UNIVERSAL QUADRATIC FORMS
OVER POLYNOMIAL RINGS

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Abstract. The Fifteen Theorem proved by Conway and Schneeberger is a criterion for quadratic forms over the rational integer ring to be universal. In this article, we give a proof of an analogy of the Fifteen Theorem for quadratic forms over polynomial rings, which is known as the Four Conjecture proposed by Gerstein.

1. Introduction

Conway and Schneeberger (see [12] and [3]) announced the so called the ‘Fifteen Theorem’ which claims that an integral positive definite quadratic form represents every positive integer, i.e., the form is universal, if it represents 1, 2, 3, 5, 6, 7, 10, 14 and 15. For example, the well-known Lagrange’s Four Square Theorem is an immediate consequence of the Fifteen Theorem. Bhargava [1] recently gave a simple proof of the theorem (see also [6], [7] and [5] for fascinating recent developments on universal forms). In [4], Gerstein studied the analogy of the Fifteen Theorem over $\mathbb{F}_q[x]$ and proposed the following conjecture.

**Four Conjecture.** An integral definite quadratic form over $\mathbb{F}_q[x]$ represents every polynomial in $\mathbb{F}_q[x]$ if it represents 1, $\delta$, $x$ and $\delta x$, where $\text{ch}(\mathbb{F}_q) \neq 2$ and $\delta$ is a non-square element in $\mathbb{F}_q$.

In this paper, we prove:

**Theorem 1.1.** The Four Conjecture is true.

Notation and terminology are standard and adopted from [11] if not explained. Particularly, $\mathbb{F}_q$ is a finite field with $q$ elements, where $q$ is an odd prime power, $\mathbb{F}_q[x]$ is the polynomial ring of one variable $x$, and $\mathbb{F}_q(x)$ is the quotient field of $\mathbb{F}_q[x]$.

We call a quadratic space $V$ over $\mathbb{F}_q(x)$ definite (resp., indefinite) if the local completion $V_\infty$ at $\infty = (1/x)$ is anisotropic (resp., isotropic). Let $L$ be a free
\[ \mathbb{F}_q[x]-\text{module in } V. \] We call \( L \) an integral lattice if
\[ B(L, L) \subseteq \mathbb{F}_q[x], \]
where \( B \) is a given symmetric bilinear form on \( V \). For any \( v \in V \), define \( Q(v) := B(v, v) \).

It should be pointed out that the Four Conjecture is not true for indefinite quadratic forms over \( \mathbb{F}_q[x] \), just like the Fifteen Theorem is not true for indefinite quadratic forms over \( \mathbb{Z} \), as the following example indicates.

**Example.** Let \( L \cong (1, -\delta, (x + a)^2, (x + a)^3, (x + a)^4) \) be a five dimensional diagonal integral lattice over \( \mathbb{F}_q[x] \), where \( a \in \mathbb{F}_q^* \) and \( \delta \in \mathbb{F}_q^* \setminus (\mathbb{F}_q^*)^2 \). Then since \( \dim(L) = 5 \), the corresponding quadratic space spanned by \( L \) is indefinite.

By strong approximation theorem for spin groups (see [10] and [8]), an element in \( \mathbb{F}_q[x] \) is represented by \( L \) if and only if it is represented by \( L \) locally at every prime of \( \mathbb{F}_q[x] \). (In fact, the class number of \( L \) is one.) It is clear that \( 1, \delta, x \) and \( \delta x \) are represented by \( L \) locally at every prime. By [OM1], however, \((x+a)\) cannot be represented by \( L \) locally at \((x+a)\).

From now on, we assume that all quadratic spaces are definite. If \( L \) is an integral lattice in a definite quadratic space \( V \), then the Four Conjecture says
\[ \{1, \delta, x, \delta x\} \subseteq Q(L) \iff Q(L) = \mathbb{F}_q[x] \]
for any given \( \delta \in \mathbb{F}_q^* \setminus (\mathbb{F}_q^*)^2 \). We choose such a \( \delta \) and fix it afterwards.

### 2. Representations of ternary lattices

In this section, we extend the last proposition in [4] to all ternary integral lattices whose determinants are linear polynomials in \( \mathbb{F}_q[x] \). This result plays a central role in our proof of the Four Conjecture.

Let \( K \) be an integral ternary lattice with \( \det(K) = ax + b \) with \( a, b \in \mathbb{F}_q, a \neq 0 \). For any \( f(x) \in \mathbb{F}_q[x] \), one can write
\[ f(x) = a_0 + a_1(ax + b) + \cdots + a_n(ax + b)^n, \]
where \( a_0, a_1, \ldots, a_n \in \mathbb{F}_q \) with \( a_n \neq 0 \). Define
\[ v_{ax+b}(f) := \min\{i : a_i \neq 0\}. \]

**Lemma 2.1.** There is only one class of ternary integral lattices with given determinant \( ax + b, a \neq 0 \), and \( f(x) \) is not represented by this class if and only if

1. \( n \) is odd and \( -a_n \in (\mathbb{F}_q^*)^2 \); or
2. \( m = v_{ax+b}(f) \) is odd and \( -a_m \in (\mathbb{F}_q^*)^2 \).

**Proof.** Let \( K \) be an integral ternary lattice with \( \det(K) = ax + b \). Since the degrees of diagonal polynomials of the Gram matrix of a reduced basis \( \{v_1, v_2, v_3\} \) (see [4] or [2]) of \( K \) are dominant, one has \( Q(v_1), Q(v_2) \in \mathbb{F}_q^* \) and hence
\[ K = \mathbb{F}_q[x]v_1 \perp \mathbb{F}_q[x]v_2 \perp \mathbb{F}_q[x]v_3. \]
From the definiteness condition it follows that $\mathbb{F}_q[x]v_1 \perp \mathbb{F}_q[x]v_2 \cong \langle 1, -\delta \rangle$ and therefore,

$$K \cong \langle 1, -\delta, -\delta(ax + b) \rangle.$$  

This proves the first assertion.

Since the class number of $K$ is one, it is a purely local problem to determine whether $f(x)$ is represented by $K$. It is clear that $K$ is unimodular and hence universal at all primes except $(ax + b)$ and $\infty$. Therefore $f(x)$ is not represented by $K$ if and only if $f(x)$ is not represented by $K$ locally at $(ax + b)$ or $\infty$.

If $f(x)$ is not represented by $K$ locally at $(ax + b)$, then one should have $a_0 = 0$, for otherwise $f(x)$ is a unit at $(ax + b)$ and is represented by the sublattice $(1, -\delta)$ of $K$ according to Hensel’s lemma. Therefore, $m = v_{ax+b}(f(x)) \geq 1$ and $f(x)$ is not represented by $K$ locally at $(ax + b)$ if and only if $f(x)$ is not represented by the quadratic space spanned by $K$ at $(ax + b)$ by [10]. This is equivalent to the fact that the quadratic space

$$[1, -\delta, -\delta(ax + b), -f(x)] \cong [1, -\delta, -\delta(ax + b), -a_m(ax + b)^m]$$

is anisotropic at $(ax + b)$. It is a standard fact that the above quadratic space is anisotropic at $(ax + b)$ if and only if $m$ is odd and $-a_m \in (\mathbb{F}_q^\times)^2$.

At $\infty$, $f(x)$ is not represented by $K$ if and only if the space

$$[1, -\delta, -\delta(ax + b), -f(x)] \cong [1, -\delta, -ax, -a_n a^n x^n]$$

is anisotropic at $\infty$. This is equivalent to the fact that $n$ is odd and $-a_n \in (\mathbb{F}_q^\times)^2$. □

3. Proof of the theorem

In [4], it was proved that any quaternary integral lattice which represents $1, \delta, x$ and $\delta x$ is isometric to

$$\langle 1, -\delta \rangle \perp \left( \begin{array}{cc} \alpha x + \beta \\
\gamma \\
-\delta ax + \eta \end{array} \right),$$

where $\alpha, \beta, \gamma, \eta \in \mathbb{F}_q$ with $\alpha \neq 0$. Therefore the theorem in §1 follows if one proves that an integral lattice

$$L \cong \langle 1, -\delta \rangle \perp \left( \begin{array}{c} x \\
\varepsilon \\
-\delta x + \xi \end{array} \right)$$

is universal for every $\xi \in \mathbb{F}_q$ and $\varepsilon \in \{0, 1\}$ (see [4]). Let $\{v_1, v_2, v_3, v_4\}$ be the corresponding basis of $L$.

Since $L$ contains a ternary sublattice $\langle 1, -\delta, x \rangle$, it suffices to consider only those $f(x)$’s which cannot be represented by this ternary sublattice. Write

$$f(x) = a_0 + a_1(-\delta x) + \cdots + a_n(-\delta x)^n,$$

where $a_0, a_1, \cdots, a_n \in \mathbb{F}_q$ with $a_n \neq 0$. Since one can easily verify that $L$ represents all linear polynomials in $\mathbb{F}_q[x]$ (see also [4]), we may assume that $n > 1$. 
3.1. Diagonal case

In this section, we prove that $L$ is universal when $\varepsilon = 0$, i.e., $L$ is diagonal.

**Proof.** It is already proved that $L$ is universal when $\xi = 0$ in [4]. So we assume that $\xi \neq 0$.

It is clear that there are $q + 1$ solutions over $\mathbb{F}_q$ of the equation

\[ y^2 - \delta z^2 = -a_0 \delta^n. \]

So, there is a solution $(y_0, z_0)$ satisfying $y_0 z_0 \neq 0$.

If $n$ is odd and $-a_n \in (\mathbb{F}_q^\times)^2$, then we put

\[ g(x) = f(x) - (-\delta x + \xi) \cdot \begin{cases} (z_0 x^{(n-1)/2})^2 & \text{if } a_0 \neq 0 \\ (z_0 x^{(n-1)/2 + 1})^2 & \text{otherwise.} \end{cases} \]

By the lemma in §2, $g(x)$ is represented by $\langle 1, -\delta, x \rangle$. Therefore $f(x)$ is represented by $L$. Otherwise, one can assume that $a_0 = 0$. Then $g(x) = f(x) - (-\delta x + \xi)$ is represented by $\langle 1, -\delta, x \rangle$ and therefore, $f(x)$ is represented by $L$. \quad \Box

3.2. Non-diagonal case, lower degree

It only remains to prove that $L$ is universal for every $\xi \in \mathbb{F}_q$ when $\varepsilon = 1$. In this section, we prove that such $L$ represents all polynomials $f(x) \in \mathbb{F}_q[x]$ of degree $n \leq 3$.

**Proof.** Let $n = 2$. One can assume that $f(x) = \alpha x (x - \xi \delta^{-1})$ for some $\alpha \in \mathbb{F}_q^\times$ by applying the lemma to the following ternary sublattices

$\langle 1, -\delta, x \rangle$ and $\langle 1, -\delta, -\delta x + \xi \rangle$.

Furthermore, one only needs to consider the case when $-\xi \alpha \in (\mathbb{F}_q^\times)^2$ since $f(x)$ is represented by the former otherwise. But if $-\xi \alpha \in (\mathbb{F}_q^\times)^2$, then $f(x)$ is represented by the latter.

Let $n = 3$ and let

\[ f(x) = a_0 + a_1 (-\delta x) + a_2 (-\delta x)^2 + a_3 (-\delta x)^3 \in \mathbb{F}_q[x] \]

with $a_3 \neq 0$.

Case (i) $-a_3 \notin (\mathbb{F}_q^\times)^2$: One can assume that $a_0 = 0$ and $-a_1 \in (\mathbb{F}_q^\times)^2$. It is clear that there are $q + 1$ solutions over $\mathbb{F}_q$ of the equation

\[ s^2 - \delta t^2 = 1. \]

Let

\[ S = \{ \xi t^2 + 2st : (s, t) \text{ is a solution of (3.2) } \}. \]

It is clear that there are exactly two solutions $(s, t)$ of (3.2) having the same ratio $s/t$ if $t \neq 0$. For any solution $(s, t)$ of (3.2) with $t \neq 0$ such that $\xi t^2 + 2st = \Delta$. 
for some $\Delta \in \mathbb{F}_q$, $s/t$ satisfies a quadratic equation over $\mathbb{F}_q$ unless $\Delta = 0$. This implies that

$$\exists S \geq \frac{q-1}{2} + 1 = \frac{q+1}{2}.$$

If $q \geq 13$, then $\exists S \geq 4$ and there is a solution $(s_0, t_0)$ of (3.2) such that

$$Q(s_0 v_3 + t_0 v_4) = (x + \Delta_0) \mid f(x)$$

over $\mathbb{F}_q[x]$ for some $\Delta_0 \in \mathbb{F}_q$. Then $f(x)$ is represented by the ternary sublattice $\mathbb{F}_q[x]v_1 \perp \mathbb{F}_q[x]v_2 \perp \mathbb{F}_q[x](s_0 v_3 + t_0 v_4) \cong \langle 1, -\delta, x + \Delta_0 \rangle$.

If $q = 11$, then one can write $-\delta = \eta^2$ for some $\eta \in \mathbb{F}_q$. Thus $S$ still contains more than three elements if $\xi \neq 0, \pm \eta$. So, it suffices to consider the exceptional cases. We have

$$S = \begin{cases} 
\{0, 3\eta^{-1}, -3\eta^{-1}\} & \text{if } \xi = 0 \\
\{0, \xi^{-1}, 6\xi^{-1}\} & \text{if } \xi = \pm \eta.
\end{cases}$$

When $\xi = 0$, one may assume that

$$f(x) = -a_3 \delta^3 x(x - 3\eta^{-1})(x + 3\eta^{-1}) = -a_3 \delta^3 x(x^2 - 2\delta^{-1})$$

by applying the lemma to the following ternary sublattices $\mathbb{F}_q[x]v_1 \perp \mathbb{F}_q[x]v_2 \perp \mathbb{F}_q[x](3v_3 \pm 5\eta^{-1}v_4) \cong \langle 1, -\delta, x \pm 3\eta^{-1} \rangle$.

It is easy to verify that $x(x^2 - 2\delta^{-1})$ is represented by

$$\begin{pmatrix} x & 1 \\
1 & -\delta x
\end{pmatrix}$$

for $\delta = 2, 6, 7, 8, 10$. So, we are done. When $\xi = \pm \eta$, one may assume that

$$f(x) = -a_3 \delta^3 x(x + \xi^{-1})(x + 6\xi^{-1}) = -a_3 \delta^3 x(x^2 - 4\xi^{-1}x + 5\delta^{-1})$$

by applying the lemma to the following ternary sublattices $\mathbb{F}_q[x]v_1 \perp \mathbb{F}_q[x]v_2 \perp \mathbb{F}_q[x](5v_3 \pm 3\eta^{-1}v_4)$.

One only needs to verify that $x(x^2 - 5\xi^{-1}x + 5\delta^{-1})$ is represented by $L$. Indeed, there are $a, c \in \mathbb{F}_q$ such that

$$a^2 - \delta c^2 = 1 \quad \text{and} \quad 4c^2 - 2\delta^{-1} \in \mathbb{F}_q^2$$

for $\delta = 2, 6, 7, 8, 10$. Let $b$ be a solution of the equation

$$b^2 + 2bc - 5\delta^{-1} = 0.$$

Then

$$x(x^2 - 4\xi^{-1}x + 5\delta^{-1}) - Q((ax + b)v_3 + cxv_4) = -(4\xi^{-1} + c^2\xi + 2ab + 2ac)x^2,$$

which is represented by $(1, -\delta)$. Therefore, $x(x^2 - 4\xi^{-1}x + 5\delta^{-1})$ is represented by $L$. 
If \( q = 9 \), then one can write \( \theta^2 = -1 \) and \( F_q = F_3(\theta) \). Then the solutions of (3.2) are
\[
\begin{cases}
\{(\pm 1, 0), (\pm \delta, \pm(1 - \delta)), (\pm(1 + \delta), \pm(1 + \delta))\} & \text{if } \delta^2 = -\delta + 1 \text{ or } \delta = 1 \pm \theta \\
\{(\pm 1, 0), (\pm \delta, \pm 1), (\pm(1 - \delta), \pm \delta)\} & \text{if } \delta^2 = \delta + 1 \text{ or } \delta = -1 \pm \theta.
\end{cases}
\]
Therefore, \( S \) contains more than three elements except when
\[
S = \begin{cases}
\{0, \xi \delta, \xi(1 - \delta)\} & \text{if } \delta = 1 \pm \theta \text{ and } \xi = \pm 1 \\
\{0, \xi, -\xi\delta\} & \text{if } \delta = 1 \pm \theta \text{ and } \xi = \pm(1 + \delta) \\
\{0, -\xi, -\xi\delta\} & \text{if } \delta = -1 \pm \theta \text{ and } \xi = \pm \delta \\
\{0, \xi\delta, -\xi(1 + \delta)\} & \text{if } \delta = -1 \pm \theta \text{ and } \xi = \pm(1 + \delta).
\end{cases}
\]

Here we only consider, for example, the case when \( \delta = 1 \pm \theta \) and \( \xi = \pm 1 \). The other cases can be proved by the same argument. It is clear that we may assume that
\[
f(x) = -a_3\delta^3 x(x + \xi\delta)(x + \xi(1 - \delta)).
\]
Since the coefficient of \(-\delta(x + \xi\delta)\) in the expansion of \( f(x) \) with respect to \(-\delta(x + \xi\delta)\) is
\[
\xi^2(2\delta^2 - \delta) = -a_3\delta^2,
\]
f(\(x\)) is represented by the sublattice
\[
\mathbb{F}_q[x]v_1 \perp \mathbb{F}_q[x]v_2 \perp \mathbb{F}_q[x](s_0v_3 + t_0v_4) \cong (1, -\delta, x + \xi\delta)
\]
for some solution \((s_0, t_0)\) of (3.2).

If \( q = 3, 7 \), then one can assume (because \(-a_3\delta^3\) is a square) that \( f(x) = x^3 + \rho x^2 + \delta\gamma^2 x \), where \( \rho \in \mathbb{F}_q \) and \( \gamma \in \mathbb{F}_q^\times \). There are \( a, c \in \mathbb{F}_q \) satisfying
\[
a^2 - \delta c^2 = 1 \quad \text{and} \quad c^2 + \delta\gamma^2 \in \mathbb{F}_q^2.
\]
Let \( b \) be a solution of the equation
\[
b^2 + 2cb - \delta\gamma^2 = 0.
\]
Then
\[
f(x) - Q((ax + b)v_3 + cxcv_4) = (\rho - 2ab - 2ac - \xi c^2)x^2
\]
is represented by \((1, -\delta)\) and therefore, \( f(x) \) is represented by \( L \).

If \( q = 5 \), then one has \( \delta^2 = -1 \) and
\[
S = \{0, 2\delta\xi \pm 2(\delta - 1)\}.
\]
When \( \xi \neq \pm(1 + \delta) \), one can assume that
\[
f(x) = -a_3\delta^3 x(x + \sigma)(x + \tau),
\]
where
\[
\sigma = 2\delta\xi + 2(\delta - 1) \quad \text{and} \quad \tau = 2\delta\xi - 2(\delta - 1).
\]
It is clear that \( \sigma \neq \tau \). By assumption, one has \( \sigma\tau \notin (\mathbb{F}_q^\times)^2 \). By applying the lemma to the following ternary sublattices
\[
(1, -\delta, x + \sigma) \quad \text{and} \quad (1, -\delta, x + \tau),
\]
one only needs to consider the case when
$$-\sigma(\tau - \sigma) \notin (F_q^\times)^2 \quad \text{and} \quad -\tau(\sigma - \tau) \notin (F_q^\times)^2.$$  
But this is impossible. When $\xi = \pm(1 + \delta)$, one has $S = \{0, -\delta \xi\}$. By the same argument as in the case of $q = 3, 7$, one may assume (because $-a_0 \delta^3$ is a square) that $f(x) = x^3 + \rho x^2 - \delta x$, where $\rho \in F_q$. By applying the lemma to the ternary sublattice $\langle 1, -\delta, x - \delta \xi \rangle$, one may further assume that
$$f(x) = x(x - \delta \xi)(x + \xi^{-1}).$$

Since $(1, -\delta) \cong (\xi, 3\xi^{-1})$, $f(x) - Q(xv_3 - \delta v_4) = \xi^{-1}(1 - \delta \xi^2)x^2 + \xi = \xi^{-1}(1 - 2\delta^2)x^2 + \xi = 3\xi^{-1}x^2 + \xi$ is represented by $(1, -\delta)$ and therefore, $f(x)$ is represented by $L$. Case (ii) $-a_0 \in (F_q^\times)^2$: Observe that the orthogonal complement of the ternary lattice $\langle 1, -\delta, x \rangle$ in $L$ is
$$\langle -\delta x^3 + \xi x^2 - x \rangle.$$  
Suppose that $a_0 \neq 0$. There are $q + 1$ solutions of the equation
$$y^2 - \delta z^2 = -a_0 \delta^3$$
over $F_q$ and at least one of them, say $(y_0, z_0)$, satisfies $y_0 z_0 \neq 0$. Then
$$f(x) = z_0^2(-\delta x^3 + \xi x^2 - x) = y_0^2 x^3 + \cdots + a_0$$
is represented by $(1, -\delta, x)$ and hence $f(x)$ is represented by $L$. So, we may assume that $a_0 = 0$.

It is clear that there are $q + 1$ solutions over $F_q$ of the equation
$$(3) \quad s^2 - \delta t^2 = \delta.$$  
Let
$$T = \{ \xi t^2 + 2st : (s, t) \text{ is a solution of (3.3)} \}.$$  
Then, as before, one has
$$\sharp T \geq \frac{q + 1}{4}.$$  
If $q \geq 13$, then $\sharp T \geq 4$ and there is a solution $(s_0, t_0)$ of (3.3) such that
$$Q(s_0 v_3 + t_0 v_4) = (\delta x + \Delta_0) \nmid f(x)$$
over $F_q[x]$ for some $\Delta_0 \in F_q$. Then $f(x)$ is represented by the ternary sublattice $F_q[x]v_1 \perp F_q[x]v_2 \perp F_q[x](s_0 v_3 + t_0 v_4) \cong \langle 1, -\delta, \delta x + \Delta_0 \rangle$.

If $q = 11$, then one can write $-\delta = \eta^2$ for some $\eta \in F_q$. Thus $T$ still contains more than three elements if $\xi \neq \pm 4\eta$. So, it suffices to consider the case:
$$T = \{ 2\xi, 5\xi, 8\xi \} \quad \text{when} \quad \xi = \pm 4\eta.$$  
By the lemma, one may assume that
$$f(x) = -a_0 (\delta x + 2\xi)(\delta x + 5\xi)(\delta x + 8\xi) = -a_3 \delta^3 x^3 - 4a_3 \xi \delta^2 x^2 + 4a_3 \xi \delta.$$  
But this contradicts the assumption that $a_0 = 0$. 

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If \( q = 9 \), then one can write \( \theta^2 = -1 \) and \( \mathbb{F}_9 = \mathbb{F}_3(\theta) \). Then \( T \) contains more than three elements except when

\[
T = \begin{cases} 
\{-\xi, \xi\delta, \xi(1 + \delta)\} & \text{if } \delta = 1 \pm \theta \text{ and } \xi = \pm 1 \\
\{-\xi, -\xi\delta, -\xi(1 + \delta)\} & \text{if } \delta = 1 \pm \theta \text{ and } \xi = \pm(1 + \delta) \\
\{-\xi, \xi\delta, -\xi(1 - \delta)\} & \text{if } \delta = -1 \pm \theta \text{ and } \xi = \pm \delta \\
\{-\xi, -\xi\delta, \xi(1 - \delta)\} & \text{if } \delta = -1 \pm \theta \text{ and } \xi = \pm(1 + \delta).
\end{cases}
\]

Here we only consider, for example, the case when \( \delta = 1 \pm \theta \) and \( \xi = \pm 1 \). The other cases can be proved by the same argument. But then, by the lemma, one may assume that

\[
f(x) = -a_3(\delta x - \xi)(\delta x + \xi\delta)(\delta x + \xi(1 + \delta)),
\]

which again contradicts the assumption that \( a_0 = 0 \).

If \( q = 7 \), then \( \delta^3 = -1 \) and

\[
T = \{4\xi \pm 2\delta^{-1}, 2\xi \pm 2\delta^{-1}\}.
\]

Therefore \( T \) contains more than three elements if \( \xi \neq 0, \pm 2\delta^{-1} \). When \( \xi = \pm 2\delta^{-1} \), one has

\[
T = \{\xi, 3\xi, 5\xi\}.
\]

By the lemma, one may assume that

\[
f(x) = -a_3(\delta x + \xi)(\delta x + 3\xi)(\delta x + 5\xi).
\]

This contradicts the assumption that \( a_0 = 0 \). When \( \xi = 0 \), one has \( T = \{\pm 2\delta^{-1}\} \). Then by the lemma, one may assume that

\[
f(x) = -a_3\delta x(\delta x - 2\delta^{-1})(\delta x + 2\delta^{-1}) = -a_3\delta x(\delta^2 x^2 + 4\delta).
\]

Then

\[
f(x) - (4a_3)\delta^2(-\delta x^3 + \xi x^3 - x) = -5a_3\delta^3 x^3 - a_3\delta^2 x = 5a_3(-\delta x)^3 + a_3\delta(-\delta x)
\]

is represented by \( \langle 1, -\delta, x \rangle \) and therefore, \( f(x) \) is represented by \( L \).

If \( q = 5 \), then

\[
T = \{-\xi, \xi \pm 2(1 + \delta)\}.
\]

When \( \xi \neq \pm(1 + \delta) \), \( T \) contains three elements

\[
\sigma = -\xi, \quad \varsigma = \xi + 2(1 + \delta) \quad \text{and} \quad \tau = \xi - 2(1 + \delta).
\]

By applying the lemma to the following ternary sublattices

\[
\langle 1, -\delta, \delta x + \sigma \rangle, \quad \langle 1, -\delta, \delta x + \varsigma \rangle \quad \text{and} \quad \langle 1, -\delta, \delta + \tau \rangle,
\]

one may assume that

\[
f(x) = -a_3(\delta x + \sigma)(\delta x + \varsigma)(\delta x + \tau)
\]

and from this it follows that

\[
(\varsigma - \sigma)(\tau - \sigma) \notin (\mathbb{F}_q^\times)^2, \quad (\sigma - \varsigma)(\tau - \varsigma) \notin (\mathbb{F}_q^\times)^2 \quad \text{and} \quad (\varsigma - \tau)(\sigma - \tau) \notin (\mathbb{F}_q^\times)^2.
\]
This, however, implies that
\[-(z - \sigma)^2(\tau - \sigma)^2(\xi - \tau)^2 \not\in (\mathbb{F}_q^\times)^2,\]
which is a contradiction. When \(\xi = \pm(1 + \delta)\), one has \(T = \{-\xi, 3\xi\}\) and may assume that
\[f(x) = -a_3\xi(x(\xi - \xi)\xi x + 3\xi).\]
Since the coefficient of \(-\delta(\delta x - \xi)\) and \((-\delta(\delta x - \xi))^3\) in the expansion of \(f(x)\) are \(-4a_3\delta^{-1}\xi^2\) and \(a_3\delta^{-3}\), respectively, whose negatives are non-squares, \(f(x)\) is represented by \((1, -\delta, \delta x - \xi)\).

If \(q = 3\), then \(T = \{\pm 1\}\). One can assume that
\[f(x) = -a_3\delta x(-x + \xi + 1)(-x + \xi - 1) = -a_3\delta^3(x^3 + \xi x^2 + (\xi^2 - 1)x).\]
When \(\xi = 0\), one has
\[f(x) = -a_3\delta^3Q((x + 1)v_3 + (x - 1)v_4) = a_3\delta^3(x^2 - x - 1),\]
which is irreducible over \(\mathbb{F}_q\) and is represented by \((1, 1, 1)\) (see [9]). Therefore, \(f(x)\) is represented by \(L\). When \(\xi = \pm 1\), consider
\[f(x) = -a_3\delta^3x^2Q(v_3 \pm v_4) = -a_4\delta x^2.\]
Since this is represented by \((1, -\delta)\), one may conclude that \(f(x)\) is represented by \(L\).

\[\square\]

### 3.3. Non-diagonal case, higher degree

In this section, we show that \(L\), when \(\varepsilon = 1\) and \(\xi \in \mathbb{F}_q\) is arbitrary, represents all polynomials \(f(x) \in \mathbb{F}_q[x]\) of degree \(n \geq 4\). Let
\[f(x) = a_0 + a_1(-\delta x) + \cdots + a_n(-\delta x)^n \in \mathbb{F}_q[x]\]
with \(a_n \neq 0\) and \(n \geq 4\).

**Proof.** Case (i) \(n\) is even or \(n\) is odd with \(a_n \not\in (\mathbb{F}_q^\times)^2\): By applying the lemma to the ternary sublattice \((1, -\delta, x)\), one can assume that \(a_0 = 0\). Suppose that \(a_1 = 0\). Then
\[f(x) = (-\delta x^3 + \xi x^2 - x)\]
is represented by \((1, -\delta, x)\) and therefore, \(f(x)\) is represented by \(L\). Assume that \(a_1 \neq 0\). One may further assume that \(-a_1 \in (\mathbb{F}_q^\times)^2\).

If \(q > 3\), then there are \(q - 1 \geq 4\) solutions over \(\mathbb{F}_q\) of the equation
\[(4)\]
\[u^2 - v^2 = -a_1\delta.\]
Then there is a solution \((u_0, v_0)\) of (3.4) satisfying \(u_0v_0 \neq 0\). Therefore
\[f(x) = v_0^2(-\delta x^3 + \xi x^2 - x)\]
is represented by \((1, -\delta, x)\) and therefore, \(f(x)\) is represented by \(L\).

If \(q = 3\) and \(n \geq 5\), then consider \(a_2\). When \(a_2 \neq 0\), one can choose a proper sign such that \(a_3 = a_2 = 0\) or 1. Then
\[f(x) = (x^3 + \xi x^2 - x)h(x)\]
is represented by \langle 1, -\delta, x \rangle by the lemma, where

\[ h(x) = \begin{cases} (x + 1)^2 & \text{if } \xi = a_2 \\ 1 & \text{if } \xi \neq a_2. \end{cases} \]

This proves that \( f(x) \) is represented by \( L \). When \( a_2 = 0 \),

\[ f(x) - (x^3 + \xi x^2 - x)k(x) \]

is represented by \( \langle 1, -\delta, x \rangle \), where

\[ k(x) = \begin{cases} (x + 1)^2 & \text{if } \xi = a_2 = 0 \\ 1 & \text{otherwise}, \end{cases} \]

which proves that \( f(x) \) is represented by \( L \).

If \( q = 3 \) and \( n = 4 \), then one only needs to consider

\[ f(x) = \begin{cases} a_4 x^4 + a_3 x^3 + \xi x^2 - x & \text{if } \xi \neq 0 \\ a_4 x^4 - x & \text{if } \xi = 0. \end{cases} \]

When \( \xi \neq 0 \), one can further assume that \( a_3 = 0 \) by the same argument as above. Since \( -x \) is represented by \( F_q[x]v_3 + F_q[x]v_4 \), one may assume that \( a_4 = -\xi \). It is clear that \( -x - \xi \) is also represented by \( F_q[x]v_3 + F_q[x]v_4 \). Then

\[ f(x) - (-x - \xi) = -\xi x^4 + \xi x^2 + \xi \]

has no roots over \( F_q \) and is represented by \( \langle 1, -\delta \rangle = \langle 1, 1 \rangle \) by [9]. When \( \xi = 0 \),

\[ -x - a_4 \]

is represented by \( F_q[x]v_3 + F_q[x]v_4 \). Then

\[ f(x) - (-x - a_4) = a_4(x^4 + 1) \]

has no roots over \( F_q \) and is represented by \( \langle 1, -\delta \rangle \) by [9]. In any case \( f(x) \) is represented by \( L \).

Case (iii) \( n \) is odd and \( -a_n \in (F_q^\times)^2 \): Suppose \( a_0 \neq 0 \) or \( -a_1 \notin F_q^2 \). Then there are \( q + 1 \) solutions of (3.1) and at least one of them, say \( (y_0, z_0) \), satisfies \( y_0 z_0 \neq 0 \). Then

\[ f(x) - (-\delta x^3 + \xi x^2 - x)(z_0 x^{(n-3)/2})^2 \]

is represented by \( \langle 1, -\delta, x \rangle \) and therefore, \( f(x) \) is represented by \( L \). Suppose \( a_0 = a_1 = 0 \). We also take a solution \( (y_0, z_0) \) of (3.1) satisfying \( y_0 z_0 \neq 0 \). Then

\[ f(x) - (-\delta x^3 + \xi x^2 - x)(z_0 x^{(n-3)/2} + 1)^2 \]

is represented by \( \langle 1, -\delta, x \rangle \), which implies that \( f(x) \) is represented by \( L \). So, we may assume that \( a_0 = 0 \) and \( -a_1 \in (F_q^\times)^2 \). Let \( (y_0, z_0) \) be a solution of (3.1) satisfying \( y_0 z_0 \neq 0 \).

If \( q > 3 \), then there is a solution of (3.4) such that \( u_0 v_0 \neq 0 \). Then

\[ f(x) - (-\delta x^3 + \xi x^2 - x)(z_0 x^{(n-3)/2} + v_0)^2 \]

is represented by \( \langle 1, -\delta, x \rangle \) and therefore, \( f(x) \) is represented by \( L \).

If \( q = 3 \) and \( n \geq 7 \), then

\[ f(x) - (x^3 + \xi x^2 - x)\ell(x) \]
is represented by \(\langle 1, 1, x \rangle\), where
\[
\ell(x) = \begin{cases} 
(x^{(n-3)/2} + 1)^2 & \text{if } \xi \neq a_2 \\
(x^{(n-3)/2} + x + 1)^2 & \text{if } \xi = a_2.
\end{cases}
\]
Therefore \(f(x)\) is represented by \(L\).

If \(q = 3\) and \(n = 5\), then one can obtain \(a_2 - \xi \pm 1 \neq 0\) by choosing a proper sign. Then
\[
f(x) - (x^3 + \xi x^2 - x)(x \mp 1)^2
\]
is represented by \(\langle 1, 1, x \rangle\), which implies that \(f(x)\) is again represented by \(L\). \(\square\)

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