implies that the SPA of W^Γ is not separable, unless W is of PPT type. This is so, even though W^Γ is still an optimal PPTES witness. We provide concrete examples of indecomposable entanglement witnesses of both copositive type and PPT type which violate the SPA conjecture. It would be very nice to characterize entanglement witnesses whose SPAs are separable.

Especially, it is still open if the SPA of an optimal decomposable entanglement witness is separable or not. Recent development[2,24] on the optimality for decomposable cases might be helpful to determine if the SPA conjecture is true for decomposable case.

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