HECKE OPERATORS AND THE SIEGEL OPERATOR

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§0. Introduction

A commuting relation between the Siegel operator and Hecke operators acting on Siegel modular forms of integral weight was proved by Zharkovskaya [1] in 1974 when the level q of the form is 1. In 1979, Andrianov [2] generalized her result to the forms with an arbitrary level. In their proof, they introduced the so called Ψ operator between Hecke rings of different degrees.

In this article, we proved the relation still holds for the forms of half integral weight (Theorem 5.1). For half integral weight forms we are to deal with the so called lifted Hecke operators, first introduced by Shimura [3] for the classical modular forms of half integral weight in 1973 and generalized for Siegel modular forms of half integral weight by Zhuravlev [4] in 1983. We used many of their ideas and results.

Zhuravlev [4] also found a Hecke ring $\hat{L}(T)$ of a moderate size, which is large enough as well for many practical purposes. We also proved the surjectivity of the Ψ operator restricted of $\mathbb{L}(T)$, which is a homomorphic image of $\hat{L}(T)$ (Theorem 5.2). An analogous result for the forms of integral weight was proved by Andrianov [2].

The theorems are expected to play important roles in decompositions of Siegel modular forms of half integral weight [9] and in the theory of theta-series associated to positive definite quadratic forms in an odd number of variables.

Let $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ be the ring of rational integers, the field of rational numbers, the field of real numbers, and the field of complex numbers, respectively.

Let $M_{m,n}(A)$ be the set of all $m \times n$ matrices over A, a commutative ring with 1, and let $M_n(A) = M_{n,n}(A)$. Let $GL_n(A)$ and $SL_n(A)$ be the group of invertible matrices in $M_n(A)$ and its subgroup consisting of matrices of determinant 1, respectively.

For $g \in M_m(A)$, $h \in M_{m,n}(A)$, let $g[h] = {}^t hgh$, where ${}^t h$ is the transpose of h. Let I_n and 0_n be the identity and the zero matrices, respectively. Let $\det g$ be the determinant of g. For $g \in M_{2n}(A)$, we let A_g, B_g, C_g, and D_g be the $n \times n$ block matrices in the upper left, upper right, lower left, and lower right corners

of g, respectively, and we write $g = (A_g, B_g; C_g, D_g)$. Let $\mathrm{diag}(N_1, N_2, \ldots, N_r)$ be the matrix with block matrices $N_1, N_2, \ldots, N_r$ on its main diagonal and zeroes outside. Let $\mathcal{N}_m$ be the set of all semi-positive definite (eigenvalues ≥ 0), semi-integral (diagonal entries and twice of nondiagonal entries are integers), symmetric $m \times m$ matrices, and $\mathcal{N}_m^+$ be its subset consisting of positive definite (eigenvalues > 0) matrices.

Let $G_n = GSp_n^+(\mathbb{R}) = \{g \in M_{2n}(\mathbb{R}); J_n[g] = rJ_n,\ r > 0\}$ where $J_n = (0_n, I_n; -I_n,\ 0_n)$ and $r = r(g)$ is a real number determined by g. Let $\Gamma^n = Sp_n(\mathbb{Z}) = \{M \in M_{2n}(\mathbb{Z}); J_n[M] = J_n\}$. Let $H_n = \{Z = X + iY \in M_n(\mathbb{C}); {}^tZ = Z,\ Y > 0\}$. For $g \in G_n$ and $Z \in H_n$, we set $g < Z >= (A_gZ+B_g)(C_gZ+D_g)^{-1} \in H_n$.

For $g \in M_n(\mathbb{C}),\ e(g) = \exp(2\pi i\sigma(g))$ where $\sigma(g)$ is the trace of g. Finally, we let $< n >= n(n+1)/2$ for $n \in \mathbb{Z}$.

For other standard terminologies and basic facts, we refer the readers [10], [11].

§1. Definitions and basic properties

Let G be a multiplicative group and let Γ be its subgroup. Let S be a semigroup of G contained in the commensurator of Γ in G, i.e., for any $g \in S$ the subgroup $\Gamma^g = g^{-1}\Gamma g \cap \Gamma$ is of finite index in both $g^{-1}\Gamma g$ and Γ. (Γ, S) is called a Hecke pair if $\Gamma S = S\Gamma = S$. Let $L(\Gamma, S)$ be the vector space over $\mathbb{C}$ spanned by left cosets $(\Gamma g),\ g \in S$. Let $D(\Gamma, S)$ be the subspace of $L(\Gamma, S)$ consisting of $X = \Sigma a_i(\Gamma g_i)$ such that $X \cdot M = X$ where $X \cdot M = \Sigma a_i(\Gamma g_i M),\ M \in \Gamma$. If we let $(\Gamma g \Gamma) = \Sigma_{i=1}^n(\Gamma g_i),\ g, g_i \in S$, where $\Gamma g \Gamma$ is the disjoint union of $\Gamma g_i,\ i = 1, \ldots, n$, then double cosets $(\Gamma_g \Gamma),\ g \in S$, form a basis for the subspace $D(\Gamma, S)$. $D(\Gamma, S)$ is in fact a ring, which is called the Hecke ring of the pair (Γ, S), with the multiplication defined by $X \cdot Y = \Sigma a_i b_j(\Gamma g_i h_i)$ for any $X = \Sigma a_i(\Gamma g_i),\ Y = \Sigma b_j(\Gamma h_j) \in D(\Gamma, S)$.

Let $(\Gamma_1, S_1),\ (\Gamma_2, S_2)$ be two Hecke pairs such that

$$\Gamma_2 \subset \Gamma_1,\ \Gamma_1 S_2 = S_1, \quad \text{and} \quad \Gamma_1 \cap S_2 S_2^{-1} \subset \Gamma_2. \tag{1.1}$$

Then the map $\varepsilon = \varepsilon(\Gamma_1, \Gamma_2) : D(\Gamma_1, S_1) \to D(\Gamma_2, S_2)$ defined by $\varepsilon(X) = \Sigma a_i(\Gamma_2 g_i) \in D(\Gamma_2, S_2)$ for any $X \in D(\Gamma_1, S_1)$, where X may be written in the form $X = \Sigma a_i(\Gamma_1 g_i)$ with $g_i \in S_2$ because of the second condition, is an injective ring homomorphism. Moreover [2] it is an isomorphism if $[\Gamma_1 : \Gamma_1^g] = [\Gamma_2 : \Gamma_2^g]$ for every $g \in S_2$.

Let $\hat{G}$ be another multiplicative group and $\gamma : \hat{G} \to G$ and $\delta : \Gamma \to \hat{G}$ be surjective and injective homomorphisms, respectively, such that $\gamma \circ \delta = 1$ on Γ and $\mathrm{Ker}\,\gamma \subset Z(\hat{G})$, the center of $\hat{G}$. For each $g \in S$, we define [4] a homomorphism

$$\rho = \rho_g : \Gamma^g \to \hat{G},\ \delta(g\alpha g^{-1}) = \zeta\alpha\zeta^{-1}\rho(\alpha), \quad \alpha \in \Gamma^g, \tag{1.2}$$

where $\zeta \in \hat{G}$ such that $\gamma(\zeta) = g$. $\rho_g(\alpha)$ is independent of the choice of ζ because $\mathrm{Ker}\,\gamma \subset Z(\hat{G})$. We call ρ_g the lifting homomorphism of g. It is known [4] that if

(Γ, S) is a Hecke pair and $[\Gamma : \operatorname{Ker} \rho_g]$ is finite for any $g \in S$, then $(\hat{\Gamma}, \hat{S})$ is also a Hecke pair where $\hat{\Gamma} = \delta(\Gamma)$ and $\hat{S} = \gamma^{-1}(S)$. It is also known [4] that if ρ_g is trivial, then $(\hat{\gamma}\zeta\hat{\Gamma}) = \Sigma_{i=1}^{\mu}(\hat{\Gamma}\zeta_i)$ if and only if $(\Gamma g \Gamma) = \Sigma_{i=1}^{\mu}(\Gamma g_i)$, where $\zeta, \zeta_i \in \hat{S}$ and $g, g_i \in S$ such that $\gamma(\zeta) = g$ and $\gamma(\zeta_i) = g_i$.

Let n, q be positive integers and p be a prime such that $(p, q) = 1$. Let $G = G_n = GSp_n^+(\mathbb{R})$, $\Gamma = \Gamma^n = Sp_n(\mathbb{Z})$. Let $S = S_p^n = \{g \in M_{2n}(\mathbb{Z}[p^{-1}]); J_n[g] = p^\delta J_n,\ \delta \in \mathbb{Z}\}$ where $\delta = \delta(g)$ is an integer depending on g. Let $\Gamma(q) = \Gamma^n(q) = \{M \in \Gamma; C_M \equiv 0 \pmod q\}$ and $S(q) = S_p^n(q) = g \in S; C_g \equiv 0 \pmod q)\}$. Let $\Gamma_0 = \Gamma_0^n = \{M \in \Gamma; C_M = 0\}$ and $S_0 = S_{p,0}^n = \{g \in S; C_g = 0\}$. Finally, let $\Lambda = \Lambda^n = SL_n(\mathbb{Z})$ and $\Delta = \Delta_p^n = \{D \in M_n(\mathbb{Z}[p^{-1}]); \det D = p^\delta,\ \delta \in \mathbb{Z}\}$. It is known [2], [5] that (Γ, S), $(\Gamma(q), S(q))$, (Γ_0, S_0), and (Λ, Δ) are Hecke pairs. We denote the corresponding Hecke rings by $L = L_p^n$, $L(q) = L_p^n(q)$, $L_0 = L_{p,0}^n$, and $H = H_p^n$, respectively. We let $S^2 = S_p^{n,2} = \{g \in S; \delta(g) \in 2\mathbb{Z}\}$, $S^2(q) = S_p^{n,2}(q) = S^2 \cap S(q)$, and $S_0^2 = S_{p,0}^{n,2} = S^2 \cap S_0$. Then (Γ, S^2), $(\Gamma(q), S^2(q))$, and (Γ_0, S_0^2) are also Hecke pairs whose corresponding Hecke rings are denoted by $L^2 = L_p^{n,2}$, $L^2(q) = L_p^{n,2}(q)$, and $L_0^2 = L_{p,0}^{n,2}$, respectively. They are the even subrings of $L, L(q)$, and L_0, respectively.

Since Hecke pairs $(\Gamma(q), S(q))$ and (Γ_0, S_0) satisfy the conditions (1.1), we have an injective homomorphism

$$\varepsilon_{0,q} : L(q) \to L_0,\ \varepsilon_{0,q}(\Sigma a_i(\Gamma(q) g_i)) = \Sigma a_i(\Gamma_0 g_i) \tag{1.3}$$

where g_i are chosen to be in S_0. Similary, we have an injective homomorphism $\varepsilon : L \to L(q)$ which is in fact an isomorphism because $[\Gamma : \Gamma^g] = [\Gamma(q) : \Gamma(q)^g]$ for any $g \in S(q)$ [2].

We define a homomorphism

$$\varphi = \varphi_n : L_0 \to \mathbb{C}(\underline{x}) = \mathbb{C}^n(\underline{x}) = \mathbb{C}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}] \tag{1.4}$$

as the following [2] : Let $X \in L_0$. X can be written in the form $X = \Sigma a_i(\Gamma_0 g_i)$, where $a_i \in \mathbb{C}$ and $g_i = (p^{\delta_i} D_i^*, B_i; 0, D_i) \in S_0$, with $\delta_i = \delta(g_i) \in \mathbb{Z}$, $B_i \in M_n(\mathbb{Z}[p^{-1}])$, $D_i \in \Delta$ and $D_i^* = ({}^tD)^{-1}$. We befine $\omega = \omega_n : L_0 \to H[t^{\pm 1}]$ by $\omega(X) = \Sigma a_i t^{\delta_i}(\Lambda D_i)$. Then ω is a surjective ring homomorphism. Let $U = \Sigma a_i t^{\delta_i}(\Lambda D_i) \in H[t^{\pm 1}]$. We may assume that each D_i is an upper triangular matrix with diagonal entries $p^{\delta_{i1}}, \ldots, p^{\delta_{in}}$. We define $\psi = \psi_n : H[t^{\pm 1}] \to \mathbb{C}(\underline{x})$ by $\psi(U) = \Sigma a_i x_0^{\delta_i} \Pi_{1 \le j \le n}(x_j p^{-j})^{\delta_{ij}}$. Then ψ is an injective ring homomorphism. Finally, we let $\varphi = \psi \circ \omega : L_0 \to \mathbb{C}(\underline{x})$.

§2. The lifted Hecke rings

Let $\hat{G} = \hat{G}_n = \{(g, \alpha(Z)); g \in G,\ \alpha(Z)$ is a holomorphic function on H_n satisfying $\alpha(Z)^2 = t(\det g)^{-1/2} \det(C_g Z + D_g)$ for some $t \in \mathbb{C}$, $|t| = 1\}$. $\hat{G}$ is a multiplicative group under the multiplication $(g, \alpha(Z)) \cdot (h, \beta(Z)) = (gh, \alpha(h < Z >)\beta(Z))$ and is called the universal covering group of G.

Let $\pi : \hat{G} \to G$ be the projection $\pi(g, \alpha(Z)) = g$. We define an action of $\hat{G}$ on H_n by $\zeta < Z >= \pi(\zeta) < Z >$ for $\zeta \in \hat{G}$, $Z \in H_n$. Note that $\mathrm{Ker}(\pi) \subset Z(\hat{G})$.

From now on, we let q be a positive integer such that $4|q$. Let

$$\theta^n(Z) = \sum_M e({}^tMMZ) = \sum_N e(Z[N]), \quad Z \in H_n, \tag{2.1}$$

where $M(N,$ resp.) runs over all the integral row (column, resp.) matrices of length n. $\theta^n(Z)$ is called the standard θ-function. For $M \in \Gamma(q)$, we define $j(M, Z) = \theta^n(M < Z >)/\theta^n(Z)$, $Z \in H_n$. It is well known [6] that $(M, j(M, Z)) \in \hat{G}$. So the map $j : \Gamma(q) \to \hat{G}$ defined by $j(M) = (M, j(M, Z))$ is a well defined injective homomorphism such that $\pi \circ j = 1$ on $\Gamma(q)$. So we can define the lifting homomorphism ρ_g for each $g \in S(q)$ and conclude that $(\hat{\Gamma}(q), \hat{S}(q))$ is a Hecke pair where $\hat{\Gamma}(q) = j(\Gamma(q))$ and $\hat{S}(q) = \pi^{-1}(S(q))$ because $[\Gamma(q) : \mathrm{Ker}\,\rho_g]$ is finite for each $g \in S(q)$ [4]. Similary $(\hat{\Gamma}_0, \hat{S}_0)$ is Hecke pair where $\hat{\Gamma}_0 = j(\Gamma_0)$ and $\hat{S}_0 = \pi^{-1}(S_0)$. We denote their corresponding Hecke rings by $\hat{L}(q) = \hat{L}_p^n(q)$ and $\hat{L}_0 = \hat{L}_{p,0}^n$, respectively. Also $(\hat{\Gamma}(q), \hat{S}^2(q))$ and $(\hat{\Gamma}_0, \hat{S}_0^2)$ are Hecke pairs, where $\hat{S}^2(q) = \pi^{-1}(S^2(q))$ and $\hat{S}_0^2 = \pi^{-1}(S_0^2)$, and we denote their corresponding Hecke rings by $\hat{L}^2(q) = \hat{L}_p^{n,2}(q)$ and $\hat{L}_0^2 = \hat{L}_{p,0}^{n,2}$, which are the even subrings of $\hat{L}(q)$ and $\hat{L}_0$, respectively.

Hecke pairs $(\hat{\Gamma}(q), \hat{S}(q))$ and $(\hat{\Gamma}_0, \hat{S}_0)$ also satisfy (1.1). So we have an injective homomorphism

$$\hat{\varepsilon}_{0,q} : \hat{L}(q) \to \hat{L}_0, \ \hat{\varepsilon}_{0,q}(\Sigma a_i(\hat{\Gamma}(q)\zeta_i)) = \Sigma a_i(\hat{\Gamma}_0\zeta_i) \tag{2.2}$$

where ζ_i are chosen to be in $\hat{S}_0$.

For each $g \in S_0$, the lifting homomorphism $\rho_g : \Gamma_0^g \to \hat{G}$ is trivial [4]. From this we obtain a surjective ring homomorphism

$$\pi_k : \hat{L}_0 \to L_0, \ \pi_k(\Gamma_0\zeta\Gamma_0) = s(\zeta)^{-2k}(\Gamma_0 g\Gamma_0) \tag{2.3}$$

where $k = m/2$, m an odd integer ≥ 1, $\zeta = (g, \alpha(Z)) \in \hat{S}_0$, and $s(\zeta) = \alpha(Z)/|\alpha(Z)|$.

Let $K_s = \mathrm{diag}(I_{n-s}, pI_s, p^2I_{n-s}, pI_s) \in S_0$ for $s = 0, 1, \ldots, n$. If we let $D_s = \mathrm{diag}(p^2I_{n-s}, pI_s)$, then $K_s = (p^2D_s^*, 0; 0, D_s)$. So $K_s \in S_0^2$. Let $T_s = (\Gamma(q)K_s\Gamma(q)) \in L(q)$ and let $\mathbb{L}(T) = L_p^n(T)$ be the subring $\mathbb{C}[T_0, \ldots, T_{n-1}, T_n, \ldots, T_{n-1}, T_n^{\pm 1}]$ of $L(q)$. Similarly, let $\hat{T}_s = (\hat{\Gamma}(q)\hat{K}_s\hat{\Gamma}(q)) \in \hat{L}(q)$, where $\hat{K}_s = (K_s, p^{(n-s)/2}) \in \hat{S}_0^2$ for $s = 0, 1, 2, \ldots, n$, and let $\hat{L}(T) = \hat{L}_p^n(T)$ be the subring $\mathbb{C}[\hat{T}_0, \ldots, \hat{T}_{n-1}, \hat{T}_n^{\pm 1}]$ of $\hat{L}(q)$. We define

$$\mathbb{L}(T) = L_p^n(T) = \pi_k \circ \hat{\varepsilon}_{0,q}(\hat{L}(T)). \tag{2.4}$$

$\mathbb{L}(T)$ is a subring of L_0^2.

Let S_n be the permutation group on $\{x_1, x_2, \ldots, x_n\}$. Let $\Omega = \{\Omega_0, \Omega_1, \ldots, \Omega_n\}$ where Ω_i are automorphisms of $\mathbb{C}(X)$ defined by : For $i = 0$, $\Omega_0(x_0) = -x_0, \Omega_0(x_j) = x_j$ for $j = 1, \ldots, n$; For $i \neq 0$, $\Omega_i(x_0) = x_0 x_i$, $\Omega_i(x_i) = x_i^{-1}$, and $\Omega_i(x_j) = x_j$ for $j \neq 0, i$. Let $W = W_n$ be the group of automorphisms of $\mathbb{C}(\underline{x})$ generated by S_n and Ω. Let $\mathbb{C}(\underline{x})^W$ be the subring of $\mathbb{C}(\underline{x})$ consisting of all W-invariant elements. Then [7]

$$\varphi : \mathbb{L}(T) \to \mathbb{C}(\underline{x})^W \tag{2.5}$$

is an isomorphism.

Let $\Delta(\underline{x}) = \Delta^n(x) = (\underline{x} x_0^2 x_1 \cdots x_n)^{\pm 1}$, and $R_i(\underline{x}) = R_i^n(\underline{x}) = s_i(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1})$ for $i = 0, \ldots, 2n$, where $s_i(\cdots)$ denotes the elementary symmetric polynomial of degree i in the variables involved. It is known [2] that $\mathbb{C}(\underline{x})^W$ is generated by $\Delta(\underline{x})^{\pm 1}$ and $R_i(\underline{x})$, $i = 1, \ldots, n$.

Let

$$r(z) = r_p^n(z) = \Pi_{1 \leq j \leq n}(1 - x_j^{-1} z)\ (1 - x_j z) = \sum_{i=0}^{2n} (-1)^i R_i(\underline{x}) z^i. \tag{2.6}$$

Obviously, W only permutes the factors of $r(z)$ and hence the coefficients $R_i(\underline{x})$ are W-invariant. So there exist $R_i(T) \in \mathbb{L}(T)$ such that $\varphi(R_i(T)) = R_i(\underline{x}), = 0, \ldots, 2n$. Let $\Delta(T) = \pi_k \circ \hat{\varepsilon}_{0,q}(\hat{T}_n) = p(\Gamma_0 I_{2n} \Gamma_0)$. Then $\varphi(\Delta(T)^{-1}) = p^{<n>} \Delta(\underline{x})^{-1}$. Therefore we obtain

$$\mathbb{L}(T) = \mathbb{C}[R_1(T), \ldots, R_n(T),\ \Delta(T)^{\pm 1}]. \tag{2.7}$$

§3. Siegel modular forms of half integral weight

Let n, q be a positive integers with $4|q$. Let χ be a Dirichlet character modulo q. Let k be a positive half integer, i.e., $k = m/2$ for some positive odd integer m. For a complex valued function F on H_n and $\zeta = (g, \alpha(Z)) \in \hat{G}$, we set

$$(F|_k \zeta)(Z) = r(g)^{(nk/2) - <n>} \alpha(Z)^{-2k} F(g < Z >), \quad Z \in H_n. \tag{3.1}$$

Since the map $Z \to g < Z >$ is an analytic automorphism of H_n and $\alpha(Z) \neq 0$ on H_n, $F|_k \zeta$ is holomorphic on H_n if F is. Also from the definition follows that $F|_k \zeta_1|_k \zeta_2 = f|_k \zeta_1 \zeta_2$ for $\zeta_1, \zeta_2 \in \hat{G}$.

A function $F : H_n \to \mathbb{C}$ is called a Siegel modular form of degree n, weght k, level q, with character χ if the following conditions hold: (i) F is holomorphic on H_n, (ii) $F|_k \hat{M} = \chi(\det D_M) \cdot F$ for every $\hat{M} = (M, j(M, Z)) \in \hat{\Gamma}(q)$, $M \in \Gamma(q)$, and (iii) $F|_k(m, \alpha(z))$ is bounded as $\operatorname{Im} z \to \infty$, $z \in H_1$, for every $(m, \alpha(z)) \in \hat{G}_1$ with $m \in \Gamma^1 = SL_2(\mathbb{Z})$ if $n = 1$. It is known [8] that the boundedness condition

(iii) follows from (i) and (ii) for $n \geq 2$. We denote the set of all such Siegel modular forms by $M_k^n(q,\chi)$. This is a finite dimensional vector space over $\mathbb{C}$.

A function $F : H_n \to \mathbb{C}$ is called an even or odd modular form of degree n if F satisfies (i), (ii)* $(\det D_M)^S F(M < Z >) = F(Z)$, $Z \in H_n$ for every $M \in \Gamma_0$, where $s = 0$ for even and $s = 1$ for odd modular forms, and (iii) $F(z)$ is bounded as $\operatorname{Im} z \to \infty$, $z \in H_1$. We denote the set of all such even modular forms by M_0^n and odd modular forms by M_1^n. They are also vector spaces over $\mathbb{C}$.

Let $F \in M_k^n(q,\chi)$ and $\chi(-1) = (-1)^s$ for $s = 0$ or 1. For $M \in \Gamma_0$, we have $\hat{M} = (M, j(M,Z)) = (M,1)$ and $\det D_M = \pm 1$. So F satisfies (ii)* and hence

$$M_k^n(q,\chi) \subset M_s^n \quad \text{if} \quad \chi(-1) = (-1)^s. \tag{3.2}$$

For $F \in M_k^n(q,\chi)$ and $\hat{X} = \Sigma a_i(\hat{\Gamma}(q)\zeta_i) \in \hat{L}^2(q)$, we set

$$F|_{k,\chi}\hat{X} = \Sigma a_i \chi(\det A_i) \cdot F|_k \zeta_i, \tag{3.3}$$

where $\zeta_i \in \hat{S}^2(q)$, $\pi(\zeta_x) = (A_i, B_i; C_i, D_i) \in S^2(q)$. There is a good reason for using the even subring $\hat{L}^2(q)$ instead of $\hat{L}(q)$; the action of double cosets in $\hat{L}(q) - \hat{L}^2(q)$ on $M_k^n(q,\chi)$ is trivial, i.e., for $F \in M_k^n(q,\chi)$ and $\hat{X} = (\hat{\Gamma}(q) g \hat{\Gamma}(q)) \in \hat{L}(q) - \hat{L}^2(q)$, $F|_{k,\chi}\hat{X} = 0$.

As for $F \in M_s^n$ and $X = \Sigma a_i(\Gamma_0 g_i) \in L_0$, we set

$$F|_{k,\chi}\chi = \Sigma a_i \chi(\det A_i) \cdot F|_k \tilde{g}_i \tag{3.4}$$

where $\tilde{g}_i = (g_i, (\det g)^{-1/4} |\det D_i|^{1/2}) \in \hat{S}_0$ with $g_i = (A_i, B_i; 0, D_i) \in S_0$, and $\chi(-1) = (-1)^s$.

$\hat{X}$ and X acting on modular spaces as above are called Hecke operators. It follows from definitions that $F|_{k,\chi}\hat{X} \in M_k^n(q,\chi)$ if $F \in M_k^n(q,x)$ and $F|_{k,\chi}\hat{X}|_{k,\chi}\hat{Y} = F|_{k,\chi}\hat{X}.\hat{Y}$ for any $\hat{X}, \hat{Y} \in \hat{L}^2(q)$. Similarly, for $F \in M_s^n$ and $X, Y \in L_0$, we have $F|_{k,\chi}X \in M_s^n$ and $F|_{k,\chi}X|_{k,\chi}Y = F|_{k,\chi}X \cdot Y$, where $\chi(-1) = (-1)^s$.

Theorem 3.1. *Let $\chi(-1) = (-1)^s$. Then for $F \in M_k^n(q,\chi) \subset M_s^n$ and $\hat{X} \in \hat{L}^2(q)$,*

$$F|_{k,\chi}\hat{X} = F|_{k,\chi}\pi_k \circ \hat{\varepsilon}_{0,q}(\hat{X}). \tag{3.5}$$

(See (2.2) and (2.3) for $\varepsilon_{0,q}$ and π_k.)

Proof. Let $\hat{X} = \Sigma a_i(\Gamma(q)\zeta_i) \in \hat{L}^2(q)$ such that $\zeta_i = (g_i, \alpha(Z)) \in \hat{S}_0^2$ where $g_i = (p^{\delta_i} D_i^*, B_i; 0, D_i)$ and $\alpha(Z) = t_i p^{-n\delta_i/4} (\det D_i)^{1/2}$ for some $t_i \in \mathbb{C}$ with $|t_i| = 1$, $\delta_i \in 2\mathbb{Z}$. We adapt the usual branch for $(\det D_i)^{1/2}$ in case that $\det D_i < 0$. Since

$j(M,Z)=1$ for $M\in\Gamma_0$, $\pi_k\circ\hat{\varepsilon}_{0,q}(\hat{X})=\Sigma a_i(t_i\varepsilon_i)^{-2k}(\Gamma_0 g_i)$ where $\varepsilon_i=1$ or $\sqrt{-1}$ according to $\det D_1>0$ or <0. So from (3.4) follows that

(3.6)

$$\begin{aligned}&F|_{k,\chi}\pi_k\circ\hat{\varepsilon}_{0,q}(\hat{X})=\Sigma a_i(t_i\varepsilon_i)^{-2k}F|_k\tilde{g}_i\\ &=\Sigma a_i\chi(\det A_i)(t_i\varepsilon_i)^{-2k}(p^{\delta_i})(nk/2)-<n>(p^{-n\delta_i/4}|\det D_i|)^{-2k}F(g_i<Z>)\\ &=\Sigma a_i\chi(\det A_i)(p^{\delta_i})^{nk-<n>}(t_i(\det D_i)^{1/2})^{-2k}F(g_i<Z>)\end{aligned}$$

where $A_i=p^{\delta_i}D_i^*$. On the other hand, from (3.1) and (3.3) follows

$$F|_{k,\chi}\hat{X}=\Sigma a_i\chi(\det A_i)(p^{\delta_i})^{nk-<n>}(t_i(\det D_i)^{1/2})^{-2k}F(g_i<Z>). \tag{3.7}$$

This completes the proof.

Note that $\pi_k\circ\hat{\varepsilon}_{0,q}(\hat{X})\in L_0^2$ if $\hat{X}\in\hat{L}^2(q)$. So this theorem enables us to discuss the action of Hecke operators in $\hat{L}^2(q)$ in terms of those in L_0^2.

§4. The Siegel operator and Ψ operator

Let n,q,χ be as above and p be a prime such that $(p,q)=1$.

Let $F\in M_s^n$. We define $\Phi:M_s^n\to M_s^{n-1}$ by

$$(\Phi F)(Z')=\lim_{\lambda\to+\infty}F\left(\begin{pmatrix}Z' & 0\\ 0 & i\lambda\end{pmatrix}\right),\ Z'\in H_{n-1}\text{ and }\lambda>0. \tag{4.1}$$

Φ is well defined and is called the Siegel operator ($M_s^0=\mathbb{C},\ H_0=\{0\}$). Every $F\in M_s^n$, hence every $F\in M_k^n(q,\chi)$, has a Fourier expansion of the form

$$F(Z)=\sum_n f(N)e(NZ),\quad Z\in H_n, \tag{4.2}$$

where N runs over the set $\mathcal{N}_n=\{N=(b_{ij})\in M_n(Q);t_{N=N}\geq 0,\ b_{ii},2b_{ij}\in\mathbb{Z}\}$. ($N\geq 0$ means that N is semi-positive definite.) Then from (4.1) follows that

$$(\Phi F)(Z')=\sum_{N'}F\left(\begin{pmatrix}N' & 0\\ 0 & 0\end{pmatrix}\right)e(N'Z'), \tag{4.3}$$

where N' runs over $\mathcal{N}_{n-1}$ ($\mathcal{N}_0=\{0\}$). It also follows from (4.1) and (4.2) that $\Phi F\in M_k^{n-1}(q,\chi)$ if $F\in M_k^n(q,\chi)$.

Let $X=\Sigma a_i(\Gamma_0^n g_i)\in L_0^n$ where $g_i=(p^{\delta_i}D_i^*,B_i;0,D_i)\in S_0^n$. By multiplying $(U_i^*,0;0,U_i)\in\Gamma_0^n$ for a suitable $U_i\in GL_n(\mathbb{Z})$ from the left of g_i, we may assume that all the D_i are of the form $D_i=\begin{pmatrix}D_i & *\\ 0 & p^{d_i}\end{pmatrix}$, $d_i\in\mathbb{Z}$, where $D_i\in\Delta^{n-1}$ is upper triangular. We set

$$\Psi(X,u)=\Sigma a_i u^{-\delta_i}(up^{-n})^{d_i}(\Gamma_0^{n-1}g_i')\in L_0^{n-1}[u^{\pm 1}] \tag{4.4}$$

where u is an independent variable and $g_i' = (p^{\delta_i}(D_i')^*, B_i'; 0, D_i')$. B_i' denotes the block of size $(n-1)\times(n-1)$ in the upper left corner of B_i. If $n=1$, we set $\Psi(X,u) = \Sigma a_i u^{-\delta_i}(up^{-1})^{d_i}$. Note that δ_i, d_i are uniquely determined by the left coset $(\Gamma_0^n g_i)$ for each i.

Ψ is a well defined ring homomorphism: $L_0^n \to L_0^{n-1}[u^{\pm 1}]$ according to the following lemma:

Lemma 4.1. *Let $g_1, g_2 \in S_0^n$ such that $(\Gamma_0^n g_1) = (\Gamma_0^n g_2)$. Then $g_1', g_2' \in S_0^{n-1}$ such that $(\Gamma_0^{n-1} g_1) = (\Gamma_0^{n-1} g_2)$, where g_i, g_i' are as above for $i = 1, 2$.*

Proof. $J_n[g_i] = p^{\delta_i} J_n$ and $(\Gamma_c^n g_i) = (\Gamma_0^n g_2)$ imply $\delta_1 = \delta_2$. Let it be δ. It follows immediately from the definition that $J_{n-1}[g_i] = p^{\delta} J_{n-1}$ so that $g_i \in S_0^{n-1}$. Clearly $J_{n-1}[g_1'(g_2')^{-1}] = J_{n-1}$. We will show that $g_1'(g_2')^{-1}$ is integral, from which follows $g_1(g_2)^{-1} \in \Gamma_0^n$ and the lemma. For $g_1 g_2^{-1} = ((D_1 D_2^{-1})^*, D_1^{*\cdot t} D_2 B_2 D^{-1} + B_1 D_2^{-1}; 0, D_1 D_2^{-1}) \in \Gamma_0^n$, we get $\begin{pmatrix} D_1'(D_2')^{-1} & * \\ 0 & 1 \end{pmatrix} \in GL_n(\mathbb{Z})$ with $D_1'(D_2')^{-1} \in GL_{n-1}(\mathbb{Z})$. Because $g_1'(g_2')^{-1} = ((D_1'(D_2'))^*, (D_1')^{*\cdot t} D_2' B_2' (D_2')^{-1} + B_1'(D_2')^{-1}; 0, D_1'(D_2')^{-1}$, it is enough to show that the upper right block $(D_1')^{*\cdot t} D_2' B_2' (D_2')^{-1} + B_1'(D_2')^{-1}$ is integral. Let $U = D_2 D_1^{-1} \in GL_n(\mathbb{Z})$ and $U' = D_2'(D_1')^{-1} \in GL_{n-1}(\mathbb{Z})$. Then we have $D_1^{*\cdot t} D_2 B_2 D_2^{-1} + B_1 D_2^{-1} = ({}^t U B_2 + B_1) D_2^{-1}$, which is integral. Similary $(D_1')^{*\cdot t} D_2' B_2' (D_2')^{-1} + B_1'(D_2')^{-1} = ({}^t U' B_2' + B_1')(D_2')^{-1}$. But we have, on the other hand, $(({}^t U B_2 + B_1) D_2^{-1})' = ({}^t U' B_2' + B_1')(D_2')^{-1}$. So $(D_1')^{*\cdot t} D_2' B_2' (D_2')^{-1} + B_1'(D_2')^{-1}$ is integral.

Let $\mathbb{C}^n(\underline{x})$ be as in (1.4). For $n \geq 2$, we define a ring homomorphism

$$\mu = \mu_{n,u} : \mathbb{C}^n(\underline{x}) \to \mathbb{C}^{n-1}(\underline{x})[u^{\pm 1}] \tag{4.5}$$

by $\mu(x_0) = x_0 u^{-1}, \mu(x_n) = u$, and $\mu(x_i) = x_i$ for $i \neq 0, n$. For $n = 1$, we define $\mu(x_0) = u^{-1}$ and $\mu(x_1) = u$. ($\mathbb{C}^0(\underline{x}) = \mathbb{C}$.) It is known [2] that

$$\mu \circ \varphi_n = (\varphi_{n-1} \times 1_u) \circ \Psi(-, u) \tag{4.6}$$

where φ_n is the homomorphism (1.4) and $\varphi_{n-1} \times 1_u$ is the ring homomorphism: $L_0^{n-1}[u^{\pm 1}] \to \mathbb{C}^{n-1}(\underline{x})[u^{\pm 1}]$ defined by $\varphi_{n-1} \times 1_u = \varphi_{n-1}$ on L_0^{n-1}, $(\varphi_{n-1} \times 1_u)(u^{\pm 1}) = u^{\pm 1}$.

§5. Main theorems

We now prove our main theorems on the commuting relation between Hecke operators and the Siegel operator (Theorem 5.1.) and the surjectivity of Ψ-operator (Theorem 5.2.). Let n, q be positive integers such that q is divisible by 4 and let p be a prime relatively prime to q. Let χ be a Dirichlet character modulo q and let k be a positive half integer.

Theorem 5.1. *Let $F \in M_k^n(q,\chi)$ and let $\hat{X} \in \hat{L}^2(q)$. Then we have*

$$\Phi(F|_{k,\chi}\hat{X}) = (\Phi F)|_{k,\chi}\Psi(Y, p^{n-k}\chi(p)^{-1}) \tag{5.1}$$

where $Y = \pi_k \circ \hat{\varepsilon}_{0,q}(\hat{X}) \in L_0^2 = L_{p,0}^{n,2}$. (If $n = 1$, then the right hand side of (5.1) is the action of $L_{p,0}^{0,2} = \mathbb{C}$ on $M_k^0(q,\chi) = \mathbb{C}$, which is just the multiplication of complex numbers.)

Proof. Let $Y = \Sigma a_i(\Gamma_0^n g_i) \in L_0^2$ with $g_i = (p^{\delta_i}D_i^*, B_i; 0, D_i) \in S_0^2$ and let $F(Z) = \Sigma_N f(N)e(NZ) \in M_k^n(q,\chi) \subset M_s^n$ where N runs over $\mathcal{N}_n$ and $\chi(-1) = (-1)^s$ (see (3.2) and (4.2)). From (3.4) we get

$$F|_{k,\chi}Y = \Sigma a_i\chi(\det(p^{\delta_i}D_i^*)) \cdot F|_k\tilde{g}_i \tag{5.2}$$

where $\tilde{g}_i = (g_i, (p^{\delta_i})^{-n/4}|\det D_i|^{1/2})$. So from (3.1) follows

(5.3)

$$\begin{aligned} F|_{k,\chi}Y &= \Sigma a_i\chi(\det(p^{\delta_i}D_i^*)(p^{\delta_i})^{nk-<n>}|\det D_i|^{-k\cdot}F(g_i < Z >) \\ &= \Sigma a_i\chi(\det(p^{\delta_i}D_i^*)(p^{\delta_i})^{nk-<n>}|detD_i|^{-k} \\ &\quad \Sigma_N f(N)e(Np^{\delta_i}D_i^*ZD_i^{-1} + NB_iD_i^{-1}) \\ &= \Sigma_{i,N}a_i\chi(\det(p^{\delta_i}D_i^*)(p^{\delta_i})^{nk-<n>}|\det D_i|^{-k} \\ &\quad f(N)e(p^{\delta_i}D_i^{-1}ND_i^*Z + NB_iD_i^{-1}). \end{aligned}$$

Let $N = \begin{pmatrix} N' & * \\ * & t \end{pmatrix}$ and $Z = \begin{pmatrix} Z' & 0 \\ 0 & i\lambda \end{pmatrix}$. Since we may assume that $D_i = \begin{pmatrix} D_i & * \\ 0 & p^{\delta_i} \end{pmatrix}$, $\mathrm{Im}(\sigma(p^{\delta_i}D_i^{-1}ND_i^*Z + NB_iD_i^{-1})) = \alpha_i + \lambda t p^{\delta_i - 2d_i} \to \infty$ as $\lambda \to \infty$ when $t > 0$, where α_i is a real number depending on i. Here $Z' \in H_{n-1}$ and $t \geq 0$ because ${}^tN = N \geq 0$. Therefore if $t > 0$, then $e(p^{\delta_i}D_i^{-1}ND_i^*Z + NB_iD_i^{-1}) \to 0$ as $\lambda \to \infty$. If $t = 0$, then $N = \begin{pmatrix} N' & 0 \\ 0 & 0 \end{pmatrix}$ and hence $N \in \mathcal{N}_{n-1}$ and $(p^{\delta_i}D_i^{-1}ND_i^*Z)' = p^{\delta_i}(D_i')^{-1}N'(D_i')^*Z'$ and $(NB_iD_i^{-1})' = N'B_i'(D_i')^{-1}$. From this we obtain

$$\begin{aligned} \Phi(F|_{k,\chi}Y) &= \sum_{i,N'} a_i\chi(\det(p^{\delta_i}D_i^*)(p^{\delta_i})^{nk-<n>}|\det D_i|^{-k} \\ &\quad \cdot f\begin{pmatrix} N' & 0 \\ 0 & 0 \end{pmatrix} e(p^{\delta_i}(D_i')^{-1}N'(D_i')^*Z' + N'B_i'(D_i')^{-1}). \end{aligned} \tag{5.4}$$

On the other hand, from (3.4) and (4.4) we obtain

$$\begin{aligned} &(\Phi F)|_{k,\chi}\Psi(Y, p^{n-k}\chi(p)^{-1}) \\ &= (\Phi F)|_{k,\chi}\Sigma a_i(p^{n-k}\chi(p)^{-1})^{-\delta_i}(p^{n-k}\chi(p)^{-1}p^{-n})^{d_i}(\Gamma_0^{n-1}g_i') \\ &= \Sigma a_i(p^{n-k}\chi(p)^{-1})^{-\delta_i}(p^{-k}\chi(p)^{-1})^{d_i}\chi(\det(p^{\delta_i}(D_i)^*)) \cdot \Phi F|_k\tilde{g}_i' \end{aligned} \tag{5.5}$$

where $\tilde{g}'_i = ((g'_i, (p^{\delta_i})^{-(n-1)/4}|\det D'_i|^{1/2}$. So from (4.3) and $\det D_i = p^{d_i}(\det D'_i)$ follows that

(5.6)
$$
\begin{aligned}
&(\Phi F)|_{k,\chi}\Psi(Y, p^{n-k}\chi(p)^{-1}) \\
&= \sum_{i,N'} a_i (p^{\delta_i})^{k-n}\chi(p)^{\delta_i - d_i}\chi(\det(p^{\delta_i}(D'_i)^*))(p^{d_i})^{-k}(p^{\delta_i})^{(n-1)k-<n-1>} \\
&\qquad \cdot |\det D'_i|^{-k} f\begin{pmatrix} N' & 0 \\ 0 & 0 \end{pmatrix} e(N' g'_i < Z' >) \\
&= \sum_{i,N'} a_i \chi(\det(p^{\delta_i} D_i^*))(p^{\delta_i})^{nk-<n>}|\det D_i|^{-k} f\begin{pmatrix} N' & 0 \\ 0 & 0 \end{pmatrix} \\
&\qquad \cdot e(p^{\delta_i}(D'_i)^{-1}N'(D'_i)^* Z' + N' B'_i (D'_i)^{-1})
\end{aligned}
$$

where N' runs over $\mathcal{N}_{n-1}$. This completes the proof.

For a space $M_k^n(1, \chi_0)$ with an integral weight k, the level $q = 1$, and the trivial character χ_0, this result was proved by Zharkovskaya [1]. Andrianov [2] generalized her result to an arbitrary level q. Theorem 5.1. is an extension of their results to half integral weight Siegel modular forms of an arbitrary level q and an arbitrary character χ mudulo q.

Theorem 5.2. *The map $\Psi(-, p^{n-k}\chi(p)^{-1}) : \mathbb{L}^n(T) \to \mathbb{L}^{n-1}(T)$ is a surjective ring homomorphism, where $\mathbb{L}^n(T) = \mathbb{L}_p^n(T)$.*

Proof. See (2.4) for $\mathbb{L}^n(T)$. According to (4.6) and (2.5), it suffices to show that $\mu(\mathbb{C}^n(\underline{x})^W) = \mathbb{C}^{-1}(\underline{x})^W$ where $\mu = \mu_{n,u}$ with $u = p^{n-k}\chi(p)^{-1}$ and $\mathbb{C}^n(\underline{x})^W = \mathbb{C}^n(\underline{x})^{W_n}$ (1.4). $\mathbb{C}^n(\underline{x})^W$ is generated by $\Delta^n(\underline{x})^{\pm 1}$ and $R_i^n(\underline{x})$, $1 \le i \le n$. Similary $\mathbb{C}^{n-1}(\underline{x})^W$ is generated by $\Delta^{n-1}(\underline{x})^{\pm 1}$ and $R_i^{n-1}(\underline{x})$, $1 \le i \le n-1$. We now consider $\mu(r(z))$ where $r(z) = r_p^n(z)$ is the polynomial (2.6) over $\mathbb{C}^n(\underline{x})^W$ and μ is applied to its coefficients. From (4.5) we get

$$
\begin{aligned}
\mu(r(z)) &= \sum_{1\le j\le n-1}(1 - x_j^{-1}z)(1 - x_j z)(1 - u^{-1}z)(1 - uz) \\
&= (1 - u^{-1}z)(1 - uz)\sum_{i=0}^{2(n-1)}(-1)^i \mu(R_i^{n-1}(\underline{x})z^i \\
&= \sum_{i=0}^{2n}(-1)^i \mu(R_i^n(\underline{x}))z^i.
\end{aligned} \tag{5.7}
$$

Let $[(1 - u^{-1}z)(1 - uz)]^{-1} = \sum_{b=0}^{\infty}(-1)^b c_b(u)z^b$ be the formal power series expan-

sion in z over $\mathbb{C}$. Then

(5.8)
$$\sum_{i=0}^{2(n-1)}(-1)^i R_i^{n-1}(\underline{x})z^i = \left(\sum_{i=0}^{2n}(-1)^i\mu(R_i^n(\underline{x}))z^i\right)\left(\sum_{b=0}^{\infty}(-1)^b c_b(u)z^b\right)$$
$$= \sum_{i=0}^{\infty}(-1)^i(\sum_{a+b=1}\mu(R_a^n(\underline{x}))c_b(u))z^i.$$

Hence $R_i^{n-1}(\underline{x}) = \mu(\sum_{a+b=i} R_a^n(\underline{x})c_b(u))$. Also from the definition of μ follows immediately that $\Delta^{n-1}(\underline{x})^{\pm 1} = \mu(u^{\pm 1}\Delta^n(\underline{x})^{\pm 1})$. Therefore we get $\mathbb{C}^{n-1}(\underline{x})^W \subset \mu(\mathbb{C}^n(\underline{x})^W)$. The other inclusion is obvious from the definition.

Theorem 5.2. is an analogue of Andrianov's result [1] that Ψ is surjective ring homomorphism from $L_0^{n,2}$ onto $L_0^{n-1,2}$ and is very useful for the decomposition of half integral weight Siegel modular forms and the representations of positive definite quadratic forms with odd number of variables.

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