2-UNIVERSAL POSITIVE DEFINITE INTEGRAL QUINARY DIAGONAL QUADRATIC FORMS

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Abstract. As a generalization of the famous four square theorem of Lagrange, Ramanujan found all positive definite integral quaternary diagonal quadratic forms that represent all positive integers. In this paper, we find all positive definite integral quinary diagonal quadratic forms that represent all positive definite integral binary quadratic forms.

§1. Introduction

The famous *four square theorem* of Lagrange [8] says that the quadratic form $x^2 + y^2 + z^2 + u^2$ represents all positive integers. In the early 20th century, Ramanujan [13] extended Lagrange's result by providing all 54 positive definite integral quaternary diagonal quadratic forms, up to equivalence, that represent all positive integers. Dickson [2] called such forms *universal* and further extended the result to the non-diagonal case, and Willerding [16] later proved that there are exactly 124 non-diagonal universal forms, up to equivalence.

Mordell [9] proved the *five square theorem* which says that the quadratic form $x^2 + y^2 + z^2 + u^2 + v^2$ represents all positive definite integral binary quadratic forms. This is certainly a very interesting new extension of Lagrange's four square theorem. (See [4],[5],[7] for further development in this direction.) If we call such a form *2-universal*, he raised the question of finding all 2-universal positive definite integral quinary diagonal quadratic forms.

1991 *Mathematics Subject Classification.* 11E20, 11E25, 11E41.

Key words and phrases. 2-universal, positive definite, quinary $\mathbb{Z}$-lattices.

The second author was partially supported by GARC-KOSEF at Seoul National University and Korean Ministry of Education (BSRI-96-1414).

In this short article, we provide a complete answer to the question. More precisely, we prove that there are just five 2-universal positive definite integral quinary diagonal quadratic forms up to equivalence, and they are :

$$x^2+y^2+z^2+u^2+v^2, \quad x^2+y^2+z^2+u^2+2v^2, \quad x^2+y^2+z^2+u^2+3v^2,$$
$$x^2+y^2+z^2+2u^2+2v^2, \quad x^2+y^2+z^2+2u^2+3v^2.$$

Remark. The five forms above coincide with the forms introduced by Peters (see [11;Satz 2]). There, the following is proved : Let K be a real quadratic field and let $\mathcal{O}_f$ be an order of K with conductor f. Then each of the five forms above, over $\mathcal{O}_f$, represents all totally positive elements of $\mathcal{O}_f$, that can be represented as sums of squares of elements of $\mathcal{O}_f$.

§2. A geometrical setting

We shall adopt lattice theoretic language. A $\mathbb{Z}$*-lattice* L is a finitely generated $\mathbb{Z}$-module in $\mathbb{R}^n$ equipped with a non-degenerate symmetric bilinear form B, such that $B(L,L) \subset \mathbb{Z}$. The corresponding quadratic map will be denoted by Q. By integral quadratic forms, we mean the forms whose non-diagonal coefficients are divisible by 2. Since these forms naturally correspond to free $\mathbb{Z}$-lattices, we assume every $\mathbb{Z}$-lattice in this article to be free.

For a $\mathbb{Z}$-lattice L with basis $\mathbf{e}_1, \cdots, \mathbf{e}_n$, i.e. $L = \mathbb{Z}\mathbf{e}_1 + \cdots + \mathbb{Z}\mathbf{e}_n$, we write $L \cong (B(\mathbf{e}_i, \mathbf{e}_j))$. We write $L = L_1 \perp L_2$ if $L = L_1 \oplus L_2$ and $B(\mathbf{e}_1, \mathbf{e}_2) = 0$ for all $\mathbf{e}_1 \in L_1, \mathbf{e}_2 \in L_2$. We call L *diagonal* if it admits an orthogonal basis and in this case, we simply write $L \cong \langle Q(\mathbf{e}_1), \cdots, Q(\mathbf{e}_n)\rangle$, where $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ is an orthogonal basis of L. We call L *non-diagonal* otherwise. L is called *positive definite* or simply *positive* if $Q(\mathbf{e}) > 0$ for any $\mathbf{e} \in L, \mathbf{e} \neq \mathbf{0}$. As usual, $dL = det(B(\mathbf{e}_i, \mathbf{e}_j))$ is called the *determinant* of L.

Let K, L be $\mathbb{Z}$-lattices. We say that L *represents* K if there is a representation σ from K into L. A positive $\mathbb{Z}$-lattice L is called k*-universal* if L represents all k-ary positive $\mathbb{Z}$-lattices.

Thus, each of Ramanujan's 54 forms above corresponds to a 1-universal positive quaternary $\mathbb{Z}$-lattice. Mordell's *five square theorem* asserts the 2-universality of $I_5 \cong \langle 1,1,1,1,1\rangle$ as Lagrange's *four square theorem* asserts the 1-universality of I_4. In fact, it is known [7] that I_n is $(n-3)$-universal for $4 \le n \le 8$.

For any unexplained terminology and basic facts about $\mathbb{Z}$-lattices, we refer the readers to O'Meara's book [11].

Let $L \cong \langle a_1, a_2, a_3, a_4, a_5\rangle$ be a 2-universal positive quinary diagonal $\mathbb{Z}$-lattice with $0 < a_1 \le a_2 \le a_3 \le a_4 \le a_5$. Since L represents $\langle 1,1\rangle$ which is unimodular, $\langle 1,1\rangle$ splits L (see [10;82:15]). So we may assume $a_1 = a_2 = 1$. If $a_3 \ge 2$, then L cannot represent $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. So we may conclude $a_3 = 1$. Similarly, if $a_4 \ge 3$, then L

cannot represent $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$. So $a_4 = 1$ or 2. Now suppose $a_5 \geq 4$. If $a_4 = 1$, then $\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$ cannot be represented by L, and if $a_4 = 2$, then $\langle 3, 3\rangle$ cannot be represented by L. So we obtain $a_5 = 1, 2,$ or 3. In summary, only five candidates for 2-universal positive quinary diagonal $\mathbb{Z}$-lattices L survive and they are $L \cong \langle 1, 1, 1, a, b\rangle$ with $(a, b) = (1, 1), (1, 2), (1, 3), (2, 2), (2, 3)$. We will prove in the next section that these five survivors are indeed 2-universal.

Remark. A $\mathbb{Z}$-lattice L is called *2-regular* if L represents all positive binary $\mathbb{Z}$-lattices which are represented by L locally everywhere. Recently, Earnest [3] proved that there are only finitely many primitive positive quaternary $\mathbb{Z}$-lattices, up to isometry, which are 2-regular. Observe, however, from the above that no positive quaternary diagonal $\mathbb{Z}$-lattice can be 2-universal.

§3 Main theorem

We state and prove our main result in the following :

Theorem. *There are exactly five 2-universal positive definite quinary diagonal $\mathbb{Z}$-lattices up to isometry, and they are*

$$\langle 1,1,1,1,1\rangle, \quad \langle 1,1,1,1,2\rangle, \quad \langle 1,1,1,1,3\rangle, \quad \langle 1,1,1,2,2\rangle, \quad \langle 1,1,1,2,3\rangle. \tag{3.1}$$

Proof. We'll denote $\langle 1, 1, 1, a, b\rangle$ by $L_{a,b}$. No positive integral quinary $\mathbb{Z}$-lattice other than the five listed in (3.1) can be 2-universal, as was shown in the previous section. So it suffices to prove that these five lattices are indeed 2-universal.

Since $L_{1,1}$ is already known to be 2-universal as was mentioned in the Introduction, we will confine ourselves to prove the 2-universality for the other four lattices.

Firstly, we'll show that all these four lattices are locally 2-universal everywhere. Obviously, they are 2-universal at the real prime. They are also 2-universal at all finite primes $p \neq 2, 3$ since they are unimodular at those p's (see [10;63:21], [9;Theorem1], and also [11;Lemma]).

At $p = 3$, both $L_{1,2}$ and $L_{2,2}$ are unimodular and hence 2-universal. Although $L_{1,3}$ is not unimodular, it is again 2-universal since its unimodular component $\langle 1, 1, 1, 1\rangle$ is already 2-universal by [10;63:21] and [9;Theorem1]. The unimodular component $\langle 1, 1, 1, 2\rangle$ of $L_{2,3}$ represents all positive binary $\mathbb{Z}_3$-lattices except $\langle 3, 3\rangle$. But this exceptional $\mathbb{Z}_3$-lattice can obviously be taken care of by $L_{2,3}$ itself. So $L_{2,3}$ is also 2-universal.

We now consider the prime $p = 2$. Since $L_{1,3}$ is unimodular with $\mathfrak{s}(L_{1,3}) = \mathfrak{g}(L_{1,3}) = \mathbb{Z}_2$, $L_{1,3}$ is 2-universal by the Third Main Theorem of [14], where $\mathfrak{s}(L)$ is the scale and $\mathfrak{g}(L)$ is the norm group of a $\mathbb{Z}_2$-lattice L (see also [9;Theorem2]). As for $L_{1,2}, L_{2,2}, L_{2,3}$, one can also conclude their 2-universality by using Theorem 3 of [10], because every positive binary $\mathbb{Z}_2$-lattice K has a lower type than any of them.

Secondly, we'll show the class numbers of $L_{1,2}, L_{1,3}, L_{2,2}, L_{2,3}$ are all one. Although this is already known [12], we will provide a short proof in the following by using Conway and Sloane's useful table of positive irreducible $\mathbb{Z}$-lattices of small determinant [1]. One may also compute the class numbers directly by using the Siegel formula (see [6] or [15] for example).

Since $\langle 2\rangle$ is the only positive irreducible lattice of determinant 2 with rank less than or equal to 5, $L_{1,2}$ is the only positive quinary $\mathbb{Z}$-lattice of determinant 2, up to isometry, and hence its class number is one.

The only positive irreducible $\mathbb{Z}$-lattices of determinant 3 with rank less than or equal to 5 are $\langle 3\rangle$ and $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, up to isometry. So, $L_{1,3}$ and $I_3 \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ are the only positive quinary $\mathbb{Z}$-lattices of determinant 3, up to isometry. But the spaces in which the two live have different Hasse symbols at $p = 3$, which implies that their respective spaces and hence their respective genera are different. So the class number of $L_{1,3}$ is one.

There are only four positive irreducible $\mathbb{Z}$-lattices of determinant 4 with rank less than or equal to 5, up to isometry. They are $\langle 4\rangle$, and root lattices

$$A_3 \cong \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \qquad D_4 \cong \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix},$$

$$D_5 \cong \begin{pmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

So, $L_{1,4}, L_{2,2}, I_2 \perp A_3, \langle 1\rangle \perp D_4$, and D_5 are the only positive quinary $\mathbb{Z}$-lattices of determinant 4 up to isometry. But locally at $p = 2$, $L_{2,2}$ is not isometric to any of the other four, which implies that the class of $L_{2,2}$ is the only class in its genus.

We have only three positive quinary $\mathbb{Z}$-lattices of determinant 6 up to isometry, namely $L_{1,6}, L_{2,3}$, and $I_2 \perp \langle 2\rangle \perp \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. This is because $\langle 6\rangle$ is the only positive irreducible $\mathbb{Z}$-lattice of determinant 6 with rank less than or equal to 5. But again the Hasse symbol of the space containing $L_{2,3}$ is different from those of the spaces containing the other two, which implies that the class number of $L_{2,3}$ is again one. This completes the proof. $\square$

Corollary. *There exist at most five k-universal positive definite $(k+3)$-ary diagonal $\mathbb{Z}$-lattices for $3 \le k \le 5$, up to isometry.*

Proof. Let L be a k-universal positive definite $(k+3)$-ary diagonal $\mathbb{Z}$-lattice. Since $\langle 1\rangle$ splits L, we may write $L \cong \langle 1\rangle \perp L_0$. Then L_0 is $(k-1)$-universal positive definite

$(k+2)$-ary diagonal $\mathbb{Z}$-lattice. The corollary follows immediately from induction on $k \geq 3$. $\square$

Remark. Note that there exists no k-universal positive definite diagonal $\mathbb{Z}$-lattice for $k \geq 6$. This follows immediately from the fact that the root lattice E_6 cannot be represented by I_n for any n.

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