ACTION ON FLAG VARIETIES : 2-DIMENSIONAL CASE*

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Abstract. Let A be a finite dimensional commutative semisimple algebra over a field k and let V be a finitely generated A-module. We examine the action of the general linear group $GL_A(V)$ on the set of flags of k-subspaces of V. Also, let (V, B) be a finitely generated symplectic module over A. We also investigate the action of the symplectic group $\mathrm{Sp}_A(V, B)$ on the set of flags of B'-isotropic k-subspaces of V, where $B' = \phi \circ B$ is the k-symplectic form induced by a nonzero k-linear map $\phi : A \to k$. In both cases, the orbits are completely classified in terms of certain integer invariants provided that $\dim_k A = 2$.

1. Introduction

In [4,7], the first author and Patrick Rabau studied the action on Grassmannians of products of general linear groups defined over extension fields of the base field.

These results were used [5,8] by the same authors to understand the analogous problem for products of symplectic groups acting on isotropic subspaces.

The present paper is a natural extension of the above mentioned works, i.e., considered are the actions of products of general linear groups and of symplectic groups over extension fields of the base field on flags of subspaces and of isotropic subspaces over the same base field in the respective cases (rather than just a single subspace or a single isotropic subspace).

More precisely, let A be a finite-dimensional commutative semisimple algebra over a field k, V a finitely generated A-module. Letting $FLAG(V, k)$ denote the set of all flags of k-subspaces of V, we consider the natural action of the group $GL_A(V)$ on $FLAG(V, k)$. Put $FLAG_{m_1,\dots,m_l}(V, k)$ for the set of all flags $(W_i)_{i=0}^{l+1}$ of k-subspaces of V with $\dim_k W_i = m_1 + \cdots + m_i$ $(i = 1, \dots, l)$, where $1 \le l \in \mathbb{Z}$, $m_1, \dots, m_l \in \mathbb{N}$ (see, section 2).

For the Theorems A and B below, assume that k is infinite and V is a faithful A-module.

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Theorem A. *If* $\dim_k A = 2$*, i.e.,* A *is a quadratic extension field of* k *or* $A = k \times k$*, then the number of orbits for the action of* $GL_A(V)$ *on* $FLAG_{m_1,\ldots,m_l}(V,k)$ *is finite and independent of the fields for every choice of* $1 \le l \in \mathbb{Z}$ *and of nonnegative integers* $m_1, \ldots, m_l$*.*

Let k, A be as before. Let (V, B) be a finitely generated A-module together with a nondegenerate A-valued symplectic form B on V. For a fixed nonzero k-linear functional $\phi : A \to k$, we form the k-valued symplectic form $B' = \phi \circ B$ and look at the action of the group $\mathrm{Sp}_A(V, B)$ on $FLAG(V, B')$ of the set of all flags of B'-isotropic k-subspaces of V. Put $FLAG_{m_1,\ldots,m_l}(V, B')$ for the set of all flags $(W_i)_{i=0}^{l+1}$ of B'-isotropic k-subspaces of V with $\dim_k W_i = m_1 + \cdots + m_i$ $(i = 1, \ldots, l)$, where $1 \le l \in \mathbb{Z},\ m_1, \ldots, m_l \in \mathbb{N}$ (see, section 2).

Theorem B. *If* $\dim_k A = 2$*, then the number of orbits for the action of* $Sp_A(V, B)$ *on* $FLAG_{m_1,\ldots,m_l}(V, B')$ *is finite and independent of the fields for every choice of* $1 \le l \in \mathbb{Z}$ *and of nonnegative integers* $m_1, \ldots, m_l$*.*

Theorems A and B follow respectively from the results in sections 4,5 and 6,7. When $\dim_k A = 2$, the orbits are actually classified in terms of certain integer invariants, which allows one to determine the precise structure of orbits, to compute the exact number of orbits (at least in principle), and to give typical representatives for each orbit.

In group theoretic terminology, this problem amounts to looking at the double coset spaces $P\backslash GL_k(V)/GL_A(V)$ and $P\backslash \mathrm{Sp}_k(V, B')/\mathrm{Sp}_A(V, B)$, where P's are parabolic subgroups of the middle groups. These double coset spaces have nice applications to the theory of automorphic forms. Indeed, in 1985 Garrett found an integral representation of the so called Rankin triple product L-functions by considering $P\backslash \mathrm{Sp}(6, k)/\mathrm{Sp}(2, k) \times \mathrm{Sp}(2, k) \times \mathrm{Sp}(2, k)$, where P is a 'bottom' maximal parabolic subgroup of $\mathrm{Sp}(6, k)$ (see, [2,6]). Also, see [1,3] for the general formalism about how these double coset space considerations can be used to construct automorphic L-functions. The present extension of previous works allows one to consider Eisenstein series attached not only to a maximal parabolic subgroup but also to any parabolic subgroup.

2. Notations and the statement of the problem

The following general notations will be used. The submodule of an R-module U generated by a subset S is written RS, or $\langle v_1, \ldots, v_n \rangle_R$ if $S = \{v_1, \ldots, v_n\}$. $\mathbb{N}$ will be used to denote the set of nonnegative integers.

(a) General linear case

Let k be a field, $A = k_1 \times \cdots \times k_m$ a finite dimensional commutative semisimple k-algebra, where each k_i/k is a finite extension of fields. The primitive idempotents of A will be denoted by $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 at the i-th place $(i = 1, \ldots, m)$.

Let V be a finitely generated A-module. The dimension vector of V is defined by

$$\mathbf{dim}_A(V) = (\dim_{k_1} e_1 V, \ldots, \dim_{k_m} e_m V).$$

If $m = 2$, then it will also be called the bidimension of V.

By a flag $(W_i)_{i=0}^{l+1}$ of k-subspaces of V we mean a chain of k-subspaces $W_0 \le W_1 \le \cdots \le W_l \le W_{l+1}$ of V with $W_0 = 0$, $W_{l+1} = V$, where $1 \le l \le \mathbb{Z}$. Write $FLAG(V, k)$ for the set of all such flags and $FLAG_{m_1,\ldots,m_l}(V, k)$ for the set of such flags $(W_i)_{i=0}^{l+1}$ with $\dim_k W_i = m_1 + \cdots + m_i$ $(i = 1, \ldots, l)$, where $m_1, \ldots, m_l \in \mathbb{N}$. Then we want to consider the action on $FLAG(V, k)$ of the group $GL_A(V) = GL_{k_1}(e_1 V) \times \cdots \times GL_{k_m}(e_m V)$ of A-linear automorphisms of V.

Assume that $[k_j : k] = r_j$, $\dim_{k_j} e_j V = n_j$, $N = \sum_j n_j r_j$. Then our problem amounts to looking at the double coset space

$$P_{m_1,\ldots,m_l} \backslash GL(N, k) / (GL(n_1, k_1) \times \cdots \times GL(n_m, k_m))$$

for any $1 \le l \in \mathbb{Z}$, $m_1, \ldots, m_l \in \mathbb{N}$ satisfying $m_1 + \cdots + m_l \le N$. Here $P_{m_1,\ldots,m_l}$ is a parabolic subgroup of $GL(N, k)$ leaving invariant a flag $(W_i)_{i=0}^{l+1} \in FLAG_{m_1,\ldots,m_l}(V, k)$.

We collect from [4,7] some notations that will be used. Let W be a k-subspace of V. The largest A-submodule of V contained in W will be called the A-component of W and written $\mathrm{comp}_A W$. For k-subspaces $W_1, \ldots, W_r$ of V, write

$$W = W_1 \oplus_A \cdots \oplus_A W_r$$

if $W = \oplus_i W_i$ and $AW = \oplus_i AW_i$. Such a decomposition will be called a direct sum over A.

(b) Symplectic case

Let $k,\ k_i,\ A,\ e_i$ be as in (a). (V, B) denotes a finitely generated A-module V together with an A-valued symplectic form B on V. Then B can be uniquely written as, for $x, y \in V$,

$$B(x, y) = \sum_{i=1}^{m} e_i B_i(e_i x, e_i y)$$

where each B_i is an k_i-valued symplectic form on $e_i V$. For each i, fix a nonzero k-linear form $\phi_i : k_i \to k$ (choose $\phi_i =$ the identity if $k_i = k$). Then $(V, B' = \phi \circ B)$ is a symplectic space over k, where $\phi(\sum \alpha_i e_i) = \sum \phi_i(\alpha_i)$ $(\alpha_i \in k_i)$.

By a flag $(W_i)_{i=0}^{l+1}$ of isotropic (= totally isotropic) k-subspaces of V we mean a chain of k-subspaces $W_0 \leq W_1 \leq \cdots \leq W_l \leq W_{l+1}$ of V, where $W_0 = 0$, $W_{l+1} = V$, and W_i $(i = 1, \ldots, l)$ are B'-isotropic k-subspaces of V. Write $FLAG(V, B')$ for the set of such flags and $FLAG_{m_1,\ldots,m_l}(V, B')$ for the set of such flags $(W_i)_{i=0}^{l+1}$ with $\dim_k W_i = m_1 + \cdots + m_i$ $(i = 1, \ldots, l)$. Then we also propose to investigate the action on $FLAG(V, B')$ of the group $\mathrm{Sp}_A(V, B) = \mathrm{Sp}_{k_1}(e_1 V, B_1) \times \cdots \times \mathrm{Sp}_{k_m}(e_m V, B_m)$ of A-linear automorphisms of V preserving the form B.

Again, if we let $[k_j; k] = r_j$, $\dim_{k_j} e_j V = 2n_j$, $N = \sum_j n_j r_j$, then in group theoretic terminology our problem amounts to considering the double coset space

$$P_{m_1,\ldots,m_l} \backslash \mathrm{Sp}(2N, k) / (\mathrm{Sp}(2n_1, k_1) \times \cdots \times \mathrm{Sp}(2n_m, k_m))$$

for any $1 \leq l \in \mathbb{Z}$, $m_1, \ldots, m_l \in \mathbb{N}$ satisfying $m_1 + \cdots + m_l \leq N$,where $P_{m_1,\ldots,m_l}$ is a parabolic subgroup of $\mathrm{Sp}(2N, k)$ leaving invariant a flag $(W_i)_{i=0}^{l+1} \in FLAG_{m_1,\ldots,m_l}(V, B')$.

The following notations in [5,8] will also be used. Orthogonality with respect to B and B' will be denoted respectively by $\perp$ and $\perp'$; for a vector x in V and a subset $S \subseteq V$, write for instance: $x \perp S$, $x \perp' S$, $S^{\perp}$, $S^{\perp'}$ with their obvious meaning. For k-subspaces $W, W_1, \ldots, W_r$ of V, write

$$W = W_1 \perp_A \cdots \perp_A W_r$$

if $W = \oplus_{iA} W_i$ and $W_1 \perp \cdots \perp W_r$. The B-radical of W is defined by $\mathrm{rad}_B W = W \cap W^{\perp}$. The subspace W is called B-isotropic if $W = \mathrm{rad}_B W$. Fix a $(a = 1, \ldots, m)$. A hyperbolic sequence in $(e_a V, B_a)$ is a sequence $v_1, v_1', \cdots, v_r, v_r'$ of vectors of $e_a V$ satisfying $B_a(v_i, v_j') = \delta_{ij}$ and $B_a(v_i, v_j) = B_a(v_i', v_j') = 0$; if that sequence is also a basis for $e_a V$, it is called a hyperbolic basis of $(e_a V, B_a)$.

In the present research, we limit ourselves to the case of $\dim_k A = 2$, i.e., $A =$ a quadratic extension field of k or $A = k \times k$.

3. Preliminaries

For the next two lemmas, V is a finitely generated A-module.

Lemma 3.1. *Assume* $\dim_k A = 2$. *For k-subspaces U, U_1, U_2 of V, suppose that $U = U_1 \oplus U_2$. If $comp_A U \leq U_1$, then $U = U_1 \oplus_A U_2$.*

Proof. Theorem 4.1(c) of [7] and Lemma 4.1. of [4]. $\square$

Lemma 3.2. (i) *For any A, let U_1, U_2 be k-subspaces of V with $U_1 \leq U_2$. Then $comp_A(U_1 + comp_A U_2) = comp_A U_2$.*

(ii) *Assume* $\dim_k A = 2$. *Let $U_1 \leq U_2 \leq \cdots \leq U_m \leq U$ be a chain of k-subspaces of V. Then there exists a k-subspace Y_j of U_j for each j such that*

$$U_j + comp_A U = comp_A U \oplus_A \oplus_{i=1}^{j}{}_A Y_i.$$

Proof. (i) is immediate. By (i), every subspace $U_j + \mathrm{comp}_A U$ has the same A-component $\mathrm{comp}_A U$ for $j = 1, \ldots, m$. Lemma 3.1 implies that there exists a k-subspace Y_j of U_j such that $U_j + \mathrm{comp}_A U = (U_{j-1} + \mathrm{comp}_A U) \oplus_A Y_j$ $(j = 1, \ldots, m)$, where $U_0 = 0$. The result follows from this. $\square$

For the following two lemmas, (V, B) is a finitely generated A-module V together with an A-valued symplectic form B on V. Also, $B' = \phi \circ B$ as in section 2. They are quoted from [8] for convenience of the reader.

Lemma 3.3. (i) *Let $x \in V$ and let U be an A-submodule of V. Then $x \perp' U \iff x \perp U$.*

(ii) *For an A-submodule U of V, $U^{\perp'} = U^{\perp}$. In particular, an A-submodule of V is B'-isotropic if and only if it is B-isotropic.*

(iii) *If W is an B'-isotropic k-subspace of V, then $comp_A W$ is B-isotropic, i.e., $comp_A W$
$\leq rad_B W$. If W is maximal isotropic in (V, B'), then $comp_A W = rad_B W = W^{\perp}$.*

Lemma 3.4. (i) *If the k-subspaces W, W_1, W_2 of V satisfy $W = W_1 \perp_A W_2$, then there exist B-nondegenerate A-submodules U_1, U_2 in (V, B) such that $W_i \leq U_i$ $(i = 1, 2)$ and $U_1 \perp U_2$.*

(ii) *Let W be an B'-isotropic k-subspace of V and let W_1 be a k-subspace of W. If there is an B-nondegenerate A-submodule U of V containing W_1 as a maximal B'-isotropic subspace, then $W = W_1 \perp_A Y$ for some k-subspace Y of W ($Y = W \cap U^{\perp}$ will do).*

(iii) *If W is an B'-isotropic k-subspace of V, then $W = comp_A W \perp_A Y$ for some k-subspace Y of W.*

4. The general linear case with $A = F$

In this section, F is a quadratic extension of k, and V is a vector space over F with $\dim_F V = n$.

For a flag $(W_i)_{i=0}^{l+1}$ of k-subspaces of V, we define the type of $(W_i)_{i=0}^{l+1}$ to be the $l(l+3)/2$-tuple of nonnegative integers

$$\begin{aligned} &\mathrm{type}(W_i)_{i=0}^{l+1} \\ = &\Big(\dim_F(\mathrm{comp}_F W_j / F(\mathrm{comp}_F W_j \cap W_{j-1}))\ (j = 1, \ldots, l); \\ &\quad \dim_k(W_i \cap \mathrm{comp}_F W_{j+1} / (W_{i-1} \cap \mathrm{comp}_F W_{j+1} + W_i \cap \mathrm{comp}_F W_j)) \\ &\quad (1 \leq i \leq j \quad \text{for} \quad j = 1, \ldots, l)\Big). \end{aligned}$$

We will write $\mathrm{type}(W_i)_{i=0}^{l+1} = (r_j; s_{ij})$ with the understanding that

$$\begin{aligned} r_j &= \dim_F(\mathrm{comp}_F W_j / F(\mathrm{comp}_F W_j \cap W_{j-1})) \text{ for } j = 1, \ldots, l \text{ and} \\ s_{ij} &= \dim_k(W_i \cap \mathrm{comp}_F W_{j+1} / (W_{i-1} \cap \mathrm{comp}_F W_{j+1} + W_i \cap \mathrm{comp}_F W_j)) \end{aligned}$$

for $1 \leq i \leq j,\ j = 1, \ldots, l$.

Remark. Note that

$$\begin{aligned} &\dim_k(W_i \cap \mathrm{comp}_F W_{j+1} / (W_{i-1} \cap \mathrm{comp}_F W_{j+1} + W_i \cap \mathrm{comp}_F W_j)) \\ = &\dim_k(W_i \cap \mathrm{comp}_F W_{j+1} + \mathrm{comp}_F W_j / W_{i-1} \cap \mathrm{comp}_F W_{j+1} + \mathrm{comp}_F W_j). \end{aligned}$$

Theorem 4.1. *Let $(W_i)_{i=0}^{l+1}$ be a flag of k-subspaces of V, with type$(W_i)_{i=0}^{l+1} = (r_j; s_{ij})$. Then there exist F-independent vectors $x_{j1}, \ldots, x_{jr_j}$. $(j = 1, \ldots, l)$ and $y_{ij1}, \ldots, y_{ijs_{ij}}$ $(1 \leq i \leq j$, for $j = 1, \ldots, l)$ in V such that each W_i $(1 \leq i \leq l)$ admits the direct sum decomposition over F*

$$W_i = \oplus_{j=1}^{i}(\oplus_{m=1}^{j-1} FY_{m,(j-1)} \oplus X_j) \oplus_F (\oplus_{j=1F}^{l} \oplus_{m=1\ F}^{i} Y_{mj})$$

with

$$X_j = \langle x_{j1}, \ldots, x_{jr_j} \rangle_F,$$
$$Y_{ij} = \langle y_{ij1}, \ldots, y_{ijs_{ij}} \rangle_k.$$

Moreover, comp$_F W_i = \oplus_{j=1}^{i}(\oplus_{m=1}^{j-1} FY_{m,(j-1)} \oplus X_j)$.

Proof. For each fixed j $(1 \leq j \leq l)$, consider the chain $W_1 \cap \mathrm{comp}_F W_{j+1} \leq W_2 \cap \mathrm{comp}_F W_{j+1} \leq \cdots \leq W_j \cap \mathrm{comp}_F W_{j+1}$.

By Lemma 3.2(ii), we can find a k-subspace Y_{ij} of $W_i \cap \mathrm{comp}_F W_{j+1}$ such that

(1) $$W_i \cap \mathrm{comp}_F W_{j+1} + \mathrm{comp}_F W_j = \mathrm{comp}_F W_j \oplus_F \oplus_{m=1F}^{i} Y_{mj},$$

for $i = 1, \ldots, j$. We see that $s_{ij} = \dim_k Y_{ij}$ by the remark before Theorem 4.1. Moreover, for $i = j$ (1) says that

(2) $$W_j \cap \mathrm{comp}_F W_{j+1} = \mathrm{comp}_F W_j \oplus_F \oplus_{m=1F}^{j} Y_{mj} \leq \mathrm{comp}_F W_{j+1}.$$

Choose for $j = 0, 1, \ldots, l-1$ an F-subspace X_{j+1} of V such that

(3) $$\mathrm{comp}_F W_{j+1} = \mathrm{comp}_F W_j \oplus \oplus_{m=1}^{j} FY_{mj} \oplus X_{j+1}.$$

Then this implies that

(4) $$\mathrm{comp}_F W_i = \oplus_{j=1}^{i}(\oplus_{m=1}^{j-1} FY_{m,(j-1)} \oplus X_j).$$

Also, (2) and (3) show that $\dim_F X_j = r_j$. Next, we want to show that $W_i = \mathrm{comp}_F W_i \oplus_F (\oplus_{j=iF}^{l} \oplus_{m=1\ F}^{i} Y_{mj})$ which together with (4) implies the desired decomposition of each W_i.

Observe that by our choice of Y_{ij}, $W_i \geq Y_{mj}$ (for $1 \leq m \leq i$, $i \leq j \leq l$). Moreover, $\mathrm{comp}_F W_i \oplus_F (\oplus_{j=i F}^{l} \oplus_{m=1\ F}^{i} Y_{mj})$ is a direct sum over F, as it can be seen from the chain

(5)

$$\begin{aligned} &(W_i \cap \mathrm{comp}_F W_{i+1}) + \mathrm{comp}_F W_i = \mathrm{comp}_F W_i \oplus_F \oplus_{m=1 F}^{i} Y_{mi} \\ \leq\ &(W_i \cap \mathrm{comp}_F W_{i+2}) + \mathrm{comp}_F W_{i+1} = \mathrm{comp}_F W_{i+1} \oplus_F \oplus_{m=1 F}^{i} Y_{m,(i+1)} \\ \leq\ &\cdots \\ \leq\ &(W_i \cap \mathrm{comp}_F W_{l+1}) + \mathrm{comp}_F W_l = \mathrm{comp}_F W_l \oplus_F \oplus_{m=1 F}^{i} Y_{ml}. \end{aligned}$$

For $i \leq j \leq l$, we have from (5)

$$\dim_k(W_i \cap \mathrm{comp}_F W_{j+1} / W_i \cap \mathrm{comp}_F W_j) = \dim_k \oplus_{m=1 F}^{i} Y_{mj}. \tag{6}$$

Summing up (6) over $j = i, \ldots, l$, we see that W_i and $\mathrm{comp}_F W_i \oplus_F (\oplus_{j=i F}^{l} \oplus_{m=1\ F}^{i} Y_{mj})$ have the same dimension over k and hence that they are the same. Pick a basis $x_{j1}, \ldots, x_{jr_j}$ of X_j over F and a basis $y_{ij1}, \ldots, y_{ijs_{ij}}$ of Y_{ij} over k, and apply [7, Theorem 4.1(c)] for the F-independence of $y_{ij1}, \ldots, y_{ijs_{ij}}$. □

The following theorem follows from the above one and can be proved just as [7, Corollary 4.3].

Theorem 4.2. *Two flags of k-subspaces of V are in the same orbit for $GL_F(V)$ if and only if they have the same type.*

Remarks. (1) If $(W_i)_{i=0}^{l+1}$ is a flag of k-subspaces of V with type $(W_i)_{i=0}^{l+1} = (r_j; s_{ij})$, then

$$\dim_F \mathrm{comp}_F W_i = \sum_{j=1}^{i} (r_j + \sum_{m=1}^{j-1} s_{m,(j-1)}),$$

$$\dim_k W_i = 2 \sum_{j=1}^{i} (r_j + \sum_{m=1}^{j-1} s_{m,(j-1)}) + \sum_{j=i}^{l} \sum_{m=1}^{i} s_{mj},$$

$$\dim_F FW_i = \sum_{j=1}^{i} (r_j + \sum_{m=1}^{j-1} s_{m,(j-1)}) + \sum_{j=i}^{l} \sum_{m=1}^{i} s_{mj}.$$

(2) From the expression for $\dim_F FW_l$, we see that an $l(l+3)/2$-tuple of nonnegative integers $(r_j(j=1,\ldots,l); s_{ij}(1 \le i \le j \text{ for } j=1,\ldots,l))$ is the type of some flag $(W_i)_{i=0}^{l+1}$ of k-subspaces of V if and only if

$$\sum_{j=1}^{l} r_j + \sum_{j=1}^{l}\sum_{m=1}^{j} s_{mj} \le n.$$

(3) Let $m_1, \ldots, m_l$ be positive integers such that $m_1+m_2+\cdots+m_l < 2n$. Then

$$\begin{aligned}
&|P_{m_1,\ldots,m_l}\backslash GL(2n,k)/GL(n,F)| \\
&= \Big|\{(r_j; s_{ij} \in \mathbb{N}^{\frac{l(l+3)}{2}} \mid 2\sum_{j=1}^{i}(r_j + \sum_{m=1}^{j-1} s_{m,(j-1)}) + \sum_{j=i}^{l}\sum_{m=1}^{i} s_{mj} \\
&= m_1+m_2+\cdots+m_i \quad \text{for} \quad i=1,\ldots,l \quad \text{and} \\
&\quad \sum_{j=1}^{l} r_j + \sum_{j=1}^{l}\sum_{m=1}^{j} s_{mj} \le n\}\Big|,
\end{aligned}$$

where $P_{m_1,\ldots,m_l}$ is a parabolic subgroup of $GL(2n,k)$ leaving invariant a flag $(W_i)_{i=0}^{l+1} \in FLAG_{m_1,\ldots,m_l}(V,k)$.

5. The general linear case with $A = k \times k$

In this section, $A = k \times k$ and V is a finitely generated A-module of bidimension (n_1, n_2).

For a flag $(W_i)_{i=0}^{l+1}$ of k-subspaces of V, we define the type of $(W_i)_{i=0}^{l+1}$ to be the $l(l+5)/2$-tuple of nonnegative integers

$$\begin{aligned}
\text{type}(W_i)_{i=0}^{l+1} = \Big(&\dim_k(e_1W_j \cap W_j/e_1W_{j-1} \cap W_j)\ (j=1,\ldots,l); \\
&\dim_k(e_2W_j \cap W_j/e_2W_{j-1} \cap W_j)\ (j=1,\ldots,l); \\
&\dim_k(W_i \cap \text{comp}_A W_{j+1}/(W_{i-1} \cap \text{comp}_A W_{j+1} + W_i \cap \text{comp}_A W_j)\Big) \\
&\qquad (1 \le i \le j \quad \text{for} \quad j=1,\ldots,l)).
\end{aligned}$$

For brevity, we will write $\text{type}(W_i)_{i=0}^{l+1} = (\alpha_j; \beta_j; s_{ij})$ with the understanding that

$$\begin{aligned}
\alpha_j &= \dim_k(e_1W_j \cap W_j/e_1W_{j-1} \cap W_j) \quad \text{for } j=1,\ldots,l, \\
\beta_j &= \dim_k(e_2W_j \cap W_j/e_2W_{j-1} \cap W_j) \quad \text{for } j=1,\ldots,l, \text{ and} \\
s_{ij} &= \dim_k(W_i \cap \text{comp}_A W_{j+1}/(W_{i-1} \cap \text{comp}_A W_{j+1} + W_i \cap \text{comp}_A W_j)) \\
&\qquad \text{for} \quad 1 \le i \le j,\ j=1,\ldots,l.
\end{aligned}$$

Remark. Note that

$$\dim_k(W_i \cap \mathrm{comp}_A W_{j+1}/(W_{i-1} \cap \mathrm{comp}_A W_{j+1} + W_i \cap \mathrm{comp}_A W_j))$$
$$= \dim_k(W_i \cap \mathrm{comp}_A W_{j+1} + \mathrm{comp}_A W_j / W_{i-1} \cap \mathrm{comp}_A W_{j+1} + \mathrm{comp}_A W_j).$$

Theorem 5.1. *Let $(W_i)_{i=0}^{l+1}$ be a flag of k-subspaces of V with type $(W_i)_{i=0}^{l+1} = (\alpha_j; \beta_j; s_{ij})$. Then there exist k-independent vectors $x_{j1}^{(1)}, \ldots, x_{j\alpha_j}^{(1)}$ $(j = 1, \ldots, l)$; $y_{ij1}^{(1)}, \ldots, y_{ijs_{ij}}^{(1)}$ $(1 \leq i \leq j,\ j = 1, \ldots, l)$ in e_1V and $x_{j1}^{(2)}, \ldots, x_{j\beta_j}^{(2)}$ $(j = 1, \ldots, l)$; $y_{ij1}^{(2)}, \ldots, y_{ijs_{ij}}^{(2)}$ $(1 \leq i \leq j,\ j = 1, \ldots, l)$ in e_2V such that each W_i $(1 \leq i \leq l)$ admits the direct sum decomposition over A*

$$W_i = \oplus_{j=1}^{i}(\oplus_{m=1}^{j-1} AY_{m,(j-1)} \oplus X_j^{(1)} \oplus X_j^{(2)}) \oplus_A (\oplus_{j=iA}^{l} \oplus_{m=1\ A}^{i} Y_{mj})$$

with

$$X_j^{(1)} = \langle x_{j1}^{(1)}, \ldots, x_{j\alpha_j}^{(1)} \rangle_k,$$
$$X_j^{(2)} = \langle x_{j1}^{(2)}, \ldots, x_{j\beta_j}^{(2)} \rangle_k,$$
$$Y_{ij} = \langle y_{ij1}^{(1)} + y_{ij1}^{(2)}, \ldots, y_{ijs_{ij}}^{(1)} + y_{ijs_{ij}}^{(2)} \rangle_k.$$

Moreover, $\mathrm{comp}_A W_i = \oplus_{j=1}^{i}(\oplus_{m=1}^{j-1} AY_{m,(j-1)} \oplus X_j^{(1)} \oplus X_j^{(2)})$.

Proof. As the proof of this theorem is parallel to that of Theorem 4.1, we will be somewhat briefer. One can find for each fixed j $(1 \leq j \leq l))$ a k-subspace Y_{ij} of $W_i \cap \mathrm{comp}_A W_{j+1}$, by using Lemma 3.2(ii), such that $W_i \cap \mathrm{comp}_A W_{j+1} + \mathrm{comp}_A W_j = \mathrm{comp}_A W_j \oplus_A \oplus_{m=1A}^{i} Y_{mj}$, for $1 \leq i \leq j$. Then necessarily $\dim_k Y_{ij} = s_{ij}$ by the remark before Theorem 5.1. Since for $i = j$ we have $W_j \cap \mathrm{comp}_A W_{j+1} = \mathrm{comp}_A W_j \oplus_A \oplus_{m=1A}^{j} Y_{mj} \leq \mathrm{comp}_A W_{j+1}$, we can choose for $j = 0, \ldots, l-1$ an A-submodule X_{j+1} of V such that

$$\mathrm{comp}_A W_{j+1} = \mathrm{comp}_A W_j \oplus \oplus_{m=1}^{j} AY_{mj} \oplus X_{j+1}.$$

Put $X_{j+1}^{(1)} = e_1 X_{j+1}$, $X_{j+1}^{(2)} = e_2 X_{j+1}$ for $j = 0, \ldots, l-1$, so that we have the direct sum of A-modules

$$\mathrm{comp}_A W_{j+1} = \mathrm{comp}_A W_j \oplus \oplus_{m=1}^{j} AY_{mj} \oplus X_{j+1}^{(1)} \oplus X_{j+1}^{(2)}. \tag{1}$$

This yields the stated expression for $\mathrm{comp}_A W_i$. The same argument as in the proof of Theorem 4.1 can be applied to show that for each i

$$W_i = \mathrm{comp}_A W_i \oplus_A (\oplus_{j=iA}^{l} \oplus_{m=1\ A}^{i} Y_{mj}). \tag{2}$$

Applying e_1 to (1) with j replaced by $j-1$, we get

$$(3)\qquad e_1W_j \cap W_j = e_1W_{j-1} \cap W_{j-1} \oplus \oplus_{m=1}^{j-1} e_1Y_{m,(j-1)} \oplus X_j^{(1)}$$

[c.f., 4, Proposition 2.5 and 2.6].

Applying e_1 to (2) with i replaced by $j-1$, we get

$$(4)\qquad e_1W_{j-1} = e_1W_{j-1} \cap W_{j-1} \oplus \oplus_{a=j-1}^{l} \oplus_{m=1}^{j-1} e_1Y_{ma}.$$

Since

$$(5)\qquad AW_l = \oplus_{j=1}^{l}(X_j^{(1)} \oplus X_j^{(2)}) \oplus (\oplus_{j=1}^{l} \oplus_{m=1}^{j} AY_{mj})$$

from the expression for W_l, the intersection of (3) and (4) is

$$(6)\qquad e_1W_{j-1} \cap W_j = e_1W_{j-1} \cap W_{j-1} \oplus \oplus_{m=1}^{j-1} e_1Y_{m,(j-1)}$$

and hence from (3) and (6) we have $\alpha_j = \dim_k X_j^{(1)}$. Similarly, $\beta_j = \dim_k X_j^{(2)}$. To finish up the proof, choose a basis $x_{j1}^{(1)}, \ldots, x_{j\alpha_j}^{(1)}$ of $X_j^{(1)}$, a basis $x_{j1}^{(2)}, \ldots, x_{j\beta_j}^{(2)}$ of $X_j^{(2)}$, and a basis $y_{ij1}, \ldots, y_{ijs_{ij}}$ of Y_{ij} (over k). Put $e_1y_{ija} = y_{ija}^{(1)}$, $e_2y_{ija} = y_{ija}^{(2)}$, for $a = 1, \ldots, s_{ij}$, so that $y_{ija} = y_{ija}^{(1)} + y_{ija}^{(2)}$. Then the required k-independence of the vectors follows from the directness of the expression for AW_l in (5). □

Theorem 5.2. *Two flags of k-subspaces of V are in the same orbit for $GL_A(V)$ if and only if they have the same type.*

Remarks. (1) Let $(W_i)_{i=0}^{l+1}$ be a flag of k-subspaces of V with type $(W_i)_{i=0}^{l+1} = (\alpha_j; \beta_j; s_{ij})$. Then

$$\mathbf{dim}_A \mathrm{comp}_A W_i = \Big(\sum_{j=1}^{i}(\alpha_j + \sum_{m=1}^{j-1} s_{m,(j-1)}), \sum_{j=1}^{i}(\beta_j + \sum_{m=1}^{j-1} s_{m,(j-1)})\Big),$$

$$\mathbf{dim}_A AW_i = \Big(\sum_{j=1}^{i}(\alpha_j + \sum_{m=1}^{j-1} s_{m,(j-1)}) + \sum_{j=i}^{l}\sum_{m=1}^{i} s_{mj},$$
$$\sum_{j=1}^{i}(\beta_j + \sum_{m=1}^{j-1} s_{m,(j-1)}) + \sum_{j=i}^{l}\sum_{m=1}^{i} s_{mj}\Big),$$

$$\dim_k W_i = \sum_{j=1}^{i}(\alpha_j + \beta_j + 2\sum_{m=1}^{j-1} s_{m,(j-1)}) + \sum_{j=i}^{l}\sum_{m=1}^{i} s_{mj}.$$

(2) From the above expression for $\mathbf{dim}_A AW_l$, we see that an $l(l+5)/2$-tuple of nonnegative integers $(\alpha_j\ (j=1,\dots,l); \beta_j\ (j=1,\dots,l); s_{ij}(1 \le i \le j \text{ for } j=1,\dots,l))$ is the type of some flag $(W_i)_{i=0}^{l+1}$ of k-subspaces of V if and only if

$$\sum_{j=1}^{l} \alpha_j + \sum_{j=1}^{l}\sum_{m=1}^{j} s_{mj} \le n_1$$

and

$$\sum_{j=1}^{l} \beta_j + \sum_{j=1}^{l}\sum_{m=1}^{j} s_{mj} \le n_2.$$

(3) Let $m_1, m_2, \dots, m_l$ be positive integers such that $m_1+m_2+\dots+m_l < n_1+n_2$. Then

$$\begin{aligned}
&|P_{m_1,m_2,\dots,m_l}\backslash GL(n_1+n_2,k)/GL(n_1,k)\times GL(n_2,k)| \\
=& \Big|\{(\alpha_j;\beta_j;s_{ij}) \in \mathbb{N}^{\frac{l(l+5)}{2}} \mid \sum_{j=1}^{l}\alpha_j + \sum_{j=1}^{l}\sum_{m=1}^{j} s_{mj} \le n_1, \\
&\sum_{j=1}^{l}\beta_j + \sum_{j=1}^{l}\sum_{m=1}^{j} s_{mj} \le n_2, \text{ and} \\
&\sum_{j=1}^{i}(\alpha_j + \beta_j + 2\sum_{m=1}^{j-1} s_{m,(j-1)}) + \sum_{j=i}^{l}\sum_{m=1}^{i} s_{mj} \\
&= m_1 + m_2 + \dots + m_i \quad \text{for} \quad i=1,\dots,l\}\Big|.
\end{aligned}$$

6. The symplectic case with $A = F$

In this section, F/k is a quadratic extension and (V,B) is a symplectic space of dimension $2n$ over F. Fix $0 \ne \alpha \in F$ such that $\phi(\alpha) = 0$.

For a flag $(W_i)_{i=0}^{l+1}$ of isotropic k-subspaces of V, we define the type of $(W_i)_{i=0}^{l+1}$ to be the $l(l+2)$-tuple of nonnegative integers

$$\begin{aligned}
&\text{type}(W_i)_{i=0}^{l+1} \\
=& \Big(\dim_F(\text{comp}_F W_j/F(\text{comp}_F W_j \cap W_{j-1}))\ (j=1,\dots,l); \\
&\dim_k(W_i \cap \text{comp}_F W_{j+1}/W_{i-1} \cap \text{comp}_F W_{j+1} + W_i \cap \text{comp}_F W_j) \\
&(1 \le i \le j \quad \text{for} \quad j=1,\dots,l-1); \\
&\dim_k(W_j \cap W_l^\perp/W_{j-1} \cap W_l^\perp + W_j \cap \text{comp}_F W_l)\ (j=1,\dots,l); \\
&\dim_k(W_i \cap W_{l-j}^\perp/W_{i-1} \cap W_{l-j}^\perp + W_i \cap W_{l-j+1}^\perp) \\
&(1 \le i \le l-j \quad \text{for} \quad j=1,\dots,l-1); \\
&\frac{1}{2}\dim_k(W_j \cap W_{j-1}^\perp/\text{rad}_B W_{j-1} + \text{rad}_B W_j)\ (j=1,\dots,l)\Big).
\end{aligned}$$

For brevity, we will write $\text{type}(W_i)_{i=0}^{l+1} = (r_j; s_{ij}; t_j; p_{ij}; q_j)$ for the above expression of the type of $(W_i)_{i=0}^{l+1}$.

We will see in the proof of the theorem below that the last subfamily of parameters are indeed integers.

Remark. Note that the fourth subfamily of parameters in the above have the following different expressions

$$\begin{aligned}&\dim_k(W_i \cap W_{l-j}^{\perp}/W_{i-1} \cap W_{l-j}^{\perp} + W_i \cap W_{l-j+1}^{\perp})\\ =&\dim_k(W_i \cap W_{l-j}^{\perp} + \sum_{m=l-j+1}^{l} \text{rad}_B W_m / W_{i-1} \cap W_{l-j}^{\perp} + \sum_{m=l-j+1}^{l} \text{rad}_B W_m),\end{aligned}$$

for $1 \le i \le l-j$, $j = 1, \ldots, l-1$.

The following lemma will be used freely in the proof of the following theorem.

Lemma 6.1. *Let $(W_i)_{i=0}^{l+1} \in FLAG(V, B')$. Put $\tilde{W}_j = W_j + rad_B W_l$, for $j = 1, \ldots, l$. Then*

(i) $rad_B \tilde{W}_j = rad_B W_j + rad_B W_l$.

(ii) $comp_F \tilde{W}_j = comp_F W_l = comp_F rad_B \tilde{W}_j$.

The proof of the following proposition is left to the reader [c.f., 5, proof of Theorem 5.1 and the following remarks].

Proposition 6.2. *Let W be a subspace of V. Then the followings are equivalent.*

(a) *W is B'-isotropic and $rad_B W = 0$.*

(b) *For every nonzero vector $u \in W$, there exists $\alpha v \in W$ with $B(u, v) = 1$.*

(c) *$W = \langle u_1, \ldots, u_r \rangle_k \oplus_F \langle \alpha v_1, \ldots, \alpha v_r \rangle_k$, where $u_1, v_1, \ldots, u_r, v_r$ form a hyperbolic sequence in (V, B).*

(d) *FW is hyperbolic and contains W as a maximal B'-isotropic subspace.*

Theorem 6.3. *Let $(W_i)_{i=0}^{l+1}$ be a flag of B'-isotropic k-subspaces of V with $\text{type}(W_i)_{i=0}^{l+1} = (r_j; s_{ij}; t_j; p_{ij}, q_j)$. Then there exists a hyperbolic sequence*

$$x_{j1}, x'_{j1}, \ldots, x_{jr_j}, x'_{jr_j} \ (j = 1, \ldots, l);$$

$$y_{ij1}, y'_{ij1}, \ldots, y_{ijs_{ij}}, y'_{ijs_{ij}} \ (1 \le i \le j \text{ for } j = 1, \ldots, l-1);$$

$$z_{j1}, z'_{j1}, \ldots, z_{jt_j}, z'_{jt_j} \ (j = 1, \ldots, l);$$

$$g_{ij1}, g'_{ij1}, \ldots, g_{ijp_{ij}}, g'_{ijp_{ij}} \ (1 \le i \le l-j \text{ for } j = 1, \ldots, l-1);$$

$$h_{j1}, h'_{j1}, \ldots, h_{jq_j}, h'_{jq_j} \ (j = 1, \ldots, l) \text{ in } (V, B)$$

such that for $1 \le i \le l$

$$\begin{aligned}W_i = {} & \oplus_{j=1}^{i}\,(\oplus_{m=1}^{j-1} FY_{m,(j-1)} \oplus X_j) \perp_F (\perp_{j=i}^{l-1}{}_F \perp_{m=1}^{i}{}_F Y_{mj}) \\ & \perp_F (\perp_{j=1}^{i}{}_F Z_j) \perp_F (\perp_{j=1}^{l-i}{}_F \perp_{m=1}^{i}{}_F G_{mj}) \\ & \perp_F (\perp_{j=l-i+1}^{l-1}{}_F \perp_{m=1}^{l-j}{}_F G_{mj} \oplus_F \alpha G'_{mj}) \\ & \perp_F (\perp_{j=1}^{i}{}_F H_j \oplus_F \alpha H'_j)\end{aligned}$$

with

$$\begin{aligned}X_j &= \langle x_{j1}, \ldots, x_{jr_j} \rangle_F, \\ Y_{ij} &= \langle y_{ij1}, \ldots, y_{ijs_{ij}} \rangle_k, \\ Z_j &= \langle z_{j1}, \ldots, z_{jt_j} \rangle_k, \\ G_{ij} &= \langle g_{ij1}, \ldots, g_{ijp_{ij}} \rangle_k, \\ G'_{ij} &= \langle g'_{ij1}, \ldots, g'_{ijp_{ij}} \rangle_k, \\ H_j &= \langle h_{j1}, \ldots, h_{jq_j} \rangle_k, \\ H'_j &= \langle h'_{j1}, \ldots, h'_{jq_j} \rangle_k.\end{aligned}$$

Moreover,

$$\begin{aligned}comp_F W_i &= \oplus_{j=1}^{i}(\oplus_{m=1}^{j-1} FY_{m,(j-1)} \oplus X_j), \\ rad_B W_i &= comp_F W_i \perp_F (\perp_{j=i}^{l-1}{}_F \perp_{m=1}^{i}{}_F Y_{mj}) \\ & \qquad \perp_F (\perp_{j=1}^{i}{}_F Z_j) \perp_F (\perp_{j=1}^{l-i}{}_F \perp_{m=1}^{i}{}_F G_{mj}).\end{aligned}$$

Remark. Let W be an B'-isotropic k-subspace of V with $\mathrm{rad}_B W = 0$. As we saw in Proposition 6.2, such a subspace W can be expressed as

$$W = \langle u_1, \ldots, u_r \rangle_k \oplus_F \langle \alpha v_1, \ldots, \alpha v_r \rangle_k$$

for some hyperbolic sequence $u_1, v_1, \ldots, u_r, v_r$ in (V, B).

As in the statement of the above theorem, such a subspace will be, without further comment, denoted by $D \oplus_F \alpha D'$, with the understanding that for some choice of hyperbolic sequence $u_1, v_1, \ldots, u_r, v_r$ in (V, B) we can write

$$\begin{aligned}D \oplus_F \alpha D' &= \langle u_1, \ldots, u_r \rangle_k \oplus_F \langle \alpha v_1, \ldots, \alpha v_r \rangle_k, \\ D &= \langle u_1, \ldots, u_r \rangle_k, \\ \alpha D' &= \langle \alpha v_1, \ldots, \alpha v_r \rangle_k.\end{aligned}$$

Proof of Theorem 6.3. The proof will involve a series of lemmas. We first start with a coarse decomposition of $\tilde{W}_j = W_j + \mathrm{rad}_B W_l$ $(j = 1, \ldots, l)$.

Lemma 6.4. *For $j = 1, \ldots, l$,*

$$\tilde{W}_j = (W_j \cap W_{j+1}^{\perp} + rad_B W_l) \perp_F G_{l-j} \perp_F (\perp_{m=l-j+1}^{l-1} {}_F G_m \oplus_F \alpha G'_m)$$
$$\perp_F (\perp_{m=1}^{j} {}_F H_m \oplus_F \alpha H'_m),$$

where $G_{l-j} \leq W_j$, $\alpha G'_{l-j+1} \leq W_j$, $H_j, \alpha H'_j \leq W_j$ for $j = 1, \ldots, l$, $G_0 = 0$, $\alpha G'_l = 0$, and

$$rad_B \tilde{W}_j = (W_j \cap W_{j+1}^{\perp} + rad_B W_l) \perp_F G_{l-j}.$$

Proof. Proceed induction on j. For $j = 1$, choose a subspace G_{l-1} of $\mathrm{rad}_B W_1$ such that

$$\mathrm{rad}_B \tilde{W}_1 = (W_1 \cap W_2^{\perp} + \mathrm{rad}_B W_l) \oplus G_{l-1}.$$

Then

$$\mathrm{rad}_B \tilde{W}_1 = (W_1 \cap W_2^{\perp} + \mathrm{rad}_B W_l) \perp_F G_{l-1}.$$

Here one can apply Lemma 3.1, since $\mathrm{comp}_F\, \mathrm{rad}_B \tilde{W}_1 = \mathrm{comp}_F W_l \leq W_1 \cap W_2^{\perp} + \mathrm{rad}_B W_l$ by Lemma 6.1. With the notational convention of the above remark in mind, we may write [5, Theorem 5.1]

$$\tilde{W}_1 = (W_1 \cap W_2^{\perp} + \mathrm{rad}_B W_l) \perp_F G_{l-1} \perp_F (H_1 \oplus_F \alpha H'_1)$$

for some $H_1, \alpha H'_1 \leq W_1$.

Assume that we have constructed $G_{l-i} \leq \mathrm{rad}_B W_i$, $\alpha G'_{l-i+1} \leq W_i$ ($\alpha G'_l = 0$), $H_i, \alpha H'_i \leq W_i$ for $i = 1, \ldots, j$ ($j \leq l-1$) so that $\tilde{W}_1, \ldots, \tilde{W}_j$ admit the stated decompositions. Put

$$R = \oplus_{m=l-j+1}^{l-1} (G_m \oplus \alpha G'_m) \oplus \oplus_{m=1}^{j} (H_m \oplus \alpha H'_m).$$

Then by Lemma 3.4(ii),

$$\tilde{W}_{j+1} = R \perp_F U$$

with $U = \tilde{W}_{j+1} \cap R^{\perp}$.

We can choose, as in the beginning of this proof, $G_{l-j-1} \leq \mathrm{rad}_B W_{j+1}$ such that

$$\mathrm{rad}_B U = \mathrm{rad}_B \tilde{W}_{j+1} = (W_{j+1} \cap W_{j+2}^{\perp} + \mathrm{rad}_B W_l) \perp_F G_{l-j-1}.$$

$\mathrm{rad}_B \tilde{W}_{j+1}$ and G_{l-j} are contained in U, and in addition $\mathrm{rad}_B \tilde{W}_{j+1} \perp_F G_{l-j}$ holds. Here one must observe that $\mathrm{comp}_F(\mathrm{rad}_B \tilde{W}_{j+1} \oplus G_{l-j}) = \mathrm{comp}_F W_l \leq \mathrm{rad}_B \tilde{W}_{j+1}$. Assume that $G_{l-j} \neq 0$, otherwise put $\alpha G'_{l-j} = 0$ and find $H_{j+1}, \alpha H'_{j+1} \leq W_{j+1}$ so that $U = \mathrm{rad}_B U \perp_F (H_{j+1} \oplus_F \alpha H'_{j+1})$. Choose $u_1 \neq 0 \in G_{l-j}$. Since u_1 is not

B-orthogonal to W_{j+1}, we can choose $\alpha v_1 \in U \cap W_{j+1}$ such that $B(u_1, v_1) = 1$. Then

$$G_{l-j} = G_{l-j} \cap \langle u_1, \alpha v_1 \rangle_k^{\perp} \perp_F \langle u_1 \rangle_k$$

and

$$U \geq (\mathrm{rad}_B U \perp_F \langle u_1, \alpha v_1 \rangle_k \perp_F G_{l-j} \cap \langle u_1, \alpha v_1 \rangle_k^{\perp}).$$

We can find by induction on $\dim_k G_{l-j}$ a subspace $\alpha G'_{l-j} \leq U \cap W_{j+1}$ such that

$$U \geq (\mathrm{rad}_B U \perp_F G_{l-j} \oplus_F \alpha G'_{l-j}).$$

Finally, by the proof of Theorem 5.1 in [5], we may write

$$U = \mathrm{rad}_B U \perp_F (G_{l-j} \oplus_F \alpha G'_{l-j}) \perp_F (H_{j+1} \oplus_F \alpha H'_{j+1})$$

for some $H_{j+1}, \alpha H'_{j+1} \leq W_{j+1}$. □

Fix j $(1 \leq j \leq l-1)$. For $1 \leq i \leq l-j$, choose $G_{ij} \leq W_i \cap W_{l-j}^{\perp}$ such that

$$W_i \cap W_{l-j}^{\perp} + \sum_{m=l-j+1}^{l} \mathrm{rad}_B W_m = (W_{i-1} \cap W_{l-j}^{\perp} + \sum_{m=l-j+1}^{l} \mathrm{rad}_B W_m) \oplus G_{ij}.$$

Then actually we have

$$(1) \quad W_i \cap W_{l-j}^{\perp} + \sum_{m=l-j+1}^{l} \mathrm{rad}_B W_m = (W_{i-1} \cap W_{l-j}^{\perp} + \sum_{m=l-j+1}^{l} \mathrm{rad}_B W_m) \perp_F G_{ij}$$

by the same reasoning used several times before. Applying (1) repeatedly for $i = l-j, l-j-1, \ldots, 1$, we see that

$$(2) \qquad \mathrm{rad}_B W_{l-j} + \sum_{m=l-j+1}^{l} \mathrm{rad}_B W_m = \sum_{m=l-j+1}^{l} \mathrm{rad}_B W_m \perp_F \perp_{i=1}^{l-j}{}_F G_{ij}.$$

Intersecting (2) with W_{l-j}, we get

$$(3) \qquad \mathrm{rad}_B W_{l-j} = W_{l-j} \cap W_{l-j+1}^{\perp} \perp_F \perp_{i=1}^{l-j}{}_F G_{ij}$$

for $j = 1, \ldots, l-1$.

In turn, (3) implies that

$$\mathrm{rad}_B \tilde{W}_j = (W_j \cap W_{j+1}^{\perp} + \mathrm{rad}_B W_l) \perp_F \perp_{i=1}^{j}{}_F G_{l,(l-j)}$$

for $j = 1, \ldots, l$.

So we may replace each G_{l-j} in the above lemma, since in addition we have $\perp_{i=1}^{j}{}_F G_{i,(l-j)}$
$\leq \mathrm{rad}_B W_j$, by $\perp_{i=1}^{j}{}_F G_{i,(l-j)}$ without destroying it. Also, note that by (1) and the remark before Lemma 6.1 we have

$$\dim_k G_{ij} = \dim_k (W_i \cap W_{l-j}^{\perp} + \sum_{m=l-j+1}^{l} \mathrm{rad}_B W_m / W_{i-1} \cap W_{l-j}^{\perp} + \sum_{m=l-j+1}^{l} \mathrm{rad}_B W_m)$$
$$= p_{ij}.$$

Lemma 6.5. *$W_i = W_i \cap W_l^{\perp} \perp_F \perp_{j=1}^{l-i}{}_F \perp_{m=1}^{i}{}_F G_{mj}$*

$$\perp_F \perp_{j=l-i+1}^{l-1}{}_F (\perp_{m=1}^{l-j}{}_F G_{mj} \oplus_F \alpha G'_j) \perp_F (\perp_{j=1}^{i}{}_F H_j \oplus_F \alpha H'_j) \tag{4}$$

for $i = 1, \ldots, l$.

Proof. It is enough to show for $i = 1, \ldots, l$

$$\mathrm{rad}_B W_i = W_i \cap W_l^{\perp} \perp_F \perp_{j=1}^{l-i}{}_F \perp_{m=1}^{i}{}_F G_{mj}, \tag{5}$$

as one can see by comparing (4) with the decomposition of W_i obtained from the above lemma (intersect both sides of the decomposition of $\tilde{W}_i$ there with W_i). From (1), for $1 \leq i \leq j \leq l-1$,

$$W_i \cap W_j^{\perp} + \sum_{m=j+1}^{l} \mathrm{rad}_B W_m = \sum_{j+1}^{l} \mathrm{rad}_B W_m \perp_F \perp_{m=1}^{i}{}_F G_{m,(l-j)}. \tag{6}$$

From this, we have

$$W_i \cap W_{j+1}^{\perp} \perp_F \perp_{m=1}^{i}{}_F G_{m,(l-j)},$$

which is contained in $W_i \cap W_j^{\perp}$.

Now,

$$W_i \cap W_j^{\perp} = W_i \cap W_{j+1}^{\perp} \perp_F \perp_{m=1}^{i}{}_F G_{m,(l-j)} \tag{7}$$

for $1 \leq i \leq j \leq l-1$, since (6) implies that

$$\dim_k \oplus_{m=1}^{i} G_{m,(l-j)} = \dim_k (W_i \cap W_j^{\perp} / W_i \cap W_{j+1}^{\perp}).$$

Applying (7) repeatedly for $j = i, i+1, \ldots, l-1$, we get

$$\mathrm{rad}_B W_i = W_i \cap W_l^{\perp} \perp_F \perp_{j=1}^{l-i}{}_F \perp_{m=1}^{i}{}_F G_{mj}$$

for $i = 1, \ldots, l$ (for $i = l$, it is trivially true). $\square$

Next, we want to decompose each $\alpha G'_j$ ($j = 1, \ldots, l-1$) according to the decomposition $G_j = \perp_{m=1}^{l-j}{}_F G_{mj}$. We need the following lemma for that.

Lemma 6.6. *For $j = 1, \ldots, l-1$,*

$$G'_j = G'_j \cap (\perp_{m=2\,F}^{l-j} G_{mj})^{\perp} \perp_F G'_j \cap G_{1j}^{\perp}. \tag{8}$$

Proof. Since the decomposition on the right hand side of (8) holds and

$$\dim_k G'_j = \dim_k \oplus_{m=1}^{l-j} G_{mj},$$

it is enough to show that

$$\dim_k (G'_j \cap (\perp_{m=2\,F}^{l-j} G_{mj})^{\perp}) = \dim_k G_{1j}$$

and

$$\dim_k (G'_j \cap G_{1j}^{\perp}) = \dim_k \oplus_{m=2}^{l-j} G_{mj}.$$

Let Y denote the hyperbolic space $F(G_j \oplus G'_j)$. In this proof, various orthogonal complements will take place in Y. Note that

$$\begin{aligned}(G'_j + G_{1j}^{\perp})^{\perp'} &= (G'_j)^{\perp'} \cap FG_{1j} \\ &= (FG'_j \oplus \alpha G_j) \cap FG_{1j} \\ &= \alpha G_{1j}.\end{aligned}$$

Thus

$$\begin{aligned}\dim_k G_{1j} &= \dim_k Y - \dim_k (G'_j + G_{1j}^{\perp}) \\ &= \dim_k Y - (\dim_k G'_j + \dim_k Y - 2\dim_k G_{1j} - \dim_k (G'_j \cap G_{1j}^{\perp})),\end{aligned}$$

which shows that $\dim_k (G'_j \cap G_{1j}^{\perp}) = \dim_k \oplus_{m=2}^{l-j} G_{mj}$. The same argument applies to the other dimensional relation. $\square$

Lemma 6.7. *With $G_j = \perp_{m=1}^{l-j} G_{mj}$ and for $j = 1, \ldots, l-1$, there exists a decomposition $G'_j = \perp_{m=1\,F}^{l-j} G'_{mj}$ so that*

$$G_j \oplus_F \alpha G'_j = \perp_{m=1\,F}^{l-j} (G_{mj} \oplus_F \alpha G'_{mj}).$$

Proof. Let u_1 be a nonzero vector in G_{1j}. Then we can find a vector $v_1 \in G'_j \cap (\perp_{m=2\,F}^{l-j} G_{mj})^{\perp}$ such that $B(u_1, v_1) = 1$ by the above lemma. As usual, we have

$$G_j \oplus_F \alpha G'_j = \langle u_1, \alpha v_1 \rangle_k \perp_F (G_j \oplus_F \alpha G'_j) \cap \langle u_1, \alpha v_1 \rangle_k^{\perp},$$

of which the latter summand is

$$(\perp_{m=2F}^{l-j} G_{mj} \perp_F G_{1j} \cap \langle u_1, \alpha v_1 \rangle_k^{\perp}) \oplus_F (\alpha G'_j \cap \langle u_1, \alpha v_1 \rangle_k^{\perp}).$$

By proceeding induction on $t = \dim_k G_{1j}$ (temporary notation), we can choose a basis $u_2, \ldots, u_t$ for $G_{1j} \cap \langle u_1, \alpha v_1 \rangle_k^{\perp}$ and vectors $v_2, \ldots, v_t$ of $G'_j \cap \langle u_1, \alpha v_1 \rangle_k^{\perp} \cap (\perp_{m=2F}^{l-j} G_{mj})^{\perp}$ so that $u_1, v_1, \ldots, u_t, v_t$ form a hyperbolic sequence.

Put $G'_{1j} = \langle v_1, \ldots, v_t \rangle_k$.

Then

$$G_j \oplus_F \alpha G'_j = G_{1j} \oplus_F \alpha G'_{1j} \perp_F (G_j \oplus_F \alpha G'_j) \cap (G_{1j} \oplus_F \alpha G'_{1j})^{\perp},$$

of which the latter summand is

$$(\perp_{m=2F}^{l-j} G_{mj}) \oplus_F \alpha G'_j \cap (G_{1j} \oplus_F \alpha G'_{1j})^{\perp}.$$

We get the desired result by continuing in this fashion. □

So far we have achieved the decomposition

$$W_i = W_i \cap W_l^{\perp} \perp_F (\perp_{j=1F}^{l-i} \perp_{m=1F}^{i} G_{mj})$$

$$\perp_F (\perp_{j=l-i+1F}^{l-1} \perp_{m=1F}^{l-j} G_{mj} \oplus_F \alpha G'_{mj}) \perp_F (\perp_{j=1F}^{i} H_j \oplus_F \alpha H'_j), \tag{9}$$

for $i = 1, \ldots, l$.

From (9), one can directly see that

$$W_i \cap W_{i-1}^{\perp} = (\mathrm{rad}_B W_{i-1} + \mathrm{rad}_B W_i) \perp_F H_i \oplus_F \alpha H'_i,$$

and hence that $\dim_k H_j = q_j$. We now want to decompose $W_i \cap W_l^{\perp}$ in the desired way. Apply the Lemma 3.2(ii) to

$$W_1 \cap W_l^{\perp} \leq W_2 \cap W_l^{\perp} \leq \cdots \leq W_l \cap W_l^{\perp} = \mathrm{rad}_B W_l.$$

Then there exists a k-subspace $Z_i \leq W_i \cap W_l^{\perp}$ for $i = 1, \ldots, l$ such that

$$W_i \cap W_l^{\perp} + \mathrm{comp}_F W_l = \mathrm{comp}_F W_l \perp_F \perp_{j=1F}^{i} Z_j.$$

This implies that

$$W_i \cap W_l^{\perp} = W_i \cap \mathrm{comp}_F W_l \perp_F \perp_{j=1F}^{i} Z_j$$

and

$$\begin{aligned}\dim_k Z_j &= \dim_k(W_j \cap W_l^{\perp} + \mathrm{comp}_F W_l / W_{j-1} \cap W_l^{\perp} + \mathrm{comp}_F W_l) \\ &= \dim_k(W_j \cap W_l^{\perp} / W_{j-1} \cap W_l^{\perp} + W_j \cap \mathrm{comp}_F W_l) \\ &= t_j.\end{aligned}$$

Consider the flag $W_0'' = 0 \leq W_1'' = W_1 \cap \mathrm{comp}_F W_l \leq \cdots \leq W_{l-1}'' = W_{l-1} \cap \mathrm{comp}_F W_l \leq W_l'' = \mathrm{comp}_F W_l$ in the vector space $\mathrm{comp}_F W_l$ over F. Then we see that

$$\begin{aligned}&\mathrm{type}(W_i'')_{i=0}^{l} \\ = &(\dim_F(\mathrm{comp}_F W_j / F(\mathrm{comp}_F W_j \cap W_{j-1}))\ (j = 1, \ldots, l-1); \\ &\dim_k(W_i \cap \mathrm{comp}_F W_{j+1} / W_{i-1} \cap \mathrm{comp}_F W_{j+1} + W_i \cap \mathrm{comp}_F W_j) \\ &(1 \leq i \leq j \text{ for } j = 1, \ldots, l-1)) = (r_j; s_{ij}),\end{aligned}$$

where the type is the one defined in section 4. By Theorem 4.1 there exist a F-subspace X_j of $\dim_F X_j = r_j$ $(j = 1, \ldots, l-1)$ and a k-subspace Y_{ij} of $\dim_k Y_{ij} = s_{ij}$ $(1 \leq i \leq j$ for $j = 1, \ldots, l-1)$, of $\mathrm{comp}_F W_l$, such that for $i = 1, \ldots, l-1$

$$(10) \qquad W_i \cap \mathrm{comp}_F W_l = \oplus_{j=1}^{i}(\oplus_{m=1}^{j-1} F Y_{m,(j-1)} \oplus X_j) \perp_F (\perp_{j=i}^{l-1}{}_F \perp_{m=1}^{i}{}_F Y_{mj}).$$

Also, choose an F-subspace of $\mathrm{comp}_F W_l$ such that

$$\mathrm{comp}_F W_l = F(\mathrm{comp}_F W_l \cap W_{l-1}) \perp_F X_l.$$

Then (10) also holds for $i = l$ and $\dim_F X_l = r_l$. Thus we have obtained the stated decomposition in the theorem. To complete the proof, choose F-independent vectors $x_{j1}, \ldots, x_{jr_j}$ of X_j $(j = 1, \ldots, l); y_{ij1}, \ldots, y_{ijs_{ij}}$ of Y_{ij} $(1 \leq i \leq j$ for $y = 1, \ldots, l-1); z_{j1}, \ldots, z_{jt_j}$ of Z_j $(j = 1, \ldots, l)$, and choose hyperbolic sequences $g_{ij1}, g'_{ij1}, \ldots, g_{ijp_{ij}}, g'_{ijp_{ij}}$ of $G_{ij} \oplus_F G'_{ij}$ $(1 \leq i \leq l-j$ for $j = 1, \ldots, l-1)$; $h_{j1}, h'_{j1}, \ldots, h_{jq_j}, h'_{jq_j}$ of $H_j \oplus_F H'_j$ so that

$$\begin{aligned}G_{ij} &= \langle g_{ij1}, \ldots, g_{ijp_{ij}} \rangle_k, \\ G'_{ij} &= \langle g'_{ij1}, \ldots, g'_{ijp_{ij}} \rangle_k, \\ H_j &= \langle h_{j1}, \ldots, h_{jq_j} \rangle_k, \\ H'_j &= \langle h'_{j1}, \ldots, h'_{jq_j} \rangle_k.\end{aligned}$$

By Lemma 3.4(i), there exist F-independent vectors $x'_{j1}, \ldots, x'_{jr_j}$ $(j = 1, \ldots, l); y'_{ij1}, \ldots, y'_{ijs_{ij}}$ $(1 \leq i \leq j$ for $j = 1, \ldots, l-1); z'_{j1}, \ldots, z'_{jt_j}$ $(j = 1, \ldots, l)$ in V so that together with already chosen vectors in the above they form a hyperbolic sequence just as in the statement of the theorem. The proof of Theorem 6.3 is now complete. □

Theorem 6.8. *Two flags of B'-isotropic k-subspaces of V are in the same orbit for $Sp_F(V,B)$ if and only if they have the same type.*

Remarks. (1) Let $(W_i)_{i=0}^{l+1}$ be a flag of B'-isotropic k-subspaces of V with type$(W_i)_{i=0}^{l+1} = (r_j; s_{ij}; t_j; p_{ij}; q_j)$. Then

$$\begin{aligned}
\dim_F \operatorname{comp}_F W_i &= \sum_{j=1}^{i}(r_j + \sum_{m=1}^{j-1} s_{m,(j-1)}),\\
\dim_F F\operatorname{rad}_B W_i &= \sum_{j=1}^{i}(r_j + \sum_{m=1}^{j-1} s_{m,(j-1)}) + \sum_{j=i}^{l-1}\sum_{m=1}^{i} s_{mj} + \sum_{j=1}^{i} t_j + \sum_{j=1}^{l-i}\sum_{m=1}^{i} p_{mj},\\
\dim_F FW_i &= \sum_{j=1}^{i}(r_j + \sum_{m=1}^{j-1} s_{m,(j-1)}) + \sum_{j=i}^{l-1}\sum_{m=1}^{i} s_{mj} + \sum_{j=1}^{i} t_j\\
&\quad + \sum_{j=1}^{l-i}\sum_{m=1}^{i} p_{mj} + 2\sum_{j=l-i+1}^{l-1}\sum_{m=1}^{l-j} p_{mj} + 2\sum_{j=1}^{i} q_j,\\
\dim_k \operatorname{rad}_B W_i &= 2\sum_{j=1}^{i}(r_j + \sum_{m=1}^{j-1} s_{m,(j-1)}) + \sum_{j=i}^{l-1}\sum_{m=1}^{i} s_{mj} + \sum_{j=1}^{i} t_j + \sum_{j=1}^{l-i}\sum_{m=1}^{i} p_{mj},\\
\dim_k W_i &= 2\sum_{j=1}^{i}(r_j + \sum_{m=1}^{j-1} s_{m,(j-1)}) + \sum_{j=i}^{l-1}\sum_{m=1}^{i} s_{mj} + \sum_{j=1}^{i} t_j + \sum_{j=1}^{l-i}\sum_{m=1}^{i} p_{mj}\\
&\quad + 2\sum_{j=l-i+1}^{l-1}\sum_{m=1}^{l-j} p_{mj} + 2\sum_{j=1}^{i} q_j.
\end{aligned}$$

(2) From the expression of W_l in Theorem 6.3, one sees that an $l(l+2)$-tuple of nonnegative integers $(r_j\ (j = 1, \ldots, l); s_{ij}\ (1 \le i \le j \text{ for } j = 1, \ldots, l-1); t_j\ (j = 1, \ldots, l); p_{ij}\ (1 \le i \le l-j \text{ for } j = 1, \ldots, l-1); q_j\ (j = 1, \ldots, l))$ is the type of some flag $(W_i)_{i=0}^{l+1}$ of B'-isotropic k-subspaces of V if and only if

$$\sum_{j=1}^{l}(r_j + \sum_{m=1}^{j-1} s_{m,(j-1)}) + \sum_{j=1}^{l} t_j + \sum_{j=1}^{l-1}\sum_{m=1}^{l-j} p_{mj} + \sum_{j=1}^{l} q_j \le n.$$

(3) Let $m_1, m_2, \ldots, m_l$ be positive integers such that $m_1 + m_2 + \cdots + m_l \le 2n$.

Then

$$
\begin{aligned}
&|P_{m_1,m_2,\ldots,m_l}\backslash \mathrm{Sp}(4n,k)/\mathrm{Sp}(2n,F)| \\
= &\Big|\Big\{(r_j;s_{ij};t_j;p_{ij};q_j)\in \mathbb{N}^{l(l+2)} \mid \\
&\sum_{j=1}^{l}(r_j+\sum_{m=1}^{j-1}s_{m,(j-1)})+\sum_{j=1}^{l}t_j+\sum_{j=1}^{l-1}\sum_{m=1}^{l-j}p_{mj}+\sum_{j=1}^{l}q_j\le n \text{ and} \\
&2\sum_{j=1}^{i}(r_j+\sum_{m=1}^{j-1}s_{m,(j-1)})+\sum_{j=i}^{l-1}\sum_{m=1}^{i}s_{mj}+\sum_{j=1}^{i}t_j+\sum_{j=1}^{l-i}\sum_{m=1}^{i}p_{mj} \\
&+2\sum_{j=l-i+1}^{l-1}\sum_{m=1}^{l-j}p_{mj}+2\sum_{j=1}^{i}q_j=m_1+m_2+\cdots+m_i \text{ for } i=1,\ldots,l\Big\}\Big|.
\end{aligned}
$$

7. The symplectic case with $A = k \times k$

In this section, $A = k\times k$ and V is an A-module of bidimension $(2n_1, 2n_2)$ with a nondegenerate A-valued simplectic form B on V. Here

$$B(x,y)=e_1B_1(e_1x,e_1y)+e_2B_2(e_2x,e_2y),$$

$$B'(x,y)=B_1(e_1x,e_1y)+B_2(e_2x,e_2y).$$

For a flag $(W_i)_{i=0}^{l+1}$ of isotropic k-subspaces of V, we define the type of $(W_i)_{i=0}^{l+1}$ to be the $l(l+3)$-tuple of nonnegative integers

$$
\begin{aligned}
&\mathrm{type}(W_i)_{i=0}^{l+1} \\
= &\Big(\dim_k(e_1W_j\cap W_j/e_1W_{j-1}\cap W_j)\quad (j=1,\ldots,l); \\
&\dim_k(e_2W_j\cap W_j/e_2W_{j-1}\cap W_j)\quad (j=1,\ldots,l); \\
&\dim_k(W_i\cap \mathrm{comp}_AW_{j+1}/W_{i-1}\cap \mathrm{comp}_AW_{j+1}+W_i\cap \mathrm{comp}_AW_j) \\
&(1\le i\le j\quad \text{for}\quad j=1,\ldots,l-1); \\
&\dim_k(W_j\cap W_l^{\perp}/W_{j-1}\cap W_l^{\perp}+W_j\cap \mathrm{comp}_AW_l)\quad (j=1,\ldots,l); \\
&\dim_k(W_i\cap W_{l-j}^{\perp}/W_{i-1}\cap W_{l-j}^{\perp}+W_i\cap W_{l-j+1}^{\perp}) \\
&(1\le i\le l-j\quad \text{for}\quad j=1,\ldots,l-1); \\
&\frac{1}{2}\dim_k(W_j\cap W_{j-1}^{\perp}/\mathrm{rad}_BW_{j-1}+\mathrm{rad}_BW_j)\quad (j=1,\ldots,l)\Big).
\end{aligned}
$$

The above expression for the type of $(W_i)_{i=0}^{l+1}$ will be simply denoted by $(\alpha_j;\beta_j;s_{ij};t_j;p_{ij};q_j)$. Also, one can show that the last subfamily of parameters are indeed integers, as in section 6.

The proof of the following theorem will be completely omitted, since we think that the reader can provide by now his own proof (c.f., proofs of Theorem 5.1 and 6.3).

Theorem 7.1. *Let $(W_i)_{i=0}^{l+1}$ be a flag of B'-isotropic k-subspaces of V with type $(W_i)_{i=0}^{l+1} = (\alpha_j; \beta_j; s_{ij}; t_j; p_{ij}; q_j)$. Then there exist hyperbolic sequences consisting of*

$$x_{j1}^{(1)}, x_{j1}^{(1)'}, \ldots, x_{j\alpha_j}^{(1)}, x_{j\alpha_j}^{(1)'} \quad (j = 1, \ldots, l);$$
$$y_{ij1}^{(1)}, y_{ij1}^{(1)'}, \ldots, y_{ijs_{ij}}^{(1)}, y_{ijs_{ij}}^{(1)'} \quad (1 \le i \le j \text{ for } j = 1, \ldots, l-1);$$
$$z_{j1}^{(1)}, z_{j1}^{(1)'}, \ldots, z_{jt_j}^{(1)}, z_{jt_j}^{(1)'} \quad (j = 1, \ldots, l);$$
$$g_{ij1}^{(1)}, g_{ij1}^{(1)'}, \ldots, g_{ijp_{ij}}^{(1)}, g_{ijp_{ij}}^{(1)'} \quad (1 \le i \le l-j \text{ for } j = 1, \ldots, l-1);$$
$$h_{j1}^{(1)}, h_{j1}^{(1)'}, \ldots, h_{jq_j}^{(1)}, h_{jq_j}^{(1)'} \quad (j = 1, \ldots, l) \quad \text{in} \quad (e_1V, B_1) \text{ and}$$
$$x_{j1}^{(2)}, x_{j1}^{(2)'}, \ldots, x_{j\beta_j}^{(2)}, x_{j\beta_j}^{(2)'} \quad (j = 1, \ldots, l);$$
$$y_{ij1}^{(2)}, y_{ij1}^{(2)'}, \ldots, y_{ijs_{ij}}^{(2)}, y_{ijs_{ij}}^{(2)'} \quad (1 \le i \le j \text{ for } j = 1, \ldots, l-1);$$
$$z_{j1}^{(2)}, z_{j1}^{(2)'}, \ldots, z_{jt_j}^{(2)}, z_{jt_j}^{(2)'} \quad (j = 1, \ldots, l);$$
$$g_{ij1}^{(2)}, g_{ij1}^{(2)'}, \ldots, g_{ijp_{ij}}^{(2)}, g_{ijp_{ij}}^{(2)'} \quad (1 \le i \le l-j \text{ for } j = 1, \ldots, l-1);$$
$$h_{j1}^{(2)}, h_{j1}^{(2)'}, \ldots, h_{jq_j}^{(2)}, h_{jq_j}^{(2)'} \quad (j = 1, \ldots, l) \quad \text{in} \quad (e_2V, B_2)$$

such that for $1 \le i \le l$

$$\begin{aligned} W_i = {} & \oplus_{j=1}^{i}(\oplus_{m=1}^{j-1} AY_{m,(j-1)} \oplus X_j^{(1)} \oplus X_j^{(2)}) \\ & \perp_A(\perp_{j=i}^{l-1}{}_A \perp_{m=1}^{i}{}_A Y_{mj}) \perp_A (\perp_{j=1}^{i}{}_A Z_j) \\ & \perp_A(\perp_{j=1}^{l-i}{}_A \perp_{m=1}^{i}{}_A G_{mj}) \\ & \perp_A(\perp_{j=l-i+1}^{l-1}{}_A \perp_{m=1}^{l-j}{}_A G_{mj} \oplus_A \tilde{G}_{mj}) \perp_A (\perp_{j=1}^{i}{}_A H_j \oplus_A \tilde{H}_j) \end{aligned}$$

with

$$\begin{aligned} X_j^{(1)} &= \langle x_{j1}^{(1)}, \ldots, x_{j\alpha_j}^{(1)} \rangle_k, \\ X_j^{(2)} &= \langle x_{j1}^{(2)}, \ldots, x_{j\beta_j}^{(2)} \rangle_k, \\ Y_{ij} &= \langle y_{ij1}^{(1)} + y_{ij1}^{(2)}, \ldots, y_{ijs_{ij}}^{(1)} + y_{ijs_{ij}}^{(2)} \rangle_k, \\ Z_j &= \langle z_{j1}^{(1)} + z_{j1}^{(2)}, \ldots, z_{jt_j}^{(1)} + z_{jt_j}^{(2)} \rangle_k, \\ G_{ij} &= \langle g_{ij1}^{(1)} + g_{ij1}^{(2)}, \ldots, g_{ijp_{ij}}^{(1)} + g_{ijp_{ij}}^{(2)} \rangle_k, \\ \tilde{G}_{ij} &= \langle g_{ij1}^{(1)'} - g_{ij1}^{(2)'}, \ldots, g_{ijp_{ij}}^{(1)'} - g_{ijp_{ij}}^{(2)'} \rangle_k, \\ H_j &= \langle h_{j1}^{(1)} + h_{j1}^{(2)}, \ldots, h_{jq_j}^{(1)} + h_{jq_j}^{(2)} \rangle_k, \\ \tilde{H}_j &= \langle h_{j1}^{(1)'} - h_{j1}^{(2)'}, \ldots, h_{jq_j}^{(1)'} - h_{jq_j}^{(2)'} \rangle_k. \end{aligned}$$

Moreover,

$$\begin{aligned} comp_A W_i &= \oplus_{j=1}^{i}(\oplus_{m=1}^{j-1} AY_{m,(j-1)} \oplus X_j^{(1)} \oplus X_j^{(2)}), \\ rad_B W_i &= comp_A W_i \perp_A (\perp_{j=i}^{l-1}{}_A \perp_{m=1}^{i}{}_A Y_{mj}) \\ & \qquad \perp_A (\perp_{j=1}^{i}{}_A Z_j) \perp_A (\perp_{j=1}^{l-i}{}_A \perp_{m=1}^{i}{}_A G_{mj}). \end{aligned}$$

Remark. Just as in the case of $A = F$, we could provide a proposition characterizing B'-isotropic subspaces W with $\mathrm{rad}_B W = 0$. In the above theorem, $G_{ij} \oplus_A \tilde{G}_{ij}$ and $H_j \oplus_A \tilde{H}_j$ are those kinds of spaces and they can be generated by such bases in standard form as in the statement of the above theorem.

Theorem 7.2. *Two flags of B'-isotropic k-subspaces of V are in the same orbit for $Sp_A(V, B)$ if and only if they have the same type.*

Remarks. (1) Let $(W_i)_{i=0}^{l+1}$ be a flag of B'-isotropic k-subspaces of V with type $(W_i)_{i=0}^{l+1} = (\alpha_j; \beta_j; s_{ij}; t_j; p_{ij}; q_j)$. Then

$$\mathbf{dim}_A \mathrm{comp}_A W_i = \Big(\sum_{j=1}^{i}(\alpha_j + \sum_{m=1}^{j-1} s_{m,(j-1)}), \sum_{j=1}^{i}(\beta_j + \sum_{m=1}^{j-1} s_{m,(j-1)})\Big),$$

$$\mathbf{dim}_A A\mathrm{rad}_B W_i = \Big(\sum_{j=1}^{i}(\alpha_j + \sum_{m=1}^{j-1} s_{m,(j-1)}) + \sum_{j=i}^{l-1}\sum_{m=1}^{i} s_{mj} + \sum_{j=1}^{i} t_j + \sum_{j=1}^{l-i}\sum_{m=1}^{i} p_{mj},$$
$$\sum_{j=1}^{i}(\beta_j + \sum_{m=1}^{j-1} s_{m,(j-1)}) + \sum_{j=i}^{l-1}\sum_{m=1}^{i} s_{mj} + \sum_{j=1}^{i} t_j + \sum_{j=1}^{l-i}\sum_{m=1}^{i} p_{mj}\Big),$$

$$\mathbf{dim}_A AW_i = \Big(\sum_{j=1}^{i}(\alpha_j + \sum_{m=1}^{j-1} s_{m,(j-1)}) + \sum_{j=i}^{l-1}\sum_{m=1}^{i} s_{mj} + \sum_{j=1}^{i} t_j$$
$$+ \sum_{j=1}^{l-i}\sum_{m=1}^{i} p_{mj} + 2\sum_{j=l-i+1}^{l-1}\sum_{m=1}^{l-j} p_{mj} + 2\sum_{j=1}^{i} q_j,$$
$$\sum_{j=1}^{i}(\beta_j + \sum_{m=1}^{j-1} s_{m,(j-1)}) + \sum_{j=i}^{l-1}\sum_{m=1}^{i} s_{mj} + \sum_{j=i}^{i} t_j$$
$$+ \sum_{j=1}^{l-i}\sum_{m=1}^{i} p_{mj} + 2\sum_{j=l-i+1}^{l-1}\sum_{m=1}^{l-j} p_{mj} + 2\sum_{j=1}^{i} q_j\Big),$$

$$\dim_k \mathrm{rad}_B W_i = \sum_{j=1}^{i}(\alpha_j + \beta_j + 2\sum_{m=1}^{j-1} s_{m,(j-1)}) + \sum_{j=i}^{l-1}\sum_{m=1}^{i} s_{mj}$$
$$+ \sum_{j=1}^{i} t_j + \sum_{j=1}^{l-i}\sum_{m=1}^{i} p_{mj},$$

$$\dim_k W_i = \sum_{j=1}^{i}(\alpha_j + \beta_j + 2\sum_{m=1}^{j-1} s_{m,(j-1)}) + \sum_{j=i}^{l-1}\sum_{m=1}^{i} s_{mj}$$
$$+ \sum_{j=1}^{i} t_j + \sum_{j=1}^{l-i}\sum_{m=1}^{i} p_{mj} + 2\sum_{j=l-i+1}^{l-1}\sum_{m=1}^{l-j} p_{mj} + 2\sum_{j=1}^{i} q_j.$$

(2) From the expression of W_l in Theorem 7.1, one can easily see that an $l(l+3)$-tuple of nonnegative integers $(\alpha_j\ (j=1,\dots,l); \beta_j\ (j=1,\dots,l); s_{ij}\ (1 \le i \le j$ for $j=1,\dots,l-1); t_j\ (j=1,\dots,l); p_{ij}\ (1 \le i \le l-j$ for $j=1,\dots,l-1); q_j\ (j=1,\dots,l))$ is the type of some flag $(W_i)_{i=0}^{l+1}$ of B'-isotropic k-subspaces of V if and only if

$$\sum_{j=1}^{l}(\alpha_j + \sum_{m=1}^{j-1} s_{m,(j-1)}) + \sum_{j=1}^{l} t_j + \sum_{j=1}^{l-1}\sum_{m=1}^{l-j} p_{mj} + \sum_{j=1}^{l} q_j \le n_1$$

and

$$\sum_{j=1}^{l}(\beta_j + \sum_{m=1}^{j-1} s_{m,(j-1)}) + \sum_{j=1}^{l} t_j + \sum_{j=1}^{l-1}\sum_{m=1}^{l-j} p_{mj} + \sum_{j=1}^{l} q_j \le n_2.$$

(3) Let $m_1, m_2, \dots, m_l$ be positive integers such that $m_1+m_2+\cdots+m_l \le n_1+n_2$. Then

$$\begin{aligned}
&|P_{m_1,m_2,\dots,m_l}\backslash \mathrm{Sp}(2(n_1+n_2),k)/\mathrm{Sp}(2n_1,k)\times \mathrm{Sp}(2n_2,k)| \\
= &\Big|\Big\{(\alpha_j;\beta_j;s_{ij};t_j;p_{ij};q_j) \in \mathbb{N}^{l(l+3)} \mid \\
&\sum_{j=1}^{l}(\alpha_j + \sum_{m=1}^{j-1} s_{m,(j-1)}) + \sum_{j=1}^{l} t_j + \sum_{j=1}^{l-1}\sum_{m=1}^{l-j} p_{mj} + \sum_{j=1}^{l} q_j \le n_1, \\
&\sum_{j=1}^{l}(\beta_j + \sum_{m=1}^{j-1} s_{m,(j-1)}) + \sum_{j=1}^{l} t_j + \sum_{j=1}^{l-1}\sum_{m=1}^{l-j} p_{mj} + \sum_{j=1}^{l} q_j \le n_2, \\
&\text{and} \\
&\sum_{j=1}^{i}(\alpha_j + \beta_j + 2\sum_{m=1}^{j-1} s_{m,(j-1)}) + \sum_{j=i}^{l-1}\sum_{m=1}^{i} s_{mj} + \sum_{j=1}^{i} t_j \\
&+ \sum_{j=1}^{l-i}\sum_{m=1}^{i} p_{mj} + 2\sum_{j=l-i+1}^{l-1}\sum_{m=1}^{l-j} p_{mj} + 2\sum_{j=1}^{i} q_j \\
&= m_1 + m_2 + \cdots + m_i \quad \text{for} \quad i = 1,2,\dots,l)\Big\}\Big|.
\end{aligned}$$

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