INVARIANCE OF THE SPACE OF THETA-SERIES UNDER THETA OPERATORS

Myung-Hwan Kim

Department of Mathematics
Seoul National University
Seoul 151-742, Korea

§0. Introduction and Notations

One of the most powerful methods of studying representations of quadratic forms by forms is via theta-series. Many authors did a great deal of work in this direction after Siegel's pioneering work[Si1]. Unfortunately, however, most of them worked in the case when the representing quadratic form has an even number of variables. The reason is simple : quadratic forms with even number of variables are associated to integral weight theta-series while those with odd number of variables to half integral weight theta-series whose transformation formulas involve branch problems.

In this article, we study the behavior of half integral weight theta-series under theta operators. Theta operators are very important in the study of theta-series in connection with Hecke operators. Andrianov[A1] proved that the space of integral weight theta-series is invariant under the action of theta operators. We prove that his statement can be extened for half integral weight theta-series with a slight modification. By using this result one can

This was partially supported by the Basic Science Research Institute programs, Ministry of Education, 1990

Submitted to Bulletin of Korean Mathematical Society

prove that the space of theta-series is invariant under the action of Hecke operators as Andrianov did for intrgral weight theta-series[A1].

Let $\mathbf{Z}, \mathbf{R}$, and $\mathbf{C}$ be the ring of rational integers, the field of real numbers, and the field of complex numbers, respectively.

For an $m \times m$ matrix g and an $m \times n$ matrix h, let $g[h] = {}^t hgh$, where ${}^t h$ is the transpose of h. For a $2n \times 2n$ matrix g, let A_g, B_g, C_g, and D_g be the $n \times n$ block matrices in the upper left, upper right, lower left, and lower right corners of g, respectively. Let $\mathrm{diag}(N_1, N_2, \ldots, N_r)$ be the matrix with block matrices $N_1, N_2, \ldots, N_r$ on its main diagonal and zeroes outside. Let $\mathcal{N}_m$ be the set of all semi-positive definite (eigenvalues ≥ 0), semi-integral (diagonal entries and twice of nondiagonal entries are integers), symmetric $m \times m$ matrices, and $\mathcal{N}_m^+$ be its subset consisting of positive definite (eigenvalues > 0) matrices.

Let

$$G_n = GSp_n^+(\mathbf{R}) = \{g \in M_{2n}(\mathbf{R}) \ ; \ J_n[g] = rJ_n, \ r > 0\}$$
$$\Gamma^n = Sp_n(\mathbf{Z}) = \{M \in M_{2n}(\mathbf{Z}) \ ; \ J_n[M] = J_n\},$$

where $J_n = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$ and $r = r(g) \in \mathbf{R}$ is determined by g. And let

$$\mathcal{H}_n = \{Z = X + iY \in M_n(\mathbf{C}) \ ; \ {}^t Z = Z, Y > 0\}$$

where $Y > 0$ means Y is positive definite. For $g \in G_n$ and $Z \in \mathcal{H}_n$, we set

$$g\langle Z \rangle = (A_g Z + B_g)(C_g Z + D_g)^{-1} \in \mathcal{H}_n.$$

For $g \in M_n(\mathbf{C})$, $e(g) = \exp(2\pi i \sigma(g))$ where $\sigma(g)$ is the trace of g. Finally, we let $\langle n \rangle = n(n+1)/2$ for $n \in \mathbf{Z}$.

For other standard terminologies and basic facts, we refer the readers [A2], [M], [O].

§1. Siegel Modular Forms

Let n, q be positive integers and p be a prime relatively prime to q. Let

$$\Gamma_0^n(q) = \{M \in \Gamma^n \ ; \ C_M \equiv 0 \pmod q\}$$
$$\Gamma_0^n = \{M \in \Gamma^n \ ; \ C_M = 0\}$$
$$L^n = L_p^n = \{g \in M_{2n}(\mathbf{Z}[p^{-1}]) \ ; \ J_n[g] = p^\delta J_n, \delta \in \mathbf{Z}\}$$
$$L_0^n = L_{0,p}^n = \{g \in L^n \ ; \ C_g = 0\}$$
$$E^n = E_p^n = \{g \in L^n \ ; \ \delta(g) \in 2\mathbf{Z}\}$$
$$E_0^n = E_{0,p}^n = E^n \cap L_0^n$$

where $\delta = \delta(g) \in \mathbf{Z}$ is determined by g. Let

$$\hat{G}_n = \{(g, \alpha(Z)) \ ; \ g \in G_n\}$$

where $\alpha(Z)$ is a holomorphic function on $\mathcal{H}_n$ satisfying $\alpha(Z)^2 = t(\det g)^{-1/2} \cdot \det(C_g Z + D_g)$ for some $t \in \mathbf{C}$, $|t| = 1$. $\hat{G}_n$ is a multiplicative group under the multiplication $(g, \alpha(Z)) \cdot (h, \beta(Z)) = (gh, \alpha(h\langle Z\rangle)\beta(Z))$ and is called the universal covering group of G_n. This group was introduced by Shimura[S] for $n = 1$ and then generalized by Zhuravlev[Z1,2] for arbitrary n. Let $\gamma : \hat{G}_n \to G_n$ be the projection $\gamma(g, \alpha(z)) = g$. We define an action of $\hat{G}_n$ on $\mathcal{H}_n$ by

$$\zeta\langle Z\rangle = \gamma(\zeta)\langle Z\rangle$$

for $\zeta \in \hat{G}_n$, $Z \in \mathcal{H}_n$.

From now on, we let q be a positive integer such that $4|q$. Let

$$\theta^n(Z) = \sum_M e({}^tMMZ) = \sum_N e(Z[N]), \ Z \in \mathcal{H}_n, \tag{1.1}$$

where $M(N,$ resp.) runs over all the integral row (column, resp.) matrices of length n. $\theta^n(Z)$ is called the standard theta-function of degree n. For $M \in \Gamma_0^n(q)$, we define

$$j(M, Z) = \theta^n(M\langle Z\rangle)/\theta^n(Z), Z \in \mathcal{H}_n. \tag{1.2}$$

It is well known [Z1] that $(M, j(M,Z)) \in \hat{G}_n$. So the map $j : \Gamma_0^n(q) \to \hat{G}_n$ defined by $j(M) = (M, j(M,Z))$ is a well defined injective homomorphism such that $\gamma \circ j$ is the identity map on $\Gamma_0^n(q)$. We denote $\hat{\Gamma}_0^n(q) = j(\Gamma_0^n(q)), \hat{\Gamma}_0^n = j(\Gamma_0^n)$ and $\hat{L}_0^n = \gamma^{-1}(L_0^n)$.

Let χ be a Dirichlet character (mod q) and k be a positive half integer, i.e., $k = m/2$ for some positive odd integer m. For a complex valued function F on $\mathcal{H}_n$ and $\zeta = (g, \alpha(Z)) \in \hat{G}_n$, we set

$$(F|_k\zeta)(Z) = r(g)^{(nk/2)-\langle n\rangle}\alpha(Z)^{-2k}F(g\langle Z\rangle),\ Z \in \mathcal{H}_n. \tag{1.3}$$

Since the map $Z \to g\langle Z\rangle$ is an analytic automorphism of $\mathcal{H}_n$ and $\alpha(Z) \neq 0$ on $\mathcal{H}_n$, $F|_k\zeta$ is holomorphic on $\mathcal{H}_n$ if F is. Also from the definition follows that $F|_k\zeta_1|_k\zeta_2 = F|_k\zeta_1\zeta_2$ for $\zeta_1, \zeta_2 \in \hat{G}_n$.

A function $F : \mathcal{H}_n \to \mathbf{C}$ is called a Siegel modular form of degree n, weight k, level q, with character χ if the following conditions hold:

(i) F is holomorphic on $\mathcal{H}_n$,

(ii)$F|_k\hat{M} = \chi(\det D_M) \cdot F$ for every $\hat{M} = (M, j(M,Z)) \in \hat{\Gamma}_0^n(q)$, and

(iii) $F|_k(M, \alpha(z))$ is bounded as $\operatorname{Im} z \to \infty$, $z \in \mathcal{H}_1$, for every $(M, \alpha(z)) \in \gamma^{-1}(\Gamma^1)$ when $n = 1$.

It is known[Kö] that the boundedness condition (iii) follows from (i) and (ii) for $n \geq 2$. We denote the set of all such Siegel modular forms by $\mathcal{M}_k^n(q, \chi)$. This is known[Si2] to be a finite dimensional vector space over $\mathbf{C}$.

A function $F : \mathcal{H}_n \to \mathbf{C}$ is called an even or odd modular form of degree n if F satisfies (i), (iii), and

(ii)$'$ $(\det D_M)^s F(M\langle Z\rangle) = F(Z),\ Z \in \mathcal{H}_n$ for every $M \in \Gamma_0^n$, where $s = 0$ for even and $s = 1$ for odd modular forms.

We denote the set of all such even modular forms by $\mathcal{M}_0^n$ and odd modular forms by $\mathcal{M}_1^n$. They are also vector spaces over $\mathbf{C}$.

Let $F \in \mathcal{M}_k^n(q,\chi)$ and $\chi(-1) = (-1)^s$ for $s = 0$ or 1. For $M \in \Gamma_0^n$, we have $\hat{M} = (M, j(M,Z)) = (M,1)$ and $\det D_M = \pm 1$. So, F satisfies (ii)$'$ and hence

$$\mathcal{M}_k^n(q,\chi) \subset \mathcal{M}_s^n \text{ if } \chi(-1) = (-1)^s. \tag{1.4}$$

Let $X = (\Gamma_0^n g \Gamma_0^n)$ with $g \in L_0^n$. Then X can be written as a disjoint union of left cosets $(\Gamma_0^n g_i), g_i \in L_0^n, i = 1, 2, \ldots, \mu$. For $F \in \mathcal{M}_s^n$ and $X = (\Gamma_0^n g \Gamma_0^n)$, we set

$$F|_{k,\chi} X = \sum_{i=1}^{\mu} \chi(\det A_i) \cdot F|_k \tilde{g}_i, \tag{1.5}$$

where

$$\tilde{g}_i = (g_i, (\det g_i)^{-1/4} |\det D_i|^{1/2}) \in \hat{L}_0^n \tag{1.6}$$

with $g_i = \begin{pmatrix} A_i & B_i \\ 0 & D_i \end{pmatrix} \in L_0^n$ and $\chi(-1) = (-1)^s$.

If we let $\mathcal{L}_0^n$ be the vector space over $\mathbf{C}$ formally spanned by double cosets $(\Gamma_0^n g \Gamma_0^n)$, $g \in L_0^n$, then we can extend the action (1.5) to $\mathcal{L}_0^n$ by linearity. It follows from definitions that for $F \in \mathcal{M}_s^n$ and $X, Y \in \mathcal{L}_0^n$, we have $F|_{k,\chi} X \in \mathcal{M}_s^n$ and $F|_{k,\chi} X|_{k,\chi} Y = F|_{k,\chi} X \cdot Y$, where $\chi(-1) = (-1)^s$. So the action (1.5) is a well defined action of $\mathcal{L}_0^n$ on $\mathcal{M}_s^n$. The elements of $\mathcal{L}_0^n$ acting on $\mathcal{M}_s^n$ in this manner are called Hecke operators. It is well known[A2] that $\mathcal{L}_0^n$ is equipped with a ring structure and is called a Hecke ring.

§2. Theta-Series and Theta Operators

Let $Q \in \mathcal{N}_m^+$. The level q_Q of Q is defined to be the smallest positive integer satisfying that $q_Q(2Q)^{-1}$ is integral with even diagonals. It is known that q_Q is divisible by 4 if m is odd. We define the theta-series of degree n associated to Q by

$$\theta^n(Z,Q) = \sum_{X \in M_{m,n}(\mathbf{Z})} e(Q[X]Z), \quad Z \in \mathcal{H}_n. \tag{2.1}$$

If we let $r(N,Q) = |\{X \in M_{m,n}(\mathbf{Z}) | Q[X] = N\}|$ for each $N \in \mathcal{N}_n$, then

$$\theta^n(Z,Q) = \sum_{N \in \mathcal{N}_n} r(N,Q) e(NZ), \quad Z \in \mathcal{H}_n. \tag{2.2}$$

It is known [K] that for odd m

$$\theta^n(Z,Q) \in \mathcal{M}_k^n(q_Q, \chi_Q) \tag{2.3}$$

where $k = \frac{m}{2}$ is a half integer and χ_Q is a Dirichlet character modulo q_Q defined by

$$\chi_Q(M) = \left(\frac{2 \det 2Q}{|\det D_M|} \right)_{Jac}, \ {}^\forall M \in \Gamma_0^n(q). \tag{2.4}$$

For even m, (2.3) was proved by Andrianov and Maloletkin [A-M].

Let Θ_m^n be the vector space over $\mathbf{C}$ spanned by $\theta^n(Z,Q)$, $Q \in \mathcal{N}_m^+$ and let $\Theta_m^n(q,d)$ be its subspace spanned by $\theta^n(Z,Q)$, $Q \in \mathcal{N}_m^+$, $\det 2Q = d$ and $q_Q = q$ for given positive integers d, q. It is easy to check that

$$\Theta_m^n \subset \mathcal{M}_0^n \ \text{ and } \ \Theta_m^n(q,d) \subset \mathcal{M}_{m/2}^n(q,\chi), \tag{2.5}$$

where $\chi(M) = (\frac{2d}{|\det D_M|})_{Jac}$ for any $M \in \Gamma_0^n(q)$.

Let $\Theta_m^n[Q]$ be the subspace of Θ_m^n spanned by $\theta^n(Z,Q')$, $Q' \in gen(Q)$ where $gen(Q)$ is the genus of Q. Clearly

$$\Theta_m^n[Q] \subset \Theta_m^n(q,d) \subset \Theta_m^n \tag{2.6}$$

if $q = q_Q$ and $d = \det 2Q$.

We now define theta operators following Andrianov[A2] : p is still a prime relatively prime to q. Let $\alpha : L_0^m \to \mathbf{C}^\times$ be a character such that $\alpha(\Gamma_0^m) = 1$. For $X = (\Gamma_0^m g_0 \Gamma_0^m) \in \mathcal{L}_0^n$ with $g_0 = \begin{pmatrix} p^\delta D_0^* & B_0 \\ 0 & D_0 \end{pmatrix} \in L_0^m$ and $\theta^n(Z,Q) \in \Theta_m^n$, we set

$$\theta^n(Z,Q) \circ_\alpha X = \alpha(g_0) \sum_{\substack{D \in \Lambda D_0 \Lambda / \Lambda \\ p^\delta Q[D^*] \in \mathcal{N}_m^+}} l_X(Q,D) \theta^n(Z, p^\delta Q[D^*]) \tag{2.7}$$

where $\Lambda = SL_m(\mathbf{Z})$ and

$$l_X(Q,D) = \sum_{B \in B_X(D)/\mathrm{mod}\, D} e(QBD^{-1}). \tag{2.8}$$

Here

$$B_X(D) = \{B \in M_m(Q)\ ;\ \begin{pmatrix} p^\delta D^* & B \\ 0 & D \end{pmatrix} \in \Gamma_0^m g_0 \Gamma_0^m\} \tag{2.9}$$

and $B_1, B_2 \in B_X(D)$ are said to be congruent modulo D on the right if $(B_1 - B_2)D^{-1} \in M_m(\mathbf{Z})$. This congruence is obviously an equivalent relation and the summation in (2.8) is over equivalent classes in $B_X(D)$ modulo D on the right. We extend (2.7) by linearity to the whole space Θ_m^n and the whole ring $\mathcal{L}_0^m$.

Let

$$\mathcal{L}_0^{m,0} = \mathcal{L}_{0,p}^{m,0} = \{\sum a_i(\Gamma_0^m g_i \Gamma_0^m) \in \mathcal{L}_0^m\ ;\ \delta_i m - 2b_i = 0\} \tag{2.10}$$

where $g_i = \begin{pmatrix} p^{\delta_i} D_i^* & B_i \\ 0 & D_i \end{pmatrix} \in L_0^m$ and $b_i = \log_p |\det D_i|$, and let

$$\mathcal{E}_0^{m,0} = \mathcal{E}_{0,p}^{m,0} = \{\sum a_i(\Gamma_0^m g_i \Gamma_0^m) \in \mathcal{L}_0^{m,0}\ ;\ \delta_i \in 2\mathbf{Z}\}. \tag{2.11}$$

§3. Main Theorem

We prove the following theorem:

Theorem 3.1. (1) *The action (2.7) is a well-defined action of $\mathcal{L}_0^m$ on Θ_m^n. The elements of $\mathcal{L}_0^m$ acting on Θ_m^n in this manner are called theta operators.*

(2) *$\Theta_m^n(q,d)$ is invariant under the theta operators of $\mathcal{L}_0^{m,0}$ if p and q are relatively prime..*

(3) *$\Theta_m^n[Q]$ is invariant under the theta operators of $\mathcal{E}_0^{m,0}$ if p and $2q_Q$ are relatively prime.*

Proof. This theorem is proved for the case m even in [A1]. So, we restrict ourselves to the case m odd here. Let

$$\varepsilon(Z,Q) = \sum_{U \in \Omega} e(Q[U]Z), \quad Z \in \mathcal{H}_m, \tag{3.1}$$

where $\Omega = GL_m(\mathbf{Z})$.

$\varepsilon(Z,Q)$ is called the ε-series of Q. For every $M = \begin{pmatrix} D^* & B \\ 0 & D \end{pmatrix} \in \Gamma_0^m$ with $D \in \Omega$, we have

$$\varepsilon(M\langle Z\rangle, Q) = \sum_{U\in\Omega} e(Q[UD^*]Z)e(Q[U]BD^{-1}) = \varepsilon(Z,Q) \tag{3.2}$$

Note that $e(Q[U]BD^{-1}) = 1$ because $Q[U] \in \mathcal{N}_m^+$ and BD^{-1} is integral symmetric [M]. From (3.2) and the definition of even modular forms follows that

$$\varepsilon(Z,Q) \in \mathcal{M}_0^m.$$

Let

$$\mathcal{A}_m = \{\sum c_i\varepsilon(Z,Q_i)\ ;\ Q_i \in \mathcal{N}_m^+\} \subset \mathcal{M}_0^m. \tag{3.3}$$

Let $k = \frac{m}{2}$ and χ be a character satisfying $\chi(-1) = 1$. Let $X = (\Gamma_0^m g_0 \Gamma_0^m) \in \mathcal{L}_0^m$ with $g_0 = \begin{pmatrix} p^\delta D_0^* & B_0 \\ 0 & D_0 \end{pmatrix} \in L_0^m$. Then

$$X = \sum_{\substack{D\in\Omega\backslash\Omega D_0\Omega \\ B\in B_X(D)/\mathrm{mod}\, D}} (\Gamma_0^m g),$$

where $g = \begin{pmatrix} p^\delta D^* & B \\ 0 & D \end{pmatrix} \in L_0^m$. See (2.9) for $B_X(D)/\mathrm{mod}\, D$. From (1.3) and (1.6) follows

$$\varepsilon(Z,Q) = \sum_{U\in\Omega} e(Q[U]Z) = \sum_{U\in\Omega} e(QZ)|_k \tilde{M}_U \tag{3.4}$$

where $M_U = \begin{pmatrix} U^* & 0 \\ 0 & U \end{pmatrix} \in \Gamma_0^m$ and $\tilde{M}_U = (M_U, 1)$. Hence

$$\varepsilon(Z,Q)|_{k,\chi} X = \sum_{U\in\Omega} \sum_{\substack{D\in\Omega\backslash\Omega D_0\Omega \\ B\in B_X(D)/\mathrm{mod}\, D}} \chi(\det p^\delta D^*) e(QZ)|_k \tilde{M}_U|_k \tilde{g}, \tag{3.5}$$

where $\tilde{g} = (g, p^{-\delta m/4}|\det D|^{\frac{1}{2}})$. Since $M_U g = \begin{pmatrix} p^\delta (UD)^* & U^*B \\ 0 & UD \end{pmatrix}$ and $U^*\{B_X(D)/\mathrm{mod}\, D\} = \{B_X(UD)/\mathrm{mod}\, UD\}$ for any $U \in \Omega$, we have

$$\varepsilon(Z,Q)|_{k,\chi} X = \sum_{\substack{D \in \Omega D_0 \Omega \\ B \in B_X(D)/\mathrm{mod}\, D}} \chi(\det p^\delta D^*) e(QZ)\,|_k\, \tilde{g}$$

So, we may rewrite (3.5) as

$$\varepsilon(Z,Q)|_{k,\chi} X = \sum_{\substack{D \in \Omega D_0 \Omega/\Omega \\ B \in B_X(D)/\mathrm{mod}\, D}} \sum_{U \in \Omega} \chi(\det p^\delta D^*) e(QZ)|_k \tilde{g}|_k \tilde{M}_U. \tag{3.6}$$

We now consider

$$\beta(Z,Q) = \sum_{B \in B_X(D)/\mathrm{mod}\, D} \chi(\det p^\delta D^*) e(QZ)|_k \tilde{g}. \tag{3.7}$$

From (1.3) follows that

$$\begin{aligned} \beta(Z,Q) &= \sum_{B \in B_X(D)/\mathrm{mod}\, D} \chi(p^{\delta m - b}) p^{\delta(mk - \langle m \rangle) - bk} e(Qp^\delta Z[D^{-1}] + QBD^{-1}) \\ &= \chi(p^{\delta m - b}) p^{\delta(mk - \langle m \rangle) - bk} e(p^\delta Q[D^*]Z) \sum_{B \in B_X(D)/\mathrm{mod}\, D} e(QBD^{-1}), \end{aligned}$$

and so,

$$\beta(Z,Q) = \alpha_{k,\chi}(g_0) e(p^\delta Q[D^*]Z) \sum_{B \in B_X(D)/\mathrm{mod}\, D} e(QBD^{-1}), \tag{3.8}$$

where $b = \log_p |\det D| = \log_p |\det D_0|$ and $\alpha_{k,\chi} : L_0^m \to \mathbf{C}^\times$ is a character defined by

$$\alpha_{k,\chi}(g) = \chi(p^{\delta m - b}) p^{\delta(mk - \langle m \rangle) - bk} \tag{3.9}$$

for any $g = \begin{pmatrix} p^\delta D^* & B \\ 0 & D \end{pmatrix} \in L_0^m$. If we take $B + AD$ instead of B as a representative in $B_X(D)/\mathrm{mod}\, D$ where ${}^tA = A \in M_m(\mathbf{Z})$, then $e(Q(B +$

$AD)D^{-1}) = e(QBD^{-1})e(QA) = e(QBD^{-1})$. So (3.8) is independent of the choice of representatives B of $B_{\chi}(D)/\mathrm{mod}\, D$. Let $g_S = \begin{pmatrix} I_m & S \\ 0 & I_m \end{pmatrix} \in \Gamma_0^m$ with ${}^tS = S \in M_m(\mathbf{Z})$. Then $\tilde{g}_S = (g_S, 1)$ and $gg_S = \begin{pmatrix} p^\delta D^* & p^\delta D^* S + B \\ 0 & D \end{pmatrix}$ so that $\{B + p^\delta D^* S\}$ is a complete set of representatives of $B_{\chi}(D)/\mathrm{mod} D$ if $\{B\}$ is. Therefore, from (3.7) follows

$$\beta(Z,Q)|_k \tilde{g}_S = \beta(Z,Q)$$

Applying $|_k \tilde{g}_S$ on the right hand side of (3.8), we obtain

$$\beta(Z,Q) = \alpha_{k,\chi}(g_0) e(p^\delta Q[D^*]Z) e(p^\delta Q[D^*]S) l_{\chi}(Q,D).$$

So, if $l_{\chi}(Q,D) \neq 0$, then $e(p^\delta Q[D^*]S) = 1$ for any ${}^tS = S \in M_m(\mathbf{Z})$. This implies that $p^\delta Q[D^*] \in \mathcal{N}_m^+$. In other words, if $p^\delta Q[D^*] \notin \mathcal{N}_m^+$, then $l_{\chi}(Q,D) = 0$. From this and (3.4), (3.6), (3.8) follows

$$\varepsilon(Z,Q)|_{k,\chi} X = \alpha_{k,\chi}(g_0) \sum_{\substack{D \in \Omega D_0 \Omega/\Omega \\ p^\delta Q[D^*] \in \mathcal{N}_m^+}} \varepsilon(Z, p^\delta Q[D^*]) l_{\chi}(Q,D) \in \mathcal{A}_m. \tag{3.10}$$

Choosing a complete set of representatives $\{D_i\}$ of $\Omega D_0 \Omega/\Omega$ such that $\det D_i = \det D_0$, we may rewrite (3.10) as follows :

$$\varepsilon(Z,Q)|_{k,\chi} X = \alpha_{k,\chi}(g_0) \sum_{\substack{D \in \Lambda D_0 \Lambda/\Lambda \\ p^\delta Q[D^*] \in \mathcal{N}_m^+}} \varepsilon(Z, p^\delta Q[D^*]) l_{\chi}(Q,D). \tag{3.11}$$

We now define a linear map $\vartheta = \vartheta_{m,n} : \mathcal{A}_m \to \Theta_m^n$ by $\vartheta(\varepsilon(Z,Q)) = \theta^n(Z_n, Q)$, $Q \in \mathcal{N}_m^+$, where $Z = \begin{pmatrix} Z_n & * \\ * & * \end{pmatrix} \in \mathcal{H}_m, Z_n \in \mathcal{H}_n$. Obviously ϑ is a well-defined epimorphism. From (3.11) and (2.7) follows

$$\vartheta(\varepsilon(Z,Q)\,|_{k,\chi} X) = \theta^n(Z,Q) \circ_\alpha X,$$

for any $X \in \mathcal{L}_0^m$, where $\alpha = \alpha_{k,\chi}$ as in (3.9).

$\varepsilon(Z,Q)|_{k,\chi}X\,|_{k,\chi}Y = \varepsilon(Z,Q)|_{k,\chi}X\cdot Y$ implies

$$\theta^n(Z,Q)\circ_\alpha X\circ_\alpha Y = \theta^n(Z,Q)\circ_\alpha X\cdot Y. \tag{3.12}$$

From the surjectivity of ϑ, (3.11) and (3.12) follows (1).

Let p be relatively prime to q and let $X\in\mathcal{L}_0^{m,0}$. We may assume $X = (\Gamma_0^m g\Gamma_0^m)$, $g = \begin{pmatrix} p^\delta D^* & B \\ 0 & D\end{pmatrix}\in L_0^m$ such that $\delta m = 2b$ with $b=\log_p|\det D|$.

To prove (2), it is enough to show that $\det 2Q' = d$ and $q_{Q'} = q$ for $Q' = p^\delta Q[D^*]\in\mathcal{N}_m^+$, where $d=\det 2Q$ and $q = q_Q$. Clearly $\det 2Q' = d$. Let q' be the level of Q'. Then $q(2Q')^{-1}p^{\delta'} = qp^{\delta'-\delta}(2Q)^{-1}[D]$ is integral for some $\delta'\geq 0$. So. $q'\,|\,qp^{\delta'}$, which implies $q'\,|\,q$. Similary $q\,|\,q'$. This proves (2).

Finally, let δ be even. Note that since we restrict ourselves to the case m odd, $q=q_Q$ is divisible by 4. Let $D' = p^{\frac{\delta}{2}}D^*$ so that $\det D' = \pm 1$. It is a well known fact[Og] that q and $d=\det(2Q)$ have the same prime factors. So $p\,|\,d$ and hence one can find $U\in M_m(\mathbf{Z})$ such that $U\equiv D'(\operatorname{mod} 8d^3)$. Since $2Q' = 2Q[D']$, we have $2Q'\equiv 2Q[U]\ (\operatorname{mod} 8d^3)$. Therefore[Si2] $Q'\in gen(Q)$ if $Q' = p^\delta Q[D^*]\in\mathcal{N}_m^+$ and this proves (3).

References

[A1] A. N. Andrianov, *The Multiplicative Arithmetic of Siegel Modular Forms*, Russian Math. Surveys, 34(1979), 75–148.

[A2] ———, *Quadratic Forms and Hecke Operators*, GMW 286, Springer-Verlag, 1987.

[A-M] A. N. Andrianov, G. N. Maloletkin, *The Behavior of Theta-series of Genus n under Modular Substitution*, Math. USSR. Izv. 9(1975), 227–241.

[K] M. H. Kim, *The Behavior of Theta-Series under Slash Operators*, Comm. KMS 2 (1987), 9–16.

[Kö] M. Köcher, *Zur Theorie der Modulformen n-ten Grades I, II*, Math. Zeit. 59(1954), 399–416, ibid., 61(1955), 455–466.

[M] H. Maass, *Siegel's Modular Forms and Dirichlet Series*, LNM 216, Springer-Verlag, 1971.

[O] O. T. O'Meara, *Introduction to Quadratic Forms*, GMW 117, Springer-Verlag, 1963.

[Og] A. Ogg, *Modular Forms and Dirichlet Series*, Benjamin, 1969.

[S] G. Shimura, *On Modular Forms of Half Integral Weight*, Ann. of Math. (2)97(1973), 440–481.

[Si1] C. L. Siegel, *Über die analytische Theorie der quadratischen Formen, I, II, III*, Ann. of Math. 36(1935), 527–606, ibid., 37(1936), 230–263, ibid., 38(1937), 212–291.

[Si2] ———, *Einführung in die Theorie der Modulfunktionen n-ten Grades*, Math. Ann. 116(1938/39), 617-657.

[Z1] V. G. Zhuravlev, *Hecke Rings for a Covering of the Symplectic Groups*, Math. USSR Sbor. 49(1984), 379–399.

[Z2] ———, *Expansions of Theta-Transformations of Siegel Modular Forms of Half Integral Weight and their Analytic Properties*, Math. USSR Sbor. 51(1985), 169–190.