Positive Definite Universal Ternary Lattices over $\mathbf{Q}(\sqrt{5})$

Myung-Hwan Kim *
Dept.of Math.,Seoul National University, Seoul 151-742, Korea

Abstract

In this article, all the universal totally positive integral ternary lattices over $\mathbf{Q}(\sqrt{5})$ are determined.

Mathematics Subject Classification (1991) :11E12, 11E20, 11E25
Short Running Title : Positive Definite Universal Ternary Lattices

*The author was partially supported by KOSEF (931-0100-003-2) and Korean Ministry of Education (BSRI-94-1414).

§1. Introduction

In 1917, Ramanujan [R] determined all 54 sets of positive integers a, b, c, d such that the quaternary quadratic form $ax^2 + by^2 + cz^2 + du^2$ represents all the positive integers (see also [D1]). The case $a = b = c = d = 1$ corresponds to the well-known Fermat-Lagrange theorem [L]. Such forms are called *universal* and later on Dickson [D2] studied non-diagonal universal forms.

On the other hand in 1945, Siegel [S1] gave a proof for the following theorem: any totally real number field K in which all the totally positive integers are sums of squares of algebraic integers in K has to be either the field $\mathbf{Q}$ of rational numbers or the real quadratic number field $\mathbf{Q}(\sqrt{5})$.

For not totally real number fields L, Siegel [S1] proved by using his 'generalized circle method', that all totally positive algebraic integers in L are sums of squares of algebraic integers in L if and only if the discriminant of L is odd. He also showed that *five* squares would suffice, possibly with *five* replaced by *four*, which was confirmed later by Peters [P]. See [H] for more recent developments.

Returning to totally real number field case, Maass' theorem [M] of 1941, that every totally positive algebraic integer in $\mathbf{Q}(\sqrt{5})$ is a sum of *three* squares of algebraic integers in $\mathbf{Q}(\sqrt{5})$ - *three* is, of course, the best possible over $\mathbf{Q}(\sqrt{5})$ as is *four* over $\mathbf{Q}$, naturally brings us to the following :

Determine all the positive ternary integral quadratic forms over the ring $\mathcal{O}$ of algebraic integers in $\mathbf{Q}(\sqrt{5})$ which are universal (i.e., represent all totally positive integers in $\mathcal{O}$).

In this article, the author found a complete answer for *classic* integral quadratic forms, i.e., those forms whose non-diagonal terms have coefficients divisible by 2, which in turn, in lattice theoretic language, correspond to free quadratic lattices whose scales are contained in the ring of integers. The answer is given by the following theorem - a kind of supplement to Maass' three square theorem.

Theorem. *There are only five universal positive definite ternary classic forms (up to equivalence) over the ring $\mathcal{O}$ of algebraic integers in $\mathbf{Q}(\sqrt{5})$. They are*

$$x^2 + y^2 + z^2, \quad x^2 + y^2 + 2z^2, \quad x^2 + y^2 + (2+\epsilon)z^2,$$

$$x^2 + 2y^2 + 2yz + (2+\epsilon)z^2, \quad x^2 + 2y^2 + 2yz + (2+\epsilon')z^2,$$

where $\epsilon = \frac{1+\sqrt{5}}{2}$ is the fundamental unit of $\mathcal{O}$, and ϵ' is its conjugate.

We prove this theorem in a lattice theoretic setting. In §2, we provide some necessary background. The proof of the theorem is given in §§3-4. We treat the diagonal case in §3, where we eliminate all the forms except the first three listed above. The universality then will be proved by showing that they represent all totally positive integers locally everywhere and have class number 1. The non-diagonal case will be treated similarly in §4 to obtain the other two forms in the list.

§2. Notations and Basic Properties.

We introduce some notations and basic properties in this section. Unexplained terminologies and simple facts will be referred to O'Meara [O1]. Let $K = \mathbf{Q}(\sqrt{5})$, $\mathcal{O} = \mathbf{Z}[\epsilon]$ be its ring of integers and $\mathcal{U} = \{\pm\epsilon^n | n \in \mathbf{Z}\}$ where ϵ is the fundamental unit defined in §1. A number ν in the field K is called *totally positive* (in symbols $\nu \succ 0$) if ν as well as its conjugate ν' over $\mathbf{Q}$ are both positive. Let $K_+ = \{\nu \in K \mid \nu \succ 0\}$ and $\mathcal{O}_+ = \mathcal{O} \cap K_+$. For any ν, η in K, we write $\nu \succ \eta$ if $\nu - \eta \succ 0$. Let $N(\nu) = \nu\nu'$ and $Tr(\nu) = \nu + \nu'$ be the norm and the trace of $\nu \in K$. Note that $\mathcal{U}_+ := \mathcal{U} \cap \mathcal{O}_+ = \{\epsilon^{2n} \mid n \in \mathbf{Z}\} = \mathcal{U}^2$.

Let L be an $\mathcal{O}$-lattice, i.e., a lattice equipped with an $\mathcal{O}$-valued non-degenerated symmetric bilinear form B. The corresponding quadratic map is denoted by Q. Since classic integral quadratic forms correspond to free $\mathcal{O}$-lattices, every $\mathcal{O}$-lattice in this paper will be assumed to be free. We say L *represents* ν in $\mathcal{O}$, if $\nu = Q(\mathbf{e})$ for some $\mathbf{e} \in L$. We call L *diagonal* if it admits an orthogonal basis and *non-diagonal* otherwise. Further, L is called *totally positive definite* or simply *positive* if $Q(\mathbf{e}) \succ 0$ for any $\mathbf{e} \in L \setminus \{\mathbf{0}\}$. A positive $\mathcal{O}$-lattice L is defined to be $\mathcal{O}_+$-*universal* or simply *universal*, if $Q(L) = \mathcal{O}_+ \cup \{0\}$.

We write $L = L_1 \perp L_2$ if $L = L_1 \oplus L_2$ and $B(\mathbf{e}_1, \mathbf{e}_2) = 0$ for all $\mathbf{e}_1 \in L_1, \mathbf{e}_2 \in L_2$. If $\mathbf{e}_1, \cdots, \mathbf{e}_n$ form a basis for L, we may write $L = \mathcal{O}\mathbf{e}_1 + \cdots + \mathcal{O}\mathbf{e}_n$ or $L \cong < B(\mathbf{e}_i, \mathbf{e}_j) >$. As usual, $dL = det(B(\mathbf{e}_i, \mathbf{e}_j))$ is called the *discriminant* of L and is well-defined modulo $\mathcal{U}^2$.

Remark. Let L be a universal positive ternary $\mathcal{O}$-lattice. Since L represents 1, there exists $\mathbf{e}_1 \in L$ with $Q(\mathbf{e}_1) = 1$ such that $\mathcal{O}\mathbf{e}_1 \cong < 1 >$ splits L, i.e., $L = \mathcal{O}\mathbf{e}_1 \perp L_0 \cong < 1 > \perp L_0$ for some positive binary sublattice L_0 of L(see [O1; 82:15]). We will use this splitting of L as well as the following lemma repeatedly in §§3-4.

Lemma. *Let $\alpha, \gamma \in K_+, \beta \in K_+ \cup \{0\}$, and $x \in K$ such that $\alpha x^2 + \beta = \gamma$. Then $N(\alpha)N(x)^2 \leq N(\gamma)$. In particular, $N(\alpha) \leq N(\gamma)$ if $0 \neq x \in \mathcal{O}$.*

Proof. $N(\alpha x^2+\beta) = N(\alpha)N(x)^2+Tr(\alpha x^2\beta')+N(\beta) = N(\gamma)$. Since $\alpha x^2\beta' \succ 0$ or $\alpha x^2\beta' = 0$, $Tr(\alpha x^2\beta') \geq 0$. So $N(\alpha)N(x)^2 \leq N(\gamma)$. The assertions follow immediately. $\square$

§3. Diagonal Case

In this section, we prove that there are only three universal positive ternary diagonal $\mathcal{O}$-lattices. Let L be a universal positive ternary diagonal $\mathcal{O}$-lattice. By the remark preceeding the Lemma in §2, we have $L \cong < 1 > \perp L_0$ for some positive binary diagonal sublattice L_0 of L.

Proposition 1. *Let L be a universal positive ternary diagonal $\mathcal{O}$-lattice. Then either (a) $L \cong < 1 > \perp < 1 > \perp < \gamma >$ with $1 \leq N(\gamma) \leq 11$, or (b) $L \cong < 1 > \perp < 2 > \perp < \delta >$ with $4 \leq N(\delta) \leq 5$.*

Proof. First, if we assume that L_0 represents a unit μ, then $\mu \in \mathcal{U}_+$ and hence L_0 represents 1. So by [O1] again, there exists $\mathbf{e}_2 \in L_0$ with $Q(\mathbf{e}_2) = 1$ such that $\mathcal{O}\mathbf{e}_2$ splits L_0, i.e., $L_0 = \mathcal{O}\mathbf{e}_2 \perp \mathcal{O}\mathbf{e}_3$ for some $\mathbf{e}_3 \in L_0$. Therefore $L \cong < 1 > \perp < 1 > \perp < \gamma >$ where $\gamma = Q(\mathbf{e}_3) \in \mathcal{O}_+$. If $L_1 = \mathcal{O}\mathbf{e}_1 \perp \mathcal{O}\mathbf{e}_2$, then L_1 can not represent $\pi_{11} = 3+\epsilon$; for, otherwise, π_{11} is a sum of two nonzero squares so that -1 is a square (mod π_{11}), which is impossible. However being universal, L represents π_{11}. Consequently, there exists $\mathbf{u} \in L_1$ such that $Q(\mathbf{u} + z\mathbf{e}_3) = Q(\mathbf{u}) + \gamma z^2 = \pi_{11}$ for some $0 \neq z \in \mathcal{O}$. By Lemma, $N(\gamma) \leq N(\pi_{11}) = 11$ because $Q(\mathbf{u}) \in \mathcal{O}_+ \cup \{0\}$; $N(\gamma) \geq 1$ is clear.

Next, let us assume that L_0 does not represent any unit. Since L represents 2, there exists $\mathbf{e} \in L_0$ such that $Q(x\mathbf{e}_1 + \mathbf{e}) = x^2 + Q(\mathbf{e}) = 2$ for some $x \in \mathcal{O}$. Since $Q(\mathbf{e}) \in \mathcal{O}_+ \cup \{0\}$, we have $0 \leq N(x^2) = N(x)^2 \leq 4$ and hence $N(x) = 0$ or 1 or 2. As 2 is not a norm, $\mathbf{e}$ must be nonzero. If $N(x) \neq 0$, then $2 \succ Q(\mathbf{e}) \succ 0$, and hence $N(Q(\mathbf{e})) = 1$ and $Q(\mathbf{e})$ is a unit since 3 cannot be a norm either. But this is impossible. So, $N(x) = 0$, or $x = 0$. Therefore, $Q(\mathbf{e}) = 2$, i.e., L_0 represents 2. On the other hand, since L_0 is diagonal, there exists an orthogonal basis $\{\mathbf{e}_2, \mathbf{e}_3\}$ of L_0 such that $L_0 \cong < \beta > \perp < \delta >$ with $Q(\mathbf{e}_2) = \beta, Q(\mathbf{e}_3) = \delta \in \mathcal{O}_+$ and $4 \leq N(\beta) \leq N(\delta)$. Then $\mathbf{e} = y\mathbf{e}_2 + z\mathbf{e}_3$ for some $y, z \in \mathcal{O}$ such that $Q(\mathbf{e}) = \beta y^2 + \delta z^2 = 2$. Again by the norm consideration, we may assume $\mathbf{e} = \mathbf{e}_2$ and $\beta = 2$. Thus $L \cong < 1 > \perp < 2 > \perp < \delta >$ where $\delta = Q(\mathbf{e}_3) \in \mathcal{O}_+$ and $N(\delta) \geq 4$. If $L_1 = \mathcal{O}\mathbf{e}_1 \perp \mathcal{O}\mathbf{e}_2$. L_1 does not represent $\pi_5 = 2 + \epsilon$; for, otherwise, -2 is a square (mod π_5), which is impossible. However, L represents π_5. So, there exists $\mathbf{v} \in L_1$ such that $Q(\mathbf{v} + z\mathbf{e}_3) = Q(\mathbf{v}) + \delta z^2 = \pi_5$ for some $0 \neq z \in \mathcal{O}$. Again by Lemma, $N(\delta) \leq N(\pi_5) = 5$ because $Q(\mathbf{v}) \in \mathcal{O}_+ \cup \{0\}$. $\square$

Note that we may now assume in Proposition 1 that $\gamma = 1, 2, 2+\epsilon, 3, 3+\epsilon, 3+\epsilon'$ and $\delta = 2, 2+\epsilon$ after multiplying by suitable unit squares if necessary. The proof of the following proposition is omitted since it is straightforward.

Proposition 2. *Over $\mathcal{O}$ we have the following:*
(a) $L \cong < 1 > \perp < 1 > \perp < 3 >$ *cannot represent* $3+\epsilon$.
(b) $L \cong < 1 > \perp < 1 > \perp < 3+\epsilon >$ *cannot represent* $7+\epsilon'$.
(c) $L \cong < 1 > \perp < 1 > \perp < 3+\epsilon' >$ *cannot represent* $7+\epsilon$.
(d) $L \cong < 1 > \perp < 2 > \perp < 2 >$ *cannot represent* $2+\epsilon$.
(e) $L \cong < 1 > \perp < 2 > \perp < 2+\epsilon >$ *cannot represent* $4+2\epsilon$.

Now we only have three candidates left, namely,

$$I = < 1 > \perp < 1 > \perp < 1 >,\ S = < 1 > \perp < 1 > \perp < 2 >,$$
$$T = < 1 > \perp < 1 > \perp < 2+\epsilon > .$$

Theorem 1. *Up to isometry, I, S, T are the only universal positive ternary diagonal $\mathcal{O}$-lattices.*
Proof. It remains to show that I, S, T are all universal. The universality of I is proved by Maass [M]. So we restrict ourselves to S and T. The proof will be finished if we can show:
(A) Both S and T represent all $\nu \in \mathcal{O}_+$ locally everywhere, and
(B) The class numbers of S and T are both 1.

Proof of (A): For any $\mathcal{O}$-lattice L, let $L_\wp$ be the localization of L at a prime $\wp$, i.e., $L_\wp = \mathcal{O}_\wp L$, where $\mathcal{O}_\wp$ is the $\wp$-adic integers. We also let $\mathcal{U}_\wp$ be the group of units in $\mathcal{O}_\wp$.
For S, we have
(1) $S_\wp$ represents all of $\mathcal{O}_\wp$ at every finite prime $\wp \neq (2)$ because $S_\wp$ is unimodular at such $\wp$. (See [O1;92:1b].)
(2) $S_\wp$ represents all positive real numbers at each real prime $\wp$.
(3) At $\wp = (2), S_\wp \cong < 1 > \perp < 1 > \perp < 2 >$. One can check that $S_\wp$ represents all $\mathcal{O}_\wp$ by using [O2]. We omit the details since it is routine.
For T, let $\pi = 2+\epsilon$. Then
(1) $T_\wp$ represents all of $\mathcal{O}_\wp$ at every finite prime $\wp \neq (2), (\pi)$ because $T_\wp$ is unimodular at such $\wp$ [O1;92:1b].
(2) $T_\wp$ represents all positive real numbers at each real prime $\wp$.
(3) At $\wp = (\pi)$, the binary unimodular component of $T_\wp$ represents all of $\mathcal{U}_\wp$, while $T_\wp$ obviously represents π. We have to show that $T_\wp$ represents $\pi\Delta$ where Δ is a nonsquare unit of $\mathcal{O}_\wp$. By [O2;Theorem 1], $T_\wp$ represents $\pi\Delta$ if

$K_\wp T_\wp$ represents $\pi\Delta$, which is equivalent as saying that $K_\wp(T_\wp \perp < -\pi\Delta >)$ is isotropic; this is true since its discriminant $-\Delta$ cannot be a square at $\wp$. So, $T_\wp$ represents all of $\mathcal{O}_\wp$. (See also [O1; 63:18].)
(4) At $\wp = (2), Q(L_\wp) = (\mathcal{O}_\wp^2 + 2\mathcal{O}_\wp) \cap Q(K_\wp L_\wp) = \mathcal{O}_\wp$ by [Ri].

To prove (B), we first note the the orthogonal groups $O(S)$ and $O(T)$ are of order 16. On the other hand the measures $m(S)$ and $m(T)$ can be computed easily by the Massformel in [K; Satz 4,6 and Hilfssatz 26]:

$$m(S) = \frac{(D_K)^{3/2}\zeta_K(2)}{4\pi^4} N(dS) \prod_{\wp|(2)} \xi(S_\wp, \wp)(1 - \frac{1}{N\wp^2}) = \frac{1}{16},$$

$$m(T) = \frac{(D_K)^{3/2}\zeta_K(2)}{4\pi^4} N(dT) \prod_{\wp|(2)} \xi(T_\wp, \wp)(1 - \frac{1}{N\wp^2}) \prod_{\wp|(\pi)} \frac{1}{2}(1 + (\frac{-1}{\wp})\frac{1}{N\wp}) = \frac{1}{16}.$$

Now applying Siegel's theorem [S2], we conclude that both lattices S and T have class number 1. □

§4. Non-diagonal Case

Let L be a universal positive ternary non-diagonal $\mathcal{O}$-lattice. We know that $L \cong < 1 > \perp L_0$ for some positive (non-diagonal) binary $\mathcal{O}$-lattice. L_0 cannot represent a unit since L would be diagonal otherwise. Then by the same argument in the proof of Proposition 1, L_0 should represent 2. Let $\mathbf{e}_2 \in L$ be such that $Q(\mathbf{e}_2) = 2$. Since $\mathbf{e}_2$ is a primitive vector, one can find $\mathbf{e}_3 \in L$ such that $\{\mathbf{e}_2, \mathbf{e}_3\}$ is a basis for L_0. Let $Q(\mathbf{e}_3) = \gamma \in \mathcal{O}_+$ and $B(\mathbf{e}_2, \mathbf{e}_3) = \beta \in \mathcal{O}$. If $2|\beta$, then $\mathcal{O}\mathbf{e}_2$ splits L_0, which contradicts to the assumption that L is non-diagonal. So $2 \not| \beta$ and hence $\beta = \mu + 2\nu$ with $\mu \in \{1, \epsilon, \epsilon'\}$ and $\nu \in \mathcal{O}$. But then, we may assume $\beta = 1$ by replacing $\mathbf{e}_3$ by $\mu^{-1}(\mathbf{e}_3 - \nu\mathbf{e}_2)$. Therefore,

$$L = \mathcal{O}\mathbf{e}_1 \perp (\mathcal{O}\mathbf{e}_2 + \mathcal{O}\mathbf{e}_3) \cong < 1 > \perp \left\langle \begin{matrix} 2 & 1 \\ 1 & \gamma \end{matrix} \right\rangle$$

with $Q(\mathbf{e}_1) = 1, Q(\mathbf{e}_2) = 2, Q(\mathbf{e}_3) = \gamma, B(\mathbf{e}_1, \mathbf{e}_2) = B(\mathbf{e}_1, \mathbf{e}_3) = 0$ and $B(\mathbf{e}_2, \mathbf{e}_3) = 1$.

Proposition 3. *Let* $L \cong < 1 > \perp \left\langle \begin{matrix} 2 & 1 \\ 1 & \gamma \end{matrix} \right\rangle$ *be a universal positive* $\mathcal{O}$*-lattice as above. Then* $1 \leq N(2\gamma - 1) \leq 20$.

Proof. Since L represents $2+\epsilon$, $Q(x\mathbf{e}_1+y\mathbf{e}_2+z\mathbf{e}_3) = 2+\epsilon$ for some $x, y, z \in \mathcal{O}$. Therefore $x^2+2y^2+2yz+\gamma z^2 = x^2+2(y+\frac{1}{2}z)^2+(\gamma-\frac{1}{2})z^2 = 2+\epsilon$. Suppose $z = 0$. Then $2x^2+(2y+z)^2 = 4+2\epsilon \equiv 0 \pmod{\pi_5}$ where $\pi_5 = 2+\epsilon$. But then this implies that -2 is a square mod π_5, which is a contradiction. So $z \neq 0$ and by Lemma, $N(\gamma-\frac{1}{2})N(z)^2 \leq N(2+\epsilon) = 5$. Consequently, $1 \leq N(2\gamma-1) \leq 20$. □

The only possible values of the norm between 1 and 20 are : 1, 4, 5, 9, 11, 13, 19 and 20. For each of them, the possible values of $2\gamma-1$ are as follows: For any $n \in Z$,

$N(2\gamma-1)$	1	4	5	9	11	
$\delta = 2\gamma-1$	ϵ^{2n}	$2\epsilon^{2n}$	$(2+\epsilon)\epsilon^{2n}$	$3\epsilon^{2n}$	$(3+\epsilon)\epsilon^{2n}$	$(3+\epsilon')\epsilon^{2n}$

$N(2\gamma-1)$	16	19		20
$\delta = 2\gamma-1$	$4\epsilon^{2n}$	$(4+\epsilon)\epsilon^{2n}$	$(4+\epsilon')\epsilon^{2n}$	$(4+2\epsilon)\epsilon^{2n}$

From the above table, one can easily eliminate the cases when $N(2\gamma-1) = 4, 16$ and 20 because 2 does not divide δ.

Proposition 4. *Let* $L_0 \cong \left\langle \begin{matrix} 2 & 1 \\ 1 & \gamma_0 \end{matrix} \right\rangle$ *and* $L_1 \cong \left\langle \begin{matrix} 2 & 1 \\ 1 & \gamma_1 \end{matrix} \right\rangle$ *be two binary* $\mathcal{O}$*-lattices such that* $dL_0 = dL_1$*. Then* L_0 *is isometric to* L_1*.*

Proof. dL_0 is well defined modulo $\mathcal{U}^2$. So we may assume $\delta_1 = \delta_0\epsilon^{2n}$ for some $n \in Z$ where $\delta_i = 2\gamma_i - 1$ for i = 0, 1. Note that $\gamma_0 = \frac{\delta_0+1}{2}, \gamma_1 = \frac{\delta_1+1}{2} = \frac{\delta_0\epsilon^{2n}+1}{2} \in \mathcal{O}$. Since δ_0+1 and $\delta_0\epsilon^{2n}+1$ are divisible by 2, so is $1-\epsilon^{2n} = (\delta_0\epsilon^{2n}+1)-(\delta_0+1)\epsilon^{2n}$. This implies that either $1+\epsilon^n$ or $1-\epsilon^n$ is divisible by 2. But $1+\epsilon^n \equiv 1-\epsilon^n \pmod 2$. So, both of them are divisble by 2. In particular, $\frac{1-\epsilon^{-n}}{2} = \frac{\epsilon^n-1}{2\epsilon^n} \in \mathcal{O}$. Let $\{\mathbf{u}_i, \mathbf{v}_i\}$ be a basis for L_i for each i = 0, 1 such that $Q(\mathbf{u}_0) = Q(\mathbf{u}_1) = 2$, $Q(\mathbf{v}_0) = \gamma_0, Q(\mathbf{v}_1) = \gamma_1$, and $B(\mathbf{u}_0, \mathbf{v}_0) = B(\mathbf{u}_1, \mathbf{v}_1) = 1$. Then $\sigma : L_0 \longrightarrow L_1$ defined by $\mathbf{u}_0 \mapsto \mathbf{u}_1$ and $\mathbf{v}_0 \mapsto \frac{1-\epsilon^{-n}}{2}\mathbf{u}_1 + \epsilon^{-n}\mathbf{v}_1$ is an isometry over $\mathcal{O}$. □

By Proposition 4, we may now choose any $\gamma \in \mathcal{O}_+$ from each possible discriminant $\delta = 2\gamma-1$ in the above table and we choose γ as follows:

$N(2\gamma-1)$	1	5	9	11		19	
$\delta = 2\gamma-1$	ϵ^{2n}	$(2+\epsilon)\epsilon^{2n}$	$3\epsilon^{2n}$	$(3+\epsilon)\epsilon^{2n}$	$(3+\epsilon')\epsilon^{2n}$	$(4+\epsilon)\epsilon^{2n}$	$(4+\epsilon')\epsilon^{2n}$
n	0	1	0	-1	1	-2	2
γ	1	$2+2\epsilon$	2	$2+\epsilon'$	$2+\epsilon$	$4+5\epsilon'$	$4+5\epsilon$

One can eliminate the case when $\gamma = 1$ since we are dealing with the non-diagonal case. The proof of the following proposition is also omitted since it is straightforward.

Proposition 5. *If* $\gamma = 2+2\epsilon$, 2, $4+5\epsilon'$, $4+5\epsilon$, *then* $L \cong < 1 > \perp \begin{pmatrix} 2 & 1 \\ 1 & \gamma \end{pmatrix}$ *cannot represent* $2+\epsilon$.

Now we only have two candidates left, namely,

$$R_1 = < 1 > \perp \begin{pmatrix} 2 & 1 \\ 1 & 2+\epsilon' \end{pmatrix} \text{ and } R_2 = < 1 > \perp \begin{pmatrix} 2 & 1 \\ 1 & 2+\epsilon \end{pmatrix}.$$

Theorem 2. *Up to isometry,* R_1 *and* R_2 *are the only universal positive ternary non-diagonal* $\mathcal{O}$*-lattices.*
Proof. As in Theorem 1, it suffices to show that
(C) Both R_1 and R_2 represent all $\nu \in \mathcal{O}_+$ locally everywhere, and
(D) The class numbers of R_1 and R_2 are both 1.

Proof of (C): Let $\pi_1 = 3+\epsilon$ and $\pi_2 = 3+\epsilon'$. Then for each i = 1, 2, we have
(1) $R_{i\wp}$ represents all of $\mathcal{O}_\wp$ at every finite prime $\wp \neq (2), (\pi_i)$ because $R_{i\wp}$ is unimodular at such $\wp$ [O1;92:1b].
(2) $R_{i\wp}$ represents all positive real numbers at each real prime $\wp$.
(3) At $\wp = (\pi_i)$, $R_{i\wp} \cong < 1 > \perp < 2 > \perp < 2\pi_i >$. Since the unimodular component of $R_{i\wp}$ is binary, it represents all $\mathcal{U}_\wp$. Obviously, $R_{i\wp}$ represents $2\pi_i$. Note that 2 is a non-square unit in $\mathcal{O}_\wp$. Since $K_\wp(R_{i\wp} \perp < -\pi_i >)$ has discriminant -1, which is again a non-square unit in $\mathcal{O}_\wp$, $K_\wp(R_{i\wp} \perp < -\pi_i >)$ is isotropic. Therefore, $R_{i\wp}$ represents π_i by [O2; Theorem 1] and hence $R_{i\wp}$ represents all of $\mathcal{O}_\wp$.
(4) At $\wp = (2)$, $Q(R_{i\wp}) = (\mathcal{O}_\wp^2 + 2\mathcal{O}_\wp) \cap Q(K_\wp R_{i\wp}) = \mathcal{O}_\wp$ [Ri; Theorem 7.4].

For (D), it can be easily shown that the orthogonal group $O(R_i)$ has order 8 for each i = 1, 2. On the other hand, by the Massformel in [K; Satz 4, 6 and Hilfssatz 26]:

$$m(R_i) = \frac{(D_K)^{3/2}\zeta_K(2)}{4\pi^4} N(dR_i) \prod_{\wp|(2)} \xi(R_{i\wp}, \wp)(1 - \frac{1}{N\wp^2}) \prod_{\wp|(\pi_i)} \frac{1}{2}(1 + (\frac{-1}{\wp})\frac{1}{N\wp}) = \frac{1}{8}.$$

So, applying Siegel's Theorem [S2], we may conclude that R_i has class number 1 for i = 1, 2, completing the proof of (D) and of Theorem 2. □

References

[D1] L.E. Dickson,'Integers represented by positive ternary quadratic forms', *Bull. Amer. Math. Soc.* 33 (1927), 63-70.

[D2] L.E. Dickson, 'Quaternary Quadratic Forms Representing All Integers', *Amer. J. Math.* 49 (1927), 39-56.

[H] J.S. Hsia, 'An application of algebraic K-theory to sums of squares', *Contemp. Math.* 55 II (1986), 479-485. 132-141.

[K] O. Körner, 'Die Maβe der Geschlechter quadratischer Formen vom Range ≤ 3 in quadratischen Zahlkörpern', *Math. Ann.*, 193 (1971), 279-314.

[M] H. Maass, 'Über die Darstellung total positiver Zahlen des Körpers $R(\sqrt{5})$ als Summe von drei Quadraten', *Abh. Math. Sem. Hamburg* 14 (1941), 185-191.

[O1] O.T. O'Meara, *Introduction to Quadratic Forms*, Springer Verlag, 1973.

[O2] O.T. O'Meara, 'The Integral Representation of Quadratic Forms over Local Fields', *Amer. J. Math.* 80 (1958), 843-878.

[P] M. Peters, 'Quadratische Formen über Zahlringen' *Acta Arith.* 24 (1973), 157-164.

[R] S. Ramanujan, 'On the Expression of a Number in the Form $ax^2 + by^2 + cz^2 + du^2$', *Proc. Cambridge Phil. Soc.* 19 (1917), 11-21.

[Ri] C. Riehm, 'On the Integral Representations of Quadratic Forms over Local Fields', *Amer. J. Math.* 86 (1964), 25-62.

[S1] C.L. Siegel, 'Sums of mth Powers of Algebraic Integers', *Ann. Math.* 46 (1945), 313-339.

[S2] C.L. Siegel, 'Über die Analytische Theorie der quadratischen Formen I, II, III', *Ann. Math.* 36 (1935), 527-606; *ibid.* 37 (1936), 230-263; *ibid.* 38 (1937), 212-291.