SLE + GFF != KPZ

- Juhan Aru
- ENS Lyon

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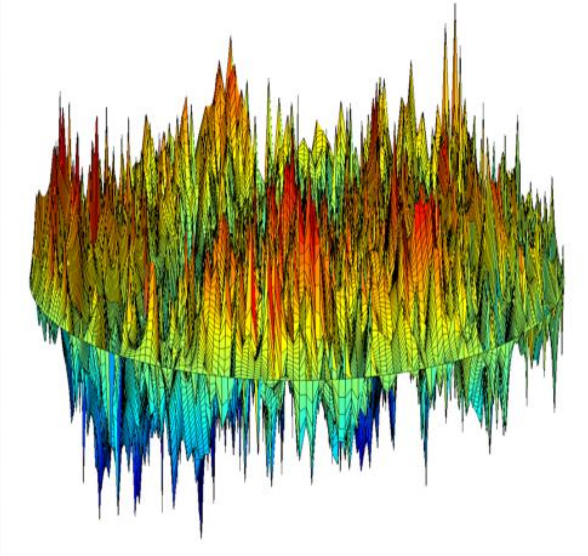

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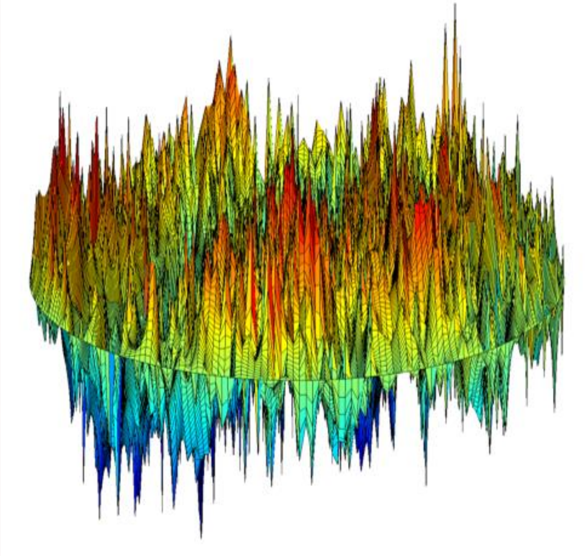

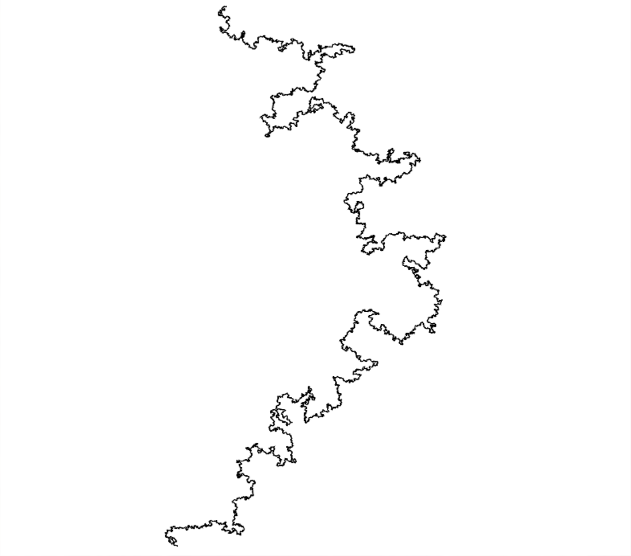

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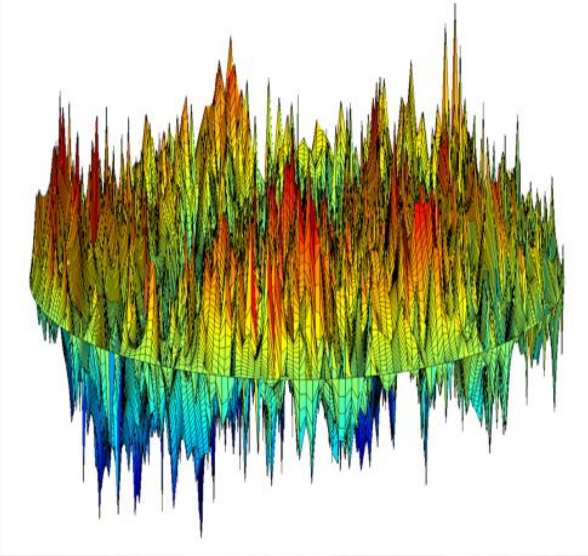

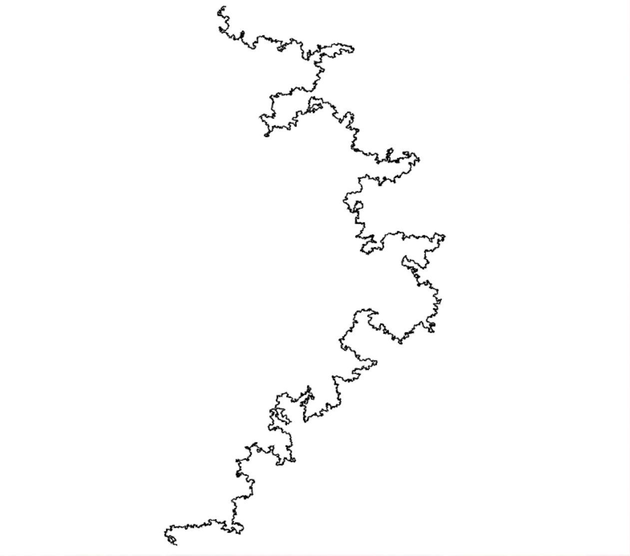

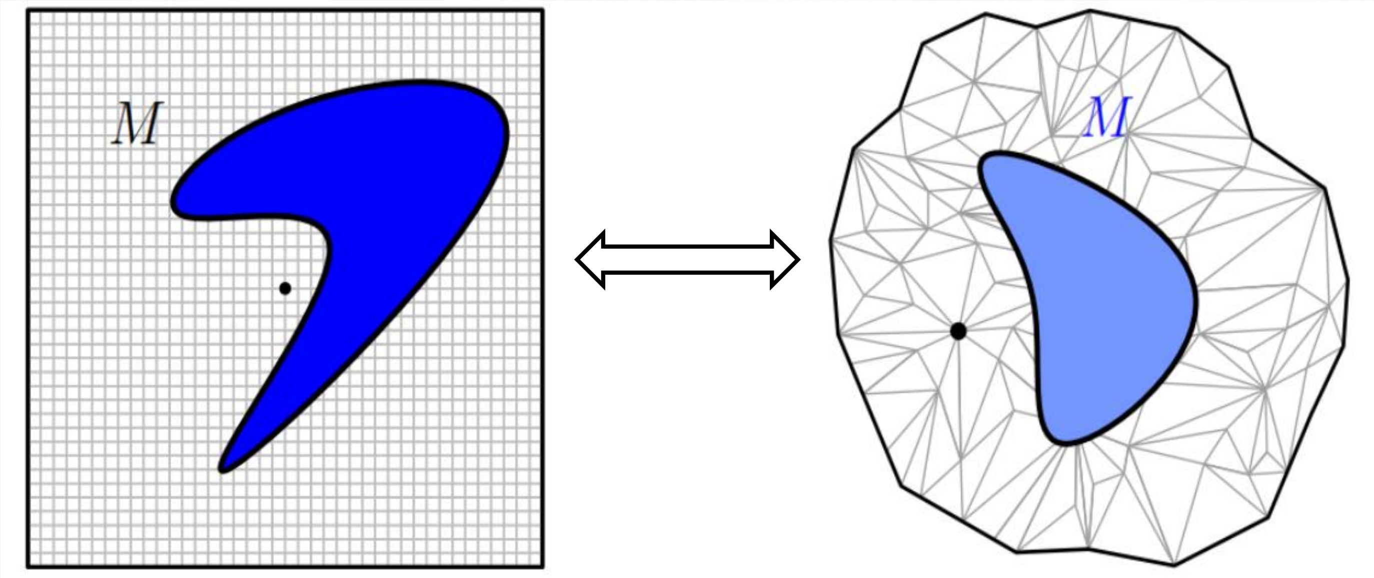

LIOUVILLE MEASURE AND THE KPZ RELATION

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- Different versions: Q-box counting **(DS)**, Hausdorff **(RV)**, we consider a Minkowski version

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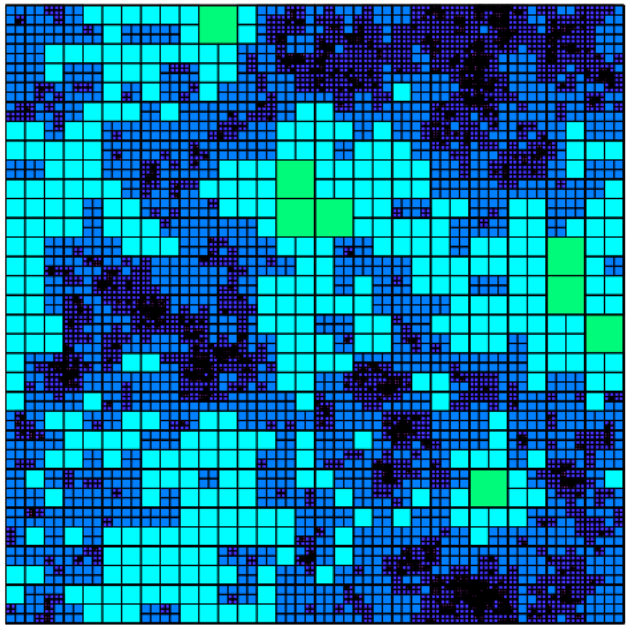

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- Also holds for field-independent sets

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- What about more natural counterexamples?

KPZ RELATION AND CONTOUR LINES

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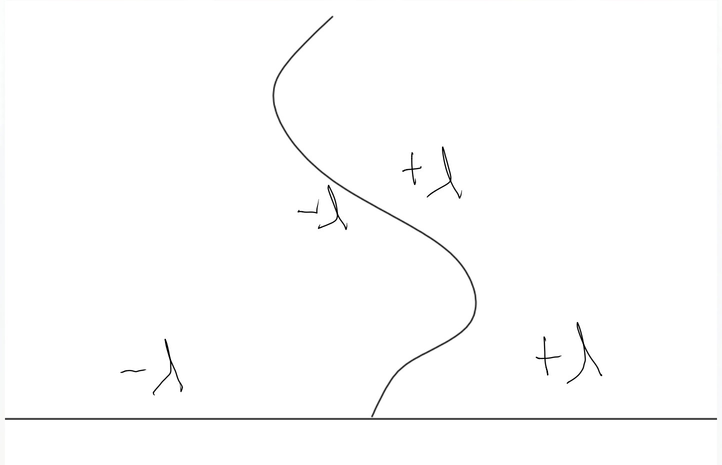

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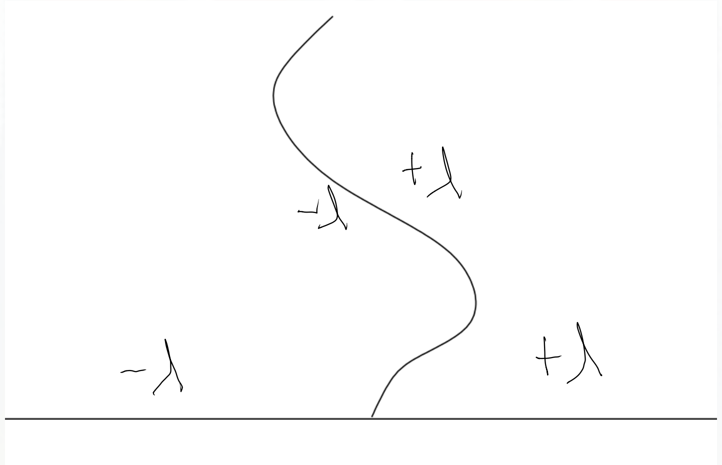

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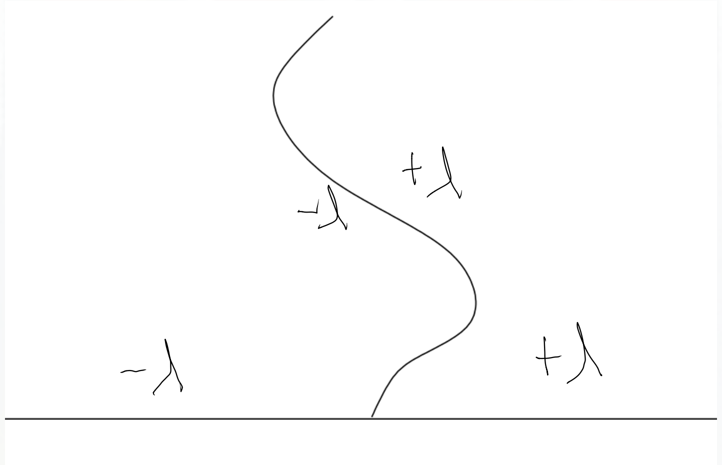

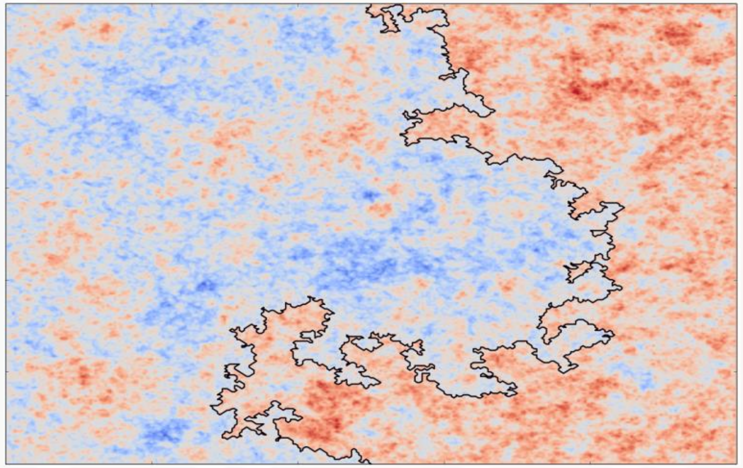

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- **PROOF:**
  - Sample the SLE4 and look at scaling of dyadic squares intresecting the SLE4
  - Use Jensen to bound: $\mathbf{E}(\mu_\gamma(S)^q) \leq \mathbf{E}\left(\mu_\gamma(S)\right)^q$
  - Regularize the field, use Fubini, control field δ-far from the curve; bound neighbourhood of the line
  - Use our knowledge that the Hausdorff (and Minkowski) dimension of the SLE4 is $\frac{3}{2}$

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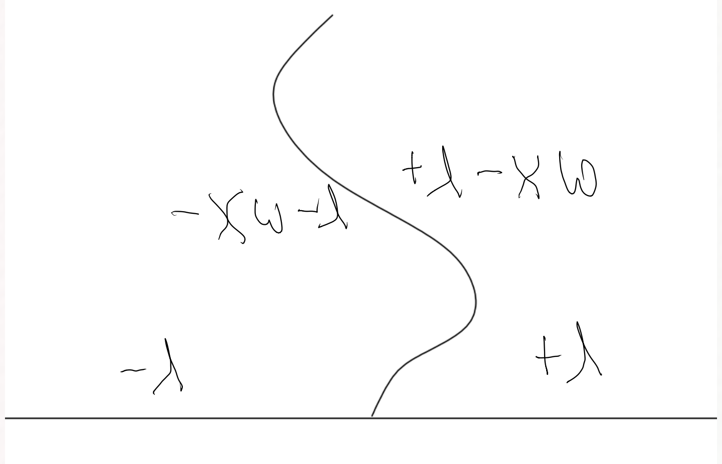
- But boundary conditions more complicated, need to also incorporate winding of the curve, given by $\mathrm{w_T(z)} := \arg {f_T}'(z)$
  - Not defined on the curve, blows up nearing the curve



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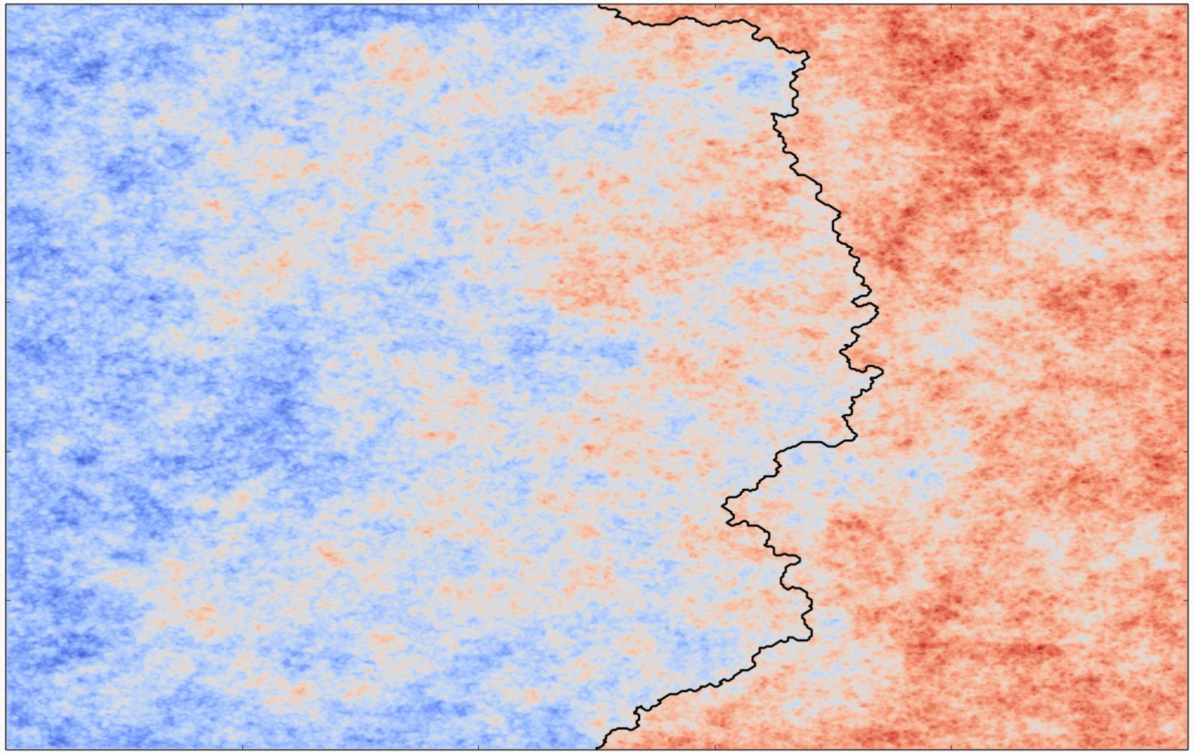

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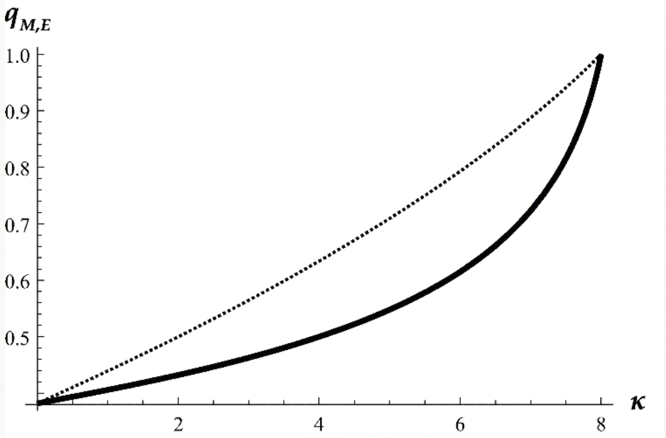

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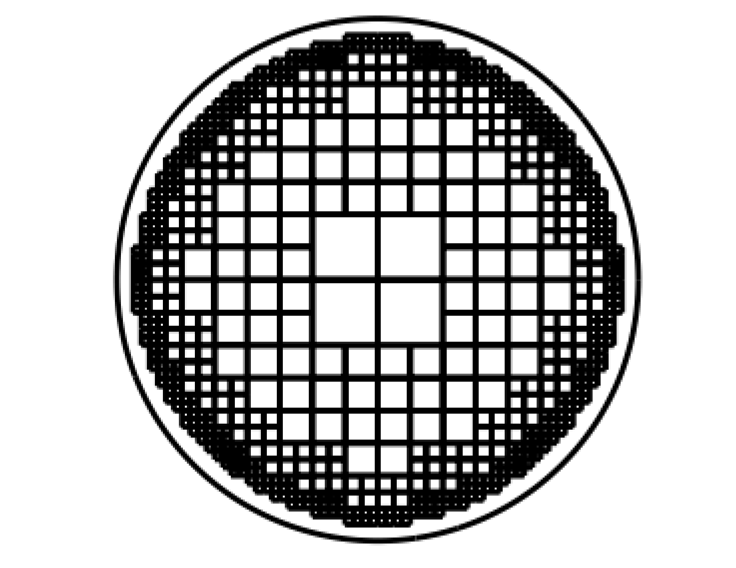

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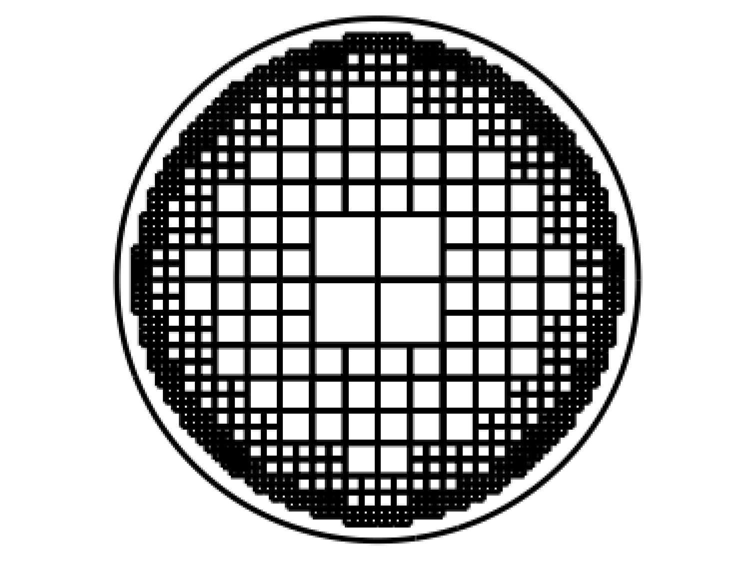

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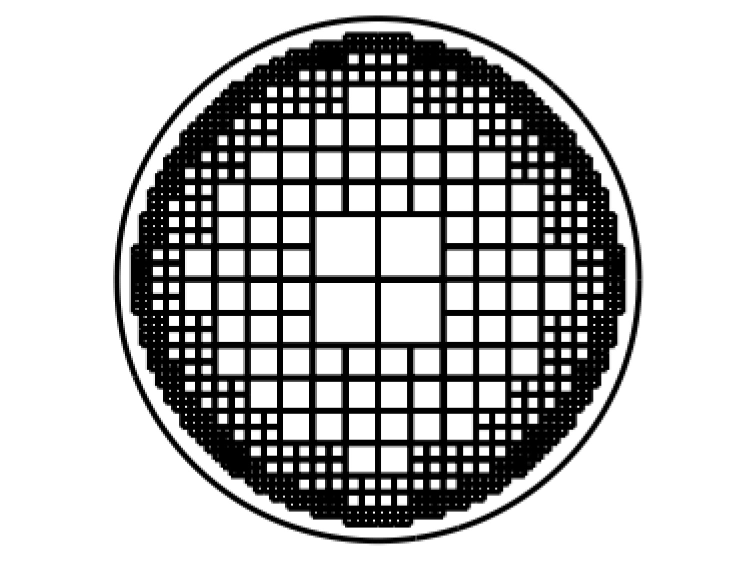

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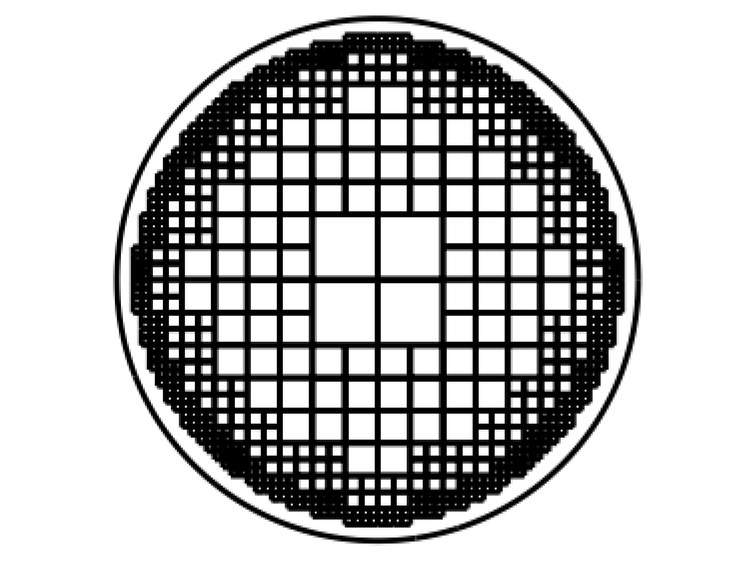
  - Use this to determine quantum fractal dimension of SLE



SCALING OF CR-WHITNEY SQUARE

- S some dyadic square, we write

$$\mathbf{E}\left(\mu_\gamma(S)^q \,|\, S \in W\right) =$$

$$\mathbf{E}\left(\left(\lim_{\delta\to 0}\int_S \delta^{\frac{\gamma^2}{2}} e^{\gamma h_\delta(z)}\,dz\right)^q |S \in W\right) =$$

$$\mathbf{E}\left(\left(\lim_{\delta\to 0}\int_S \delta^{\frac{\gamma^2}{2}} e^{\gamma h_{\delta,H_t}(z)+\gamma w(z)}\,dz\right)^q |S \in W\right) =$$

SCALING OF CR-WHITNEY SQUARE

- Incorporate winding:
  - CR-Whitney: condition the centre of the square to satisfy $\mathrm{CR}(z, H_{SLE}) \asymp \epsilon$
  - show that winding is up to an additive error constant in each CR-Whitney square
  - determine exponential moments of winding under this specific conditioning
- Use Kahane convexity inequalities for lower bound

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  - more or less a Gaussian of variance $-\frac{\kappa}{4} \log \epsilon$
- The notions of windings are different, but (should) agree asymptotically near the curve

WINDING OF CHORDAL SLE:

- **THEOREM:**
  Fix $z \in \mathbf{H}$; $0 < \kappa < 8$. Let τ be the first time that the SLEκ cuts z from infinity.

  For ϵ small enough, conditioned on $\mathrm{CR}(z, H_{SLE}) \asymp \epsilon$ the exponential moments of winding $w(z) \coloneqq \lim_{t\to\tau} \arg f_t{}'(z)$ are given by

$$\mathbf{E}(e^{\lambda w(z)} | \mathrm{CR}(z, H_{SLE}) \asymp \epsilon) \asymp \epsilon^{-\frac{\lambda^2 \kappa}{8}}$$

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  - Same diffusion studied in papers of **(L), (SSW)**

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  - If this was the case for our conditioning, we could calculate:

$$\mathbf{E}e^{\lambda w(z)} |\, \tau \in [T, T+c] =$$
$$\mathbf{E}e^{\lambda \int_0^\tau \cot\frac{X_s}{2}ds} |\, \tau \in [T, T+c] =$$
$$\mathbf{E}e^{-\lambda\frac{\sqrt{\kappa}}{2}B_\tau + \lambda X_\tau} |\, \tau \in [T, T+c] = O(1)e^{\frac{\lambda^2\kappa}{8}T}$$

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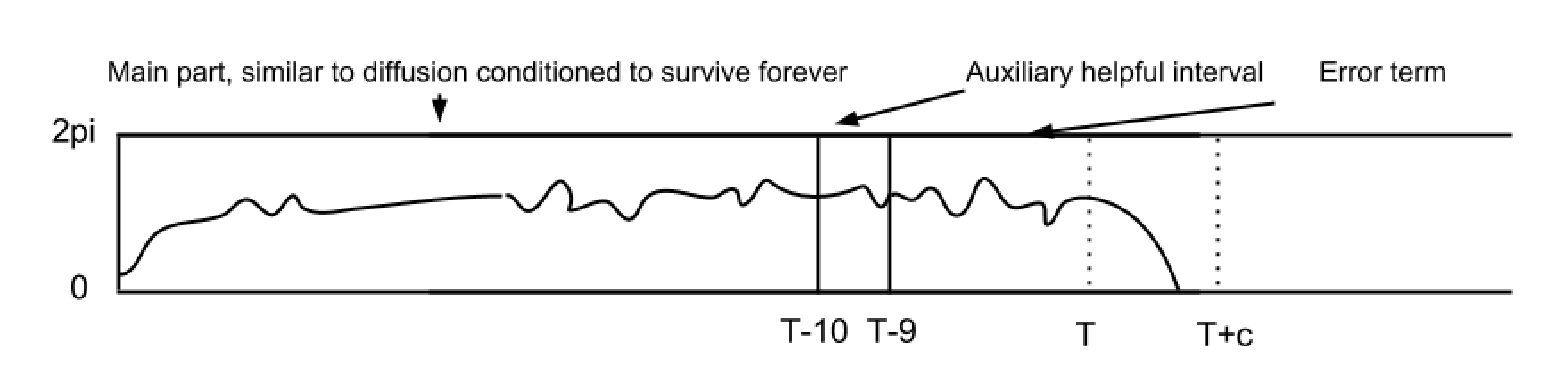

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KEY LEMMA: $\left(\frac{\phi_i(x)}{\phi_0(x)}\right)' \leq \mathrm{O}(\lambda_i^m)$

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$$e(x,s) = \frac{{\phi_0}'(x)}{\phi_0(x)} - \frac{{\boldsymbol{P}_x}'(\tau \in [T-s, T-s+c])}{\boldsymbol{P}_x(\tau \in [T-s, T-s+c])}$$

  - To bound the error for the main contribution, we need some hands-on analysis to control the boundary behaviour of all eigenfunctions:

KEY LEMMA: $\left(\frac{\phi_i(x)}{\phi_0(x)}\right)' \leq \mathrm{O}(\lambda_i^m)$

  - To bound the rest term, we use more probabilistic arguments

THANK YOU!