REPRESENTATIONS OF BINARY FORMS BY QUINARY QUADRATIC FORMS

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§1. Introduction

(1) As a generalization of the famous four square theorem of Lagrange, Ramanujan and Willerding found all quaternary positive integral quadratic forms that represent all positive integers (1-universal). We discuss on quinary positive integral quadratic forms that represent all positive integral binary quadratic forms (2-universal).

(2) We discuss on almost 2-universal quinary forms and representations of binary forms by the classes in 2-universal genera of quinary positive integral quadratic forms of class number 2.

We adopt lattice theoretic language.

(*) A $\underline{\mathbb{Z}\text{-lattice}}$ L is a finitely generated free $\mathbb{Z}$-module equipped with a non-degenerate symmetric bilinear form B such that $B(L, L) \subseteq \mathbb{Z}$.

Q is the corresponding quadratic map ($Q(\mathbf{e}) := B(\mathbf{e}, \mathbf{e})$).

$\mathbb{Z}$-lattices correspond to classic integral quadratic forms.

L is $\underline{\text{positive definite}}$ or simply $\underline{\text{positive}}$ if $Q(L^{\times}) > 0$.

We'll assume every $\mathbb{Z}$-lattice is positive unless $\cdots$.

(*) $V := \mathbb{Q}L = \mathbb{Q} \otimes L$ is the quadratic space spanned by L.

$L_p := \mathbb{Z}_p L = \mathbb{Z}_p \otimes L$ is the localization of L at p.

$V_p := \mathbb{Q}_p V = \mathbb{Q}_p \otimes V$ is the localization of V at p.

(∗) We write $L \cong (B(\mathbf{e}_i, \mathbf{e}_j))$ for $L = R\mathbf{e}_1 + R\mathbf{e}_2 + \cdots + R\mathbf{e}_n$, where $R = \mathbb{Z}$ or $\mathbb{Z}_p$.

L is <u>even</u> if $Q(L) \subset 2R$, and <u>odd</u> otherwise, $R = \mathbb{Z}$ or $\mathbb{Z}_2$.

$\det L := \det(B(\mathbf{e}_i, \mathbf{e}_j))$ is the <u>discriminant</u> of L.

(∗) If $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ is an orthogonal basis, we write
$\langle Q(\mathbf{e}_1), Q(\mathbf{e}_2), \cdots, Q(\mathbf{e}_n)\rangle$ for the lattice;
$(Q(\mathbf{e}_1), Q(\mathbf{e}_2), \cdots, Q(\mathbf{e}_n))$ for the space.

(∗) L <u>represents</u> K ($K \to L$) if $\exists$ an isometry $\sigma : FK \to FL$ such that $\sigma(K) \subset L$, where $F = \mathbb{Q}$ or $\mathbb{Q}_p$ according to K, L are $\mathbb{Z}$ or $\mathbb{Z}_p$-lattices, respectively.

L is <u>equivalent</u> to K ($L \cong K$) if $\sigma(K) = L$.

(∗) For a $\mathbb{Z}$-lattice L, $cls(L)$ (the <u>class</u> of L) is the set of all $\mathbb{Z}$-lattices K such that $K \cong L$.

$gen(L)$ (the <u>genus</u> of L) is the set of all $\mathbb{Z}$-lattices K such that $K_p \cong L_p$ at every p (including ∞).

$K \to gen(L)$ if $K \to L'$ for some $L' \in gen(L)$, i.e., L_p represents K_p at every p (including ∞).

The number of inequivalent classes in a genus is finite and is called the <u>class number</u> of L or of $gen(L)$.

(∗) A positive $\mathbb{Z}$-lattice L is <u>k-universal</u> <u>(almost k-universal)</u> if L represents all (all but finitely many, respectively) positive $\mathbb{Z}$-lattices K of rank k.

§2. 2-universal $\mathbb{Z}$-lattices

2-1. (1-)universal $\mathbb{Z}$-lattices

(∗) Lagrange's four square theorem (1770): $I_4 = \langle 1, 1, 1, 1\rangle$ is universal.

(∗) Ramanujan (1917) claimed: ∃ exactly 55 universal quaternary diagonal $\mathbb{Z}$-lattices up to equivalence.

$\langle 1,1,1,1 \rangle$	$\langle 1,1,1,2 \rangle$	$\langle 1,1,2,2 \rangle$	$\langle 1,2,2,2 \rangle$
$\langle 1,1,1,3 \rangle$	$\langle 1,1,2,3 \rangle$	$\langle 1,2,2,3 \rangle$	$\langle 1,1,3,3 \rangle$
$\langle 1,2,3,3 \rangle$	$\langle 1,1,1,4 \rangle$	$\langle 1,1,2,4 \rangle$	$\langle 1,2,2,4 \rangle$
$\langle 1,1,3,4 \rangle$	$\langle 1,2,3,4 \rangle$	$\langle 1,2,4,4 \rangle$	$\langle 1,1,1,5 \rangle$
$\langle 1,1,2,5 \rangle$	$\langle 1,2,2,5 \rangle$	$\langle 1,1,3,5 \rangle$	$\langle 1,2,3,5 \rangle$
$\langle 1,2,4,5 \rangle$	$\langle 1,1,1,6 \rangle$	$\langle 1,1,2,6 \rangle$	$\langle 1,2,2,6 \rangle$
$\langle 1,1,3,6 \rangle$	$\langle 1,2,3,6 \rangle$	$\langle 1,2,4,6 \rangle$	$\langle 1,2,5,6 \rangle$
$\langle 1,1,1,7 \rangle$	$\langle 1,1,2,7 \rangle$	$\langle 1,2,2,7 \rangle$	$\langle 1,2,3,7 \rangle$
$\langle 1,2,4,7 \rangle$	$\langle 1,2,5,7 \rangle$	$\langle 1,1,2,8 \rangle$	$\langle 1,2,3,8 \rangle$
$\langle 1,2,4,8 \rangle$	$\langle 1,2,5,8 \rangle$	$\langle 1,1,2,9 \rangle$	$\langle 1,2,3,9 \rangle$
$\langle 1,2,4,9 \rangle$	$\langle 1,1,5,9 \rangle$	$\langle 1,1,2,10 \rangle$	$\langle 1,2,3,10 \rangle$
$\langle 1,2,4,10 \rangle$	$\langle 1,2,5,10 \rangle$	$\langle 1,1,2,11 \rangle$	$\langle 1,2,4,11 \rangle$
$\langle 1,1,2,12 \rangle$	$\langle 1,2,4,12 \rangle$	$\langle 1,1,2,13 \rangle$	$\langle 1,2,4,13 \rangle$
$\langle 1,1,2,14 \rangle$	$\langle 1,2,4,14 \rangle$	$\langle 1,2,5,5 \rangle^*$	

Dickson (1927) confirmed Ramanujan's claim except $\langle 1, 2, 5, 5\rangle$, which fails to represent 15

(∗) Willerding (1947) found all 124 universal quaternary non-diagonal $\mathbb{Z}$-lattices up to equivalence.

So, ∃ exactly 178 universal quaternary $\mathbb{Z}$-lattices.

There exists no universal ternary $\mathbb{Z}$-lattice.

(∗) Conway and Schneebeger recently proved so called the 15-Theorem: A $\mathbb{Z}$-lattice L is universal if and only if L represents $1, 2, 3, 5, 6, 7, 10, 14, 15$.

They listed all 204(?) universal quaternary $\mathbb{Z}$-lattices.

(∗) Mordell's five square theorem (1930): I_5 is 2-universal, i.e., I_5 represents all binary $\mathbb{Z}$-lattices.

(∗) The following questions arise naturally:

(Q1) How many 2-universal quinary $\mathbb{Z}$-lattices? Can you determine all of them up to equivalence?

(Q2) An ananlogy of the 15-Theorem for 2-universality?

(Q3) How about 3-universal, 4-universal, $\cdots$ $\mathbb{Z}$-lattices?

2-2. 2-universal quninary $\mathbb{Z}$-lattices

(∗) We write for convenience

$$[a,b,c] := \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad [a,b,c,d,e] \cong \begin{pmatrix} a & b & 0 \\ b & c & d \\ 0 & d & e \end{pmatrix}.$$

(∗) Diagonal case:

Let $L \cong \langle a_1, a_2, a_3, a_4, a_5 \rangle$ be 2-universal with $0 < a_1 \le a_2 \le a_3 \le a_4 \le a_5$.

$\langle 1,1 \rangle$ splits $L \Rightarrow a_1 = a_2 = 1$.

If $a_3 \ge 2$, then $[2,1,2] \not\to L \Rightarrow a_3 = 1$.

If $a_4 \ge 3$, then $[2,1,3] \not\to L \Rightarrow a_4 = 1, 2$.

If $a_5 \ge 4$, then $[2,1,4] \not\to L$ when $a_4 = 1$ and $\langle 3,3 \rangle \not\to L$ when $a_4 = 2 \Rightarrow a_4 \le a_5 \le 3$.

In summary, only five candidates survive:

$$L \cong \langle 1,1,1 \rangle \perp \langle a,b \rangle \text{ with}$$
$$\langle a,b \rangle = \langle 1,1 \rangle,\ \langle 1,2 \rangle,\ \langle 1,3 \rangle,\ \langle 2,2 \rangle,\ \langle 2,3 \rangle.$$

(∗) Non-diagonal case:

Let L be a 2-universal nondiagonal quinary $\mathbb{Z}$-lattice.

$\langle 1,1\rangle$ splits $L \Rightarrow L \cong \langle 1,1\rangle \perp L_0$, where L_0 is ternary non-diagonal.

(Subcase 1) 1 is represented by L_0, i.e., $1 \in Q(L_0)$:

$L_0 \cong \langle 1\rangle \perp L_1$, where $L_1 \cong [a,b,c]$ is <u>Minkowski reduced</u>, i.e., $2 \le a \le c,\ 0 < 2b \le a$.

If $a \ge 3$, then $[2,1,3] \not\to L \Rightarrow a = 2, b = 1$.

$\langle 3,3\rangle \to L \Rightarrow c = 2,3$.

Therefore:

$$L_0 \cong \langle 1\rangle \perp L_1 \text{ with } L_1 \cong [2,1,2], \quad [2,1,3].$$

(Subcase 2) 1 is not represented by L_0, i.e., $1 \notin Q(L_0)$:

Since $[2,1,2], \langle 2,3\rangle \to L$,

$$L_0 \cong [2,1,2] \perp \langle c\rangle, \quad [2,1,2,1,c] \text{ with } c = 2,3.$$

(∗) A $\mathbb{Z}$-lattice L is <u>2-regular</u> if L represents all binary $\mathbb{Z}$-lattices that are represented by L at every p.

Earnest proved: there are only finitely many 2-regular primitive quaternary $\mathbb{Z}$-lattices, up to equivalence.

(∗) No quaternary $\mathbb{Z}$-lattice can be 2-universal.

Theorem 1. (Kim-Oh-K) There are exactly eleven quinary 2-universal positive $\mathbb{Z}$-lattices, up to equivalence. They are:

$$\langle 1,1,1,1,1\rangle,\ \langle 1,1,1,1,2\rangle$$
$$\langle 1,1,1,1,3\rangle,\ \langle 1,1,1,2,2\rangle,\ \langle 1,1,1,2,3\rangle$$
$$\langle 1,1,1\rangle \perp [2,1,2],\ \langle 1,1,1\rangle \perp [2,1,3],$$
$$\langle 1,1,2\rangle \perp [2,1,2],\ \langle 1,1,3\rangle \perp [2,1,2],$$
$$\langle 1,1\rangle \perp [2,1,2,1,2],\ \langle 1,1\rangle \perp [2,1,2,1,3].$$

PROOF. The five diagonal forms are all 2-universal by local-global principle, i.e., each of them is <u>locally 2-universal</u> at every prime p (including ∞) and has class number 1.

The non-diagonal forms except $\langle 1,1,2\rangle \perp [2,1,2]$ are also 2-universal by local-global principle.

$K := \langle 1,1,2\rangle \perp [2,1,2]$ is locally 2-universal at every p. Although its class number is 2, it is still 2-universal globally. The other class in $\text{gen}(K)$ is the class of $M := \langle 1,1,1,1,6\rangle$, which is not 2-universal because $[2,1,4] \not\to M$.

Since $\langle 2,2,2,2,6\rangle$ is a sublattice of both K and M and $\langle 1,1,1,1,3\rangle$ is 2-universal, both K and M represent all binary lattices $[a,b,c]$ with a,b,c even.

Let $\ell \cong [a,b,c]$ be a Minkowski reduced binary $\mathbb{Z}$-lattice, i.e., $0 \le 2b \le a \le c$.

If ℓ is not even unimodular over $\mathbb{Z}_2$ ($\det \ell \not\equiv 1 \pmod 2$), then $\ell \to K$ over $\mathbb{Z}$.

If $\det \ell \not\equiv 1 \pmod 3$, then $\ell \to K$ over $\mathbb{Z}$.
(pf) Assume $\sigma : \ell \to M = I_4 \perp \langle 6 \rangle$ such that
$\sigma(\ell) = \mathbb{Z}(x_1+y_1)+\mathbb{Z}(x_2+y_2),\ x_1, x_2 \in I_4,\ y_1, y_2 \in \langle 6 \rangle$.
Let $\tilde{\ell} := \mathbb{Z}x_1 + \mathbb{Z}x_2 \subset I_4$. Then $\tilde{\ell} \to \langle 1,3 \rangle \perp [2,1,2]$.
So, $\ell \to \tilde{\ell} \perp \langle 6 \rangle \to \langle 1,3 \rangle \perp [2,1,2] \perp \langle 6 \rangle \subset K$.

We may assume: $Q(\ell) \subseteq 2\mathbb{Z},\ \det \ell \equiv 1 \pmod 6$.
Then $a, c \equiv 2, 4 \pmod 6$ and b is odd.

Define for integers s, α, β

$$\ell_s :\cong [a, sa+b, s^2a + 2sb + c],$$
$$\ell(\alpha,\beta) :\cong [a - 2\alpha^2, b - 2\alpha\beta, c - 2\beta^2].$$

Note that $\ell_s \cong \ell = \ell_0$ and that
$\ell_s(\alpha,\beta) \to N := \langle 1,1 \rangle \perp [2,1,2] \implies \ell \to N \perp \langle 2 \rangle = K$.

Indeed, there exists s, α, β such $\ell_s(\alpha,\beta) \to N$ under the assumption that $\ell \not\to N$ and that a is large so that $\ell_s(\alpha,\beta)$ is positive. Note that N is of class number 1.

The case when a is not large enough to make $\ell_s(\alpha,\beta)$ positive can be proved by direct computation. □

2-3. Characterization of 2-universality

Theorem 2. (Kim-Oh-K) A positive $\mathbb{Z}$-lattice is 2-universal if it represents the following six binary $\mathbb{Z}$-lattices:

$$\langle 1,1\rangle,\ \langle 2,3\rangle,\ \langle 3,3\rangle,\ [2,1,2],\ [2,1,3],\ [2,1,4].$$

Moreover, this is a minimal set.

PROOF. $L \cong \langle 1,1\rangle \perp L_0$.

$[2,1,2] \to L \Rightarrow L_0$ represents 1 or $[2,1,2]$.

(Case 1) L_0 represents 1: Then $L \cong \langle 1,1,1\rangle \perp L_1$.

$[2,1,3] \to L \Rightarrow L_1$ represent 1 or 2.

L_1 represent $1 \Rightarrow L \cong \langle 1,1,1,1\rangle \perp L_2$.
$[2,1,4] \to L \Rightarrow L_2$ represents 1, 2 or 3.
So, L has a sublattice isometric to:

$$\langle 1,1,1,1,1\rangle,\ \langle 1,1,1,1,2\rangle,\ \langle 1,1,1,1,3\rangle.$$

$\therefore L$ is 2-universal.

L_1 does not represent $1 \Rightarrow L_1$ represents 2.
$\langle 3,3\rangle \to L \Rightarrow L$ has a sublattice isometric to:

$$\langle 1,1,1,2,2\rangle,\ \langle 1,1,1,2,3\rangle,$$
$$\langle 1,1,1\rangle \perp [2,1,2],\ \langle 1,1,1\rangle \perp [2,1,3].$$

$\therefore L$ is 2-universal.

(Case 2) Let L_0 does not represent 1: $[2,1,2] \to L_0$.

$\langle 2,3\rangle \to L \Rightarrow L_0$ represents:
$\langle 2\rangle \perp [2,1,2]$, $\langle 3\rangle \perp [2,1,2]$, $[2,1,2,1,2]$, $[2,1,2,1,3]$.
So L has a sublattice isometric to:

$$\langle 1,1,2\rangle \perp [2,1,2],\ \langle 1,1,3\rangle \perp [2,1,2],$$
$$\langle 1,1\rangle \perp [2,1,2,1,2],\ \langle 1,1\rangle \perp [2,1,2,1,3].$$

$\therefore$ L is 2-universal.

($*$) For the second assertion, we let S be the set of six binary $\mathbb{Z}$-lattices listed in the theorem.

One can easily show that:

$\langle 1,2\rangle \perp [2,1,2]$ represents all of S but $\langle 1,1\rangle$

$\langle 1,1\rangle \perp [2,1,2]$ represents all of S but $\langle 2,3\rangle$.

$\langle 1,1,1,2\rangle$ represents all of S but $\langle 3,3\rangle$.

$\langle 1,1,2,2\rangle$ represents all of S but $[2,1,2]$.

$\langle 1,1,1,3\rangle$ represents all of S but $[2,1,3]$.

$\langle 1,1,1,1\rangle$ represents all of S but $[2,1,4]$. $\square$

2-4. k-universal $\mathbb{Z}$-lattices for $k \geq 3$

(∗) $u_{\mathbb{Z}}(k) = \min\{\, \mathrm{rank}(L) \mid L \text{ is } k\text{-universal}\,\}$.

$u_{\mathbb{Z}}(k)$ exists for every $k \geq 1$: $u_{\mathbb{Z}}(1) = 4, u_{\mathbb{Z}}(2) = 5, \cdots$.
Ko proved (1937): $u_{\mathbb{Z}}(k) = k + 3$ for $k = 3, 4, 5$.

(∗) For every given positive integer k, there are only finitely many k-universal $\mathbb{Z}$-lattices of rank $u_{\mathbb{Z}}(k)$.

(proof) Suppose there are infinitely many k-universal $\mathbb{Z}$-lattices, say $L_1, L_2, \ldots,$ of rank $u_{\mathbb{Z}}(k)$.
Define $\mathfrak{L} = \{\ \ell \mid \ell \xrightarrow{\sigma_i} L_i \ \text{ for infinitely many } i\ \}$,
where σ_i is a representation from ℓ to L_i.
Let ℓ be a $\mathbb{Z}$-lattice of maximal rank in $\mathfrak{L}$.
Then $k \leq \mathrm{rank}(\ell) < u_{\mathbb{Z}}(k)$.
We may assume: $\sigma_i(\ell)$ is a primitive sublattice of L_i.
Since $\mathrm{rank}(\ell) < u_{\mathbb{Z}}(k)$, ℓ cannot be k-universal.
So, there exist a lattice ℓ' of rank k such that
$\ell' \not\to \ell$ but $\ell' \xrightarrow{\sigma_i'} L_i$ for all i.
ℓ_i: the $\mathbb{Z}$-sublattice of L_i generated by $\sigma_i(\ell)$ and $\sigma_i'(\ell')$.
Since $\sigma_i(\ell)$ is primitive, $\mathrm{rank}(\ell) < \mathrm{rank}(\ell_i) \leq u_{\mathbb{Z}}(k)$.
So, ℓ_i has a bounded discriminant and a bounded rank.
Thus, the number of ℓ_i is finite up to equivalence .
But then, we can take $\hat{\ell}$ with $\mathrm{rank}(\hat{\ell}) > \mathrm{rank}(\ell)$ such that $\hat{\ell} \to L_i$ for infinitely many i. □

Theorem 3. (Kim-Oh-K)

$$u_{\mathbb{Z}}(k) = \begin{cases} k+3 & \text{if } 1 \le k \le 5, \\ 13 & \text{if } k = 6, \\ 15 & \text{if } k = 7, \\ 16 & \text{if } k = 8, \\ 28 & \text{if } k = 9, \\ 30 & \text{if } k = 10. \end{cases}$$

($*$) For each $k, 1 \le k \le 8$, we know all k-universal $\mathbb{Z}$-lattices of rank $u_{\mathbb{Z}}(k)$, up to equivalnce, except for $k = 3, 5$.

For $k = 3$, $I_3 \perp \langle 2 \rangle \perp [2,1,2], I_3 \perp \langle 3 \rangle \perp [2,1,2]$; and for $k = 5$, $I_6 \perp [2,1,2]$ to be determined.

($*$) For $k = 8$, there is a unique 8-universal $\mathbb{Z}$-lattice of rank $u_{\mathbb{Z}}(8) = 16$, up to equivalence, namely

$$I_8 \perp E_8.$$

Furthermore, we have an analogy of the 15-Theorem:

L is 8-universal $\iff$ L represents I_8 and E_8.

($*$) There exists no k-universal diagonal $\mathbb{Z}$-lattices $\forall\, k \ge 6$.

($*$) $u_{\mathbb{Z}}(k)$ seems to grow very quickly.

2-5. Fourier coefficients of theta-series

(∗) Consider the following theta-sereis of degree k attached to a positive $\mathbb{Z}$-lattice L of rank $m \geq k$:

$$\Theta^k(L, Z) := \sum_X \exp(2\pi i \mathrm{Tr}({}^t XMXZ))$$
$$= \sum_N r(N, M) \exp(2\pi i \mathrm{Tr}(NZ))$$

$Z \in \mathcal{H}_k$, X: all $m \times k$ integer matrices,
M: the integral symmetric matrix corresponding to L,
N: all $k \times k$ semi-positive integral symmetric matrices
$r(N, M) := \#\{\, X \mid {}^t XMX = N \,\} < \infty$.

(∗) If $k = 1$ and $r(n, M) > 0$ for $n = 1, 2, 3, 5, 6, 7, 10, 14, 15$, then $r(n, M) > 0$ for all nonnegative integers n.

(∗) If $k = 2$ and $r(N, M) > 0$ for $N = \langle 1, 1\rangle, \langle 2, 3\rangle, \langle 3, 3\rangle$, $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$, then $r(N, M) > 0$ for all 2×2 semi-positive integral symmetric matrices N.

(∗) If $k = 8$ and $r(I_8, M), r(E_8, M) > 0$, then $r(N, M) > 0$ for all 8×8 semi-positive integral symmetric N.

§3. Almost 2-universal quinary $\mathbb{Z}$-lattices

(∗) An <u>almost k-universal</u> $\mathbb{Z}$-lattice is a $\mathbb{Z}$-lattice that represents all but finitely many $\mathbb{Z}$-lattices of rank k.

3-1. Almost (1-)universal quaternary $\mathbb{Z}$-lattices

(∗) Kloosterman (1926) determined all quadruples (a, b, c, d) of positive integers with $a \le b \le c \le d$ for which $\langle a, b, c, d\rangle$ are almost universal.

He left the following four $\mathbb{Z}$-lattices undetermined: $\langle 1, 2, 11, 38\rangle$ $\langle 1, 2, 17, 33\rangle$ $\langle 1, 2, 19, 22\rangle$ $\langle 1, 2, 19, 38\rangle$.

Pall (1946) showed the almost universality for the remaining $\mathbb{Z}$-lattices above.

Pall and Ross (1946) extended Kloosterman's result to the non-diagonal case.

Halmos (1938) found all 88 quaternay diagonal $\mathbb{Z}$-lattices that represent all positive integers except exacltly one (the largest exception is 15)

3-2. Almost 2-universal quinary $\mathbb{Z}$-lattices

$(*)$ Hwang (1997): In order for a quinary diagonal $\mathbb{Z}$-lattice L to be almost 2-universal, L should be one of the following forms:

$\langle 1,1,1,1,a\rangle$ $\langle 1,1,1,2,a\rangle$ $\langle 1,1,1,3,a\rangle$ $\langle 1,1,2,2,a\rangle$,
$\langle 1,1,2,3,a\rangle$ $\langle 1,1,2,4,a\rangle$ $\langle 1,1,2,5,a\rangle$,
where a's are yet to be determined.

$\exists$ only three positive quinary diagonal $\mathbb{Z}$-lattices that represent all but one positive binary $\mathbb{Z}$-lattice :

$\langle 1,1,1,2,4\rangle$, $\langle 1,1,1,2,5\rangle$, $\langle 1,1,2,2,3\rangle$
whose exceptions are $\langle 3,3\rangle, \langle 3,3\rangle, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, resp.

$(*)$ Oh (2000): There exist only finitely many almost 2-universal quinary $\mathbb{Z}$-lattices.

Furthermore, he improved Hwang's result: Other than 2-universal quinary $\mathbb{Z}$-lattices, there are at most nine almost 2-universal quinary diagonal $\mathbb{Z}$-lattices.

The following six are almost 2-universal:

$\langle 1,1,1,1,5\rangle$, $\langle 1,1,1,2,4\rangle$, $\langle 1,1,1,2,5\rangle$,
$\langle 1,1,2,2,3\rangle$, $\langle 1,1,1,2,7\rangle$, $\langle 1,1,2,2,5\rangle$.

But for each of the following three, almost 2-universality is not known yet:
$\langle 1,1,1,3,7\rangle$, $\langle 1,1,2,3,5\rangle$, $\langle 1,1,2,3,8\rangle$.

3-3. Kitaoka's question

(∗) In his book, Kitaoka raised a question on representations of binary integral quadratic forms by the following two quinary even integral quadratic forms:

$$f = 2\Big(\sum_{i=1}^{4} x_i^2 + 2x_5^2 - x_1x_2 - x_2x_3 - x_3x_4\Big),$$

$$g = 2\Big(\sum_{i=1}^{4} x_i^2 + 3x_5^2 - x_1x_2 - x_1x_3 - x_1x_4 - x_1x_5 + x_2x_5 + x_3x_5\Big).$$

$f \leftrightarrow L := A_4 \perp \langle 4 \rangle,\ g \leftrightarrow K := D_4(5)$, where

$$D_4(k) = D_4 4k[2\frac{1}{2}] \cong \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 & -1 \\ 0 & -1 & 0 & 2 & 1 \\ 0 & 0 & -1 & 1 & k+1 \end{pmatrix}.$$

These two $\mathbb{Z}$-lattices are indistinguishable locally at every p but inequivalent globally, i.e., $K \in gen(L),\ K \not\cong L$.

$gen(L) = gen(K) = cls(L) \cup cls(K)$; $\det L = \det K = 20$ is the smallest among genera of quinary (positive) even $\mathbb{Z}$-lattices with class number > 1

(∗) Since each $\mathbb{Z}$-lattice above represents all binary even integral quadratic forms locally at every p, the genus is <u>even 2-universal</u>, i.e., $\ell \to gen(L)$ for every binary positive even $\mathbb{Z}$-lattices ℓ.

Hence either $\ell \to L$ or $\ell \to K$.

Kitaoka conjectured each of the two $\mathbb{Z}$-lattices above represents all but finitely many binary even $\mathbb{Z}$-lattices

(∗) Koo-Oh-K : L represents all but one even $\mathbb{Z}$-lattice, which is $[4, 2, 4]$; K represents all but three even $\mathbb{Z}$-lattices, which are $[2, 1, 4]$, $[4, 1, 4]$, $[8, 1, 8]$.

We further found examples of even 2-universal genera of quinary $\mathbb{Z}$-lattices with class number 2, whose classes have no exceptions, finitely many exceptions or infinitely many exceptions.

(∗) From these examples, we may conclude that all kinds of combination is possible concerning representations of binary $\mathbb{Z}$-lattices by such genera:

Both classes may represent all binary forms;

Both may have finitely many exceptions;

One may represent all while the other has finitely or infinitely many exceptions;

One may have finitely many exceptions while the other has infinitely many.

(∗) $[2,1,2] \perp \langle 2,2,2\rangle$ is even 2-universal;

$D_4 \perp \langle 6\rangle$ has infinitely many exceptions, which are
$[4,1,c]$, $[8,1,8]$, $[8,1,12]$, $[8,1,16]$, $[8,3,8]$,
where c is a positive even integer.

(∗) $[2,1,2,1,2] \perp [2,1,4]$ is even 2-universal;

$D_4(7)$ has eleven exceptions, which are
$[2,1,4]$, $[4,1,4]$, $[4,1,6]$, $[6,3,8]$, $[8,1,8]$, $[8,1,16]$
$[8,1,24]$, $[8,1,32]$, $[8,1,40]$, $[8,3,8]$, $[8,3,16]$.

(∗) Both $A_3 \perp \langle 2,6\rangle$ and $A_2 \perp \langle 2,2,4\rangle$ are even 2-universal.

(∗) $J \perp \langle 2\rangle$ has one exception, which is $[2,1,2]$;

$D_4 \perp \langle 10\rangle$ has infinitely many exceptions, which are
$[16,1,16]$, $[16,3,16]$, $[16,1,20]$, $[12,1,36]$, $[12,1,40]$
$[12,1,44]$, $[12,1,48]$, $[10,5,12]$, $[8,1,c_1]$, $[8,3,c_1]$
$[6,1,8]$, $[6,3,8]$, $[4,1,c_2]$, $[2,1,8]$, $[12,b_1,12]$
$[12,b_1,16]$, $[12,b_1,20]$, $[12,b_2,24]$, $[12,b_2,28]$
$[12,b_2,32]$, $[4,2,8]$, $[8,2,8]$, $[16,2,16]$,

where c_1, c_2 are positive even integers with $c_1 \geq 8$
$b_1 = 1,3,5$ and $b_2 = 1,3$, while J above is defined by

$$J := \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix}.$$

(∗) We know that there are only finitely many almost 2-universal quinary $\mathbb{Z}$-lattices. Observe that this implies that almost all 2-universal genera have the property that all classes in the genera have infinitely many exceptions.

One such example is $gen(\langle 1,1,1,2,6\rangle)$, which consists of

$$cls(\langle 1,1,1,2,6\rangle) \text{ and } cls(\langle 1,2,2\rangle \perp [2,1,2]).$$

Note that for all nonnegative integers t,

$$\langle 3, 3^{2t+1}\rangle \not\to \langle 1,1,1,2,6\rangle;$$
$$\langle 3, 3^{2t}\rangle \not\to \langle 1,2,2\rangle \perp [2,1,2].$$

In this vein, although we couldn't find an even 2-universal genus of quinary $\mathbb{Z}$-lattices of class number 2 such that both classes in the genus have infinitely many exceptions, we believe that there are only finitely many quinary almost even 2-universal $\mathbb{Z}$-lattices.