

# MULTIFRACTALITY OF WHOLE-PLANE SLE

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Let  $f(z) = \sum_{n \geq 0} a_n z^n$  be a holomorphic function in the unit disc  $\mathbb{D}$ . Further assume that  $f$  is injective. Then  $a_1 \neq 0$  and *Bieberbach* proved in 1916 that  $|a_2| \leq 2|a_1|$ . In the same paper, he famously conjectured that  $\forall n \geq 2, |a_n| \leq n|a_1|$ , guided by the intuition that the *Koebe function*

$$\mathcal{K}(z) := - \sum_{n \geq 1} n(-z)^n = \frac{z}{(1+z)^2},$$

which is a holomorphic bijection between  $\mathbb{D}$  and  $\mathbb{C} \setminus [1/4, +\infty)$ , should be *extremal*. This conjecture was finally proven in 1984 by *de Branges*. The earliest important contribution to the proof of Bieberbach's conjecture is that by *Loewner* in 1923 that  $|a_3| \leq 3|a_1|$ . *Oded Schramm* revived Loewner's method in 1999, introducing *randomness* into it, as driven by *standard Brownian motion*.

## Whole-Plane SLE & LLE

$$\frac{\partial f_t}{\partial t} = z \frac{\partial f_t}{\partial z} \frac{\lambda(t) + z}{\lambda(t) - z}, \quad z \in \mathbb{D},$$

$$\lambda(t) = e^{i\sqrt{\kappa}B_t} [e^{i\xi L_t}].$$

The characteristic function of a Lévy process  $L_t$  has the form

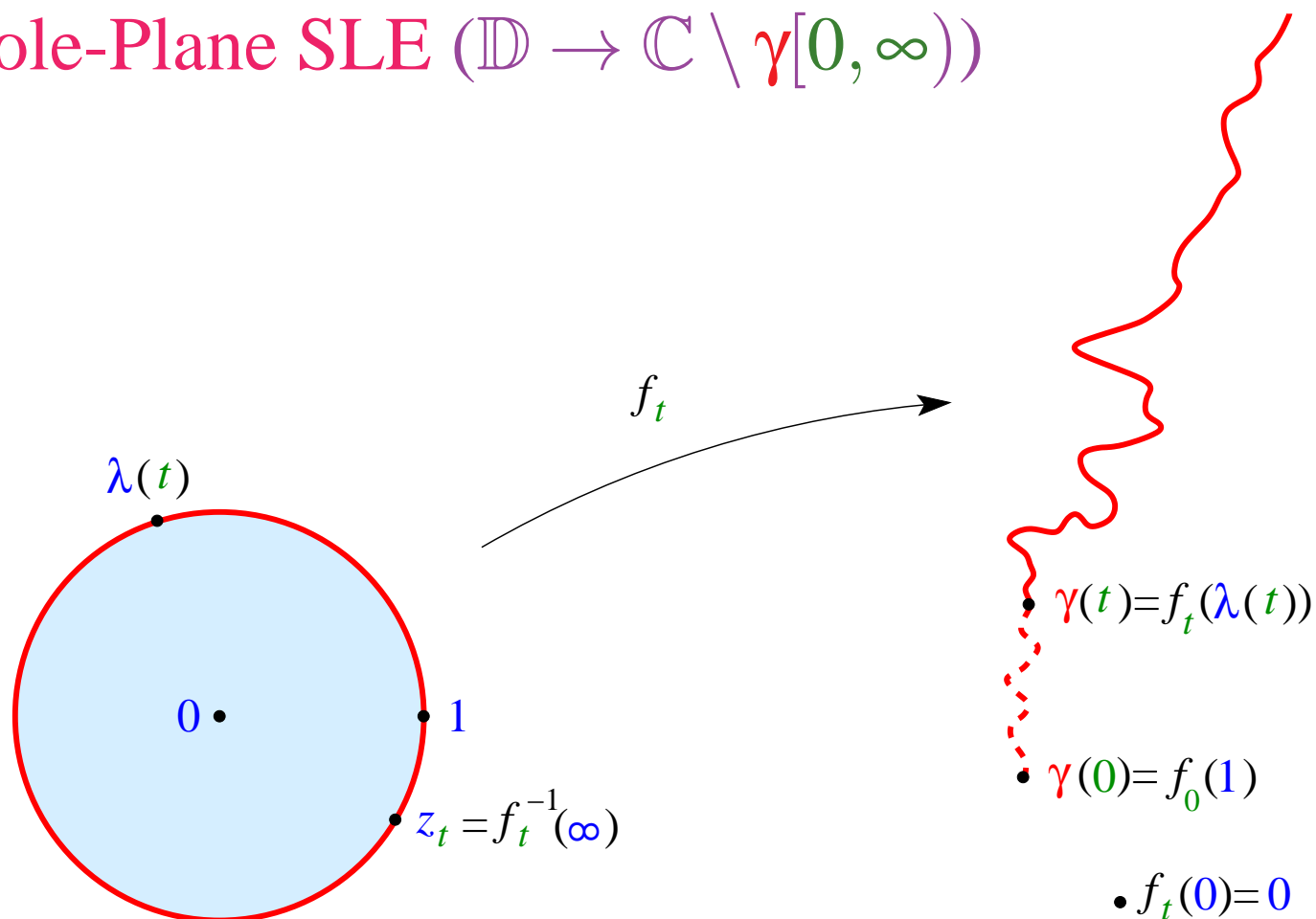
$$\mathbb{E}(e^{i\xi L_t}) = e^{-t\eta(\xi)},$$

where  $\eta$  the Lévy symbol. The function

$$\eta(\xi) = \kappa|\xi|^\alpha/2, \quad \alpha \in (0, 2]$$

is the Lévy symbol of the  $\alpha$ -stable process. The normalization here is chosen so that it is  $\text{SLE}_\kappa$  for  $\alpha = 2$ .

# Whole-Plane SLE ( $\mathbb{D} \rightarrow \mathbb{C} \setminus \gamma[0, \infty)$ )



Loewner map  $z \mapsto f_t(z)$  from the unit disk  $\mathbb{D}$  to the slit domain  $\Omega_t = \mathbb{C} \setminus \gamma([t, \infty))$ . One has  $f_t(0) = 0, \forall t \geq 0$ . At  $t = 0$ ,  $\lambda(0) = 1$ , so that the image of  $z = 1$  is at the tip  $\gamma(0) = f_0(1)$  of the curve.

## Series expansions

Let  $f_t$  be the whole-plane evolution generated by the Lévy process  $(L_t)$  with Lévy symbol  $\eta$ . We write

$$e^{-t} f_t(z) = z + \sum_{n=2}^{\infty} a_n(t) z^n; \quad e^{-t/2} h_t(z) = z + \sum_{n \geq 1} b_{2n+1}(t) z^{2n+1}.$$

Then the *conjugate* whole-plane LLE  $e^{-iL_t} f_t(e^{iL_t} z)$  has the same law as  $f_0(z)$ , i.e.,  $e^{i(n-1)L_t} a_n(t) \stackrel{(\text{law})}{=} a_n(0)$ . Similarly, the *conjugate of the oddified* whole-plane LLE  $h_t(z) := z \sqrt{f_t(z^2)/z^2}$ ,  $e^{-(i/2)L_t} h_t(e^{(i/2)L_t} z)$ , has the same law as  $h_0(z)$ , i.e.,  $e^{inL_t} b_{2n+1}(t) \stackrel{(\text{law})}{=} b_{2n+1}(0)$ .

## Loewner's method

Recall that

$$f_t(z) = e^t \left( z + \sum_{n \geq 2} a_n(t) z^n \right).$$

Expanding both sides of Loewner's equation as power series and identifying coefficients yields the set of *recursion equations for  $n \geq 2$*

$$\dot{a}_n(t) - (n-1)a_n(t) = 2 \sum_{k=1}^{n-1} k a_k(t) \bar{\lambda}^{n-k}(t),$$

with  $a_1 = 1$ ; *the dot means a  $t$ -derivative*, and  $\bar{\lambda}(t) = 1/\lambda(t)$ , with  $\lambda(t) = e^{i\sqrt{\kappa}B_t} [e^{i\xi L_t}]$ .

## Expected coefficients

**Theorem 1.** Setting  $a_n := a_n(0)$  and  $b_{2n+1} := b_{2n+1}(0)$ , we have

$$\mathbb{E}(a_n) = \prod_{k=0}^{n-2} \frac{\eta_k - k - 2}{\eta_{k+1} + k + 1}, \quad n \geq 2,$$

$$\mathbb{E}(b_{2n+1}) = \prod_{k=0}^{n-1} \frac{\eta_k - k - 1}{\eta_{k+1} + k + 1}, \quad n \geq 1.$$

**Corollary 1.** If  $\eta_1 = 3$ ,  $\mathbb{E}(f'_0(z)) = 1 - z$  ( $SLE_6$ );

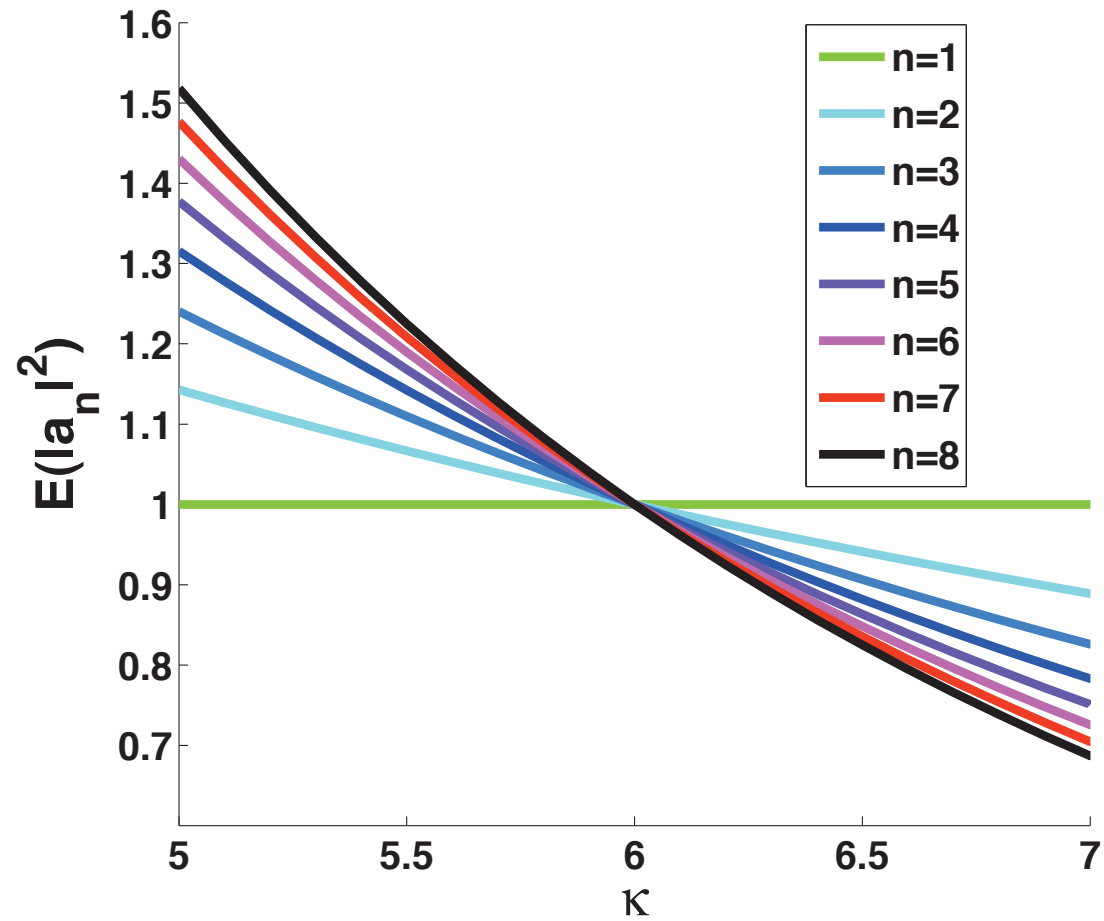
if  $\eta_1 = 1$  and  $\eta_2 = 4$ ,  $\mathbb{E}(f'_0(z)) = (1 - z)^2$  ( $SLE_2$ );

if  $\eta_1 = 2$ ,  $\mathbb{E}(h'_0(z)) = 1 - z^2$  ( $SLE_4$ ).

[See also [Kemppainen '10](#) for expectations of SLE coefficient moments.]

# The Surprise: Expected Square Coefficients $\mathbb{E}(|a_n|^2)$

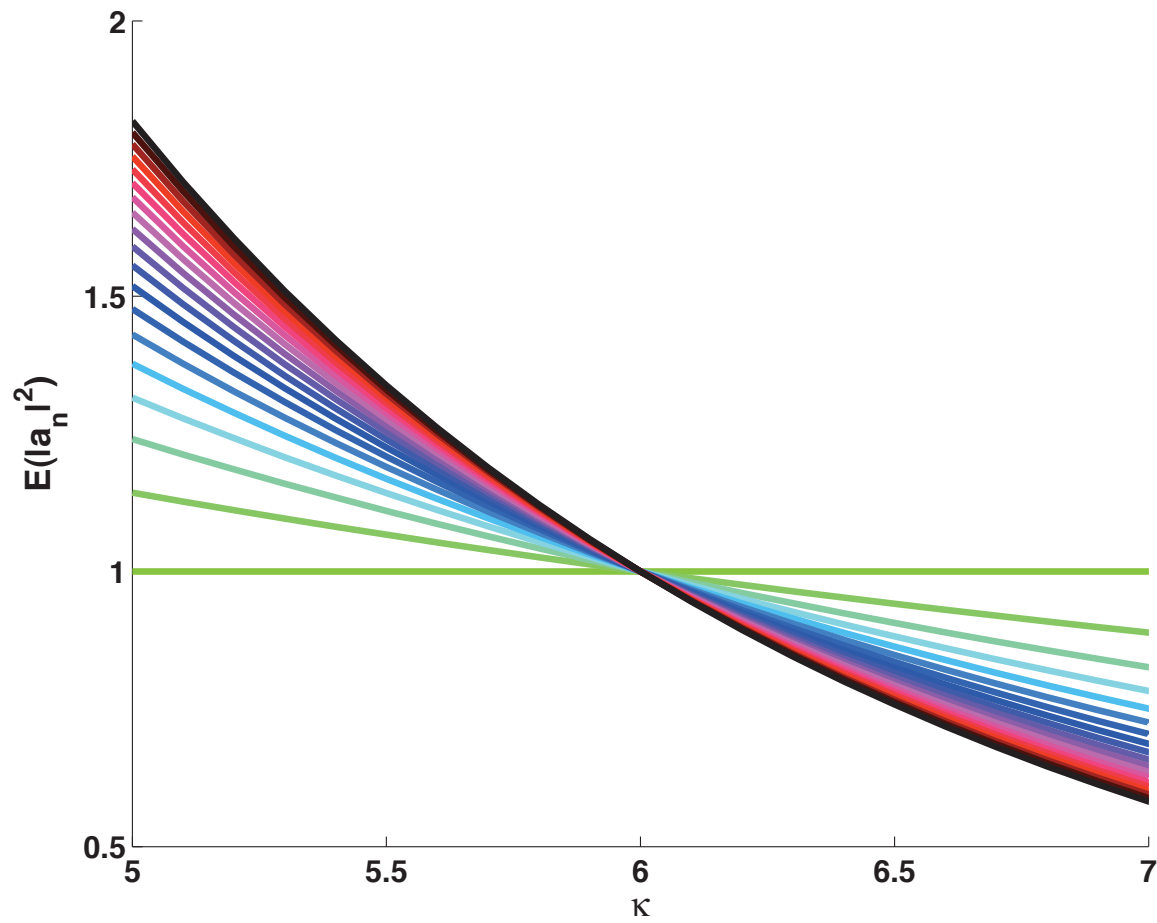
*Examples: For  $SLE_6$*





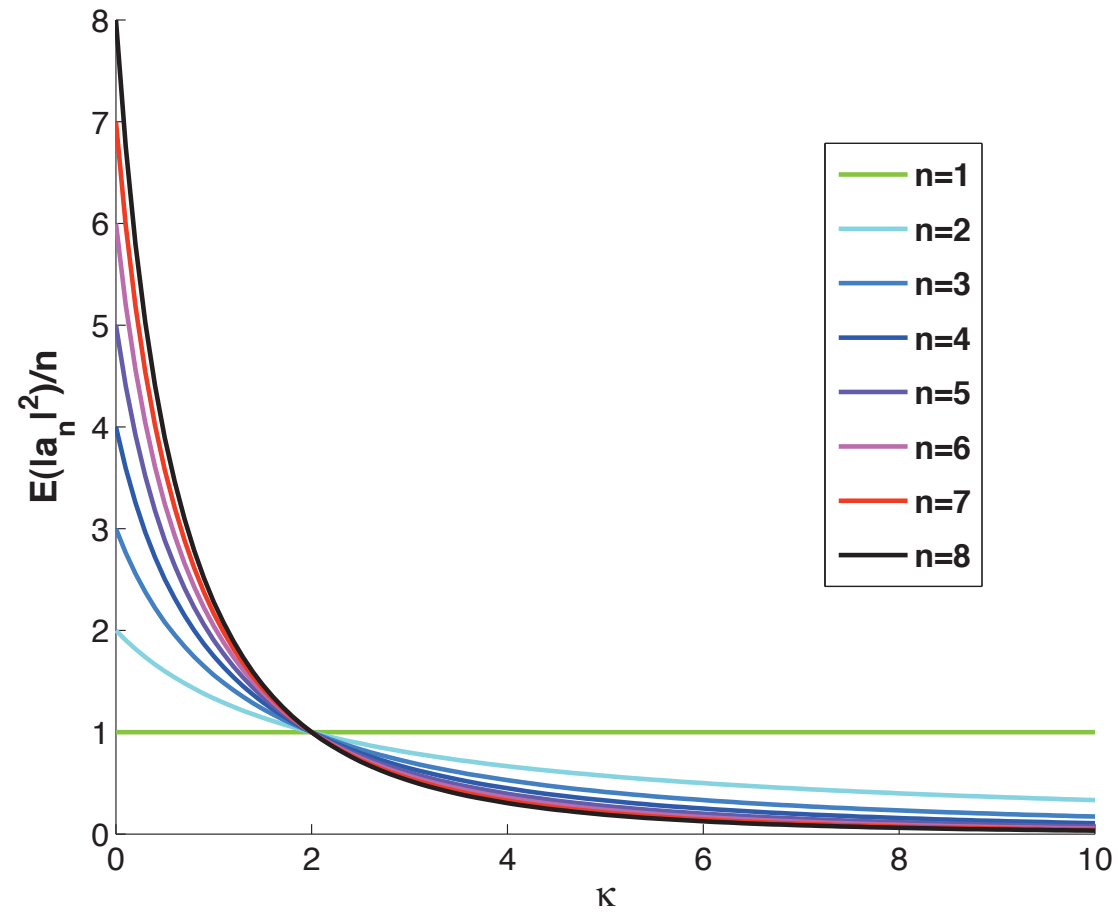
# Expected Square Coefficients $\mathbb{E}(|a_n|^2)$

For  $SLE_6$

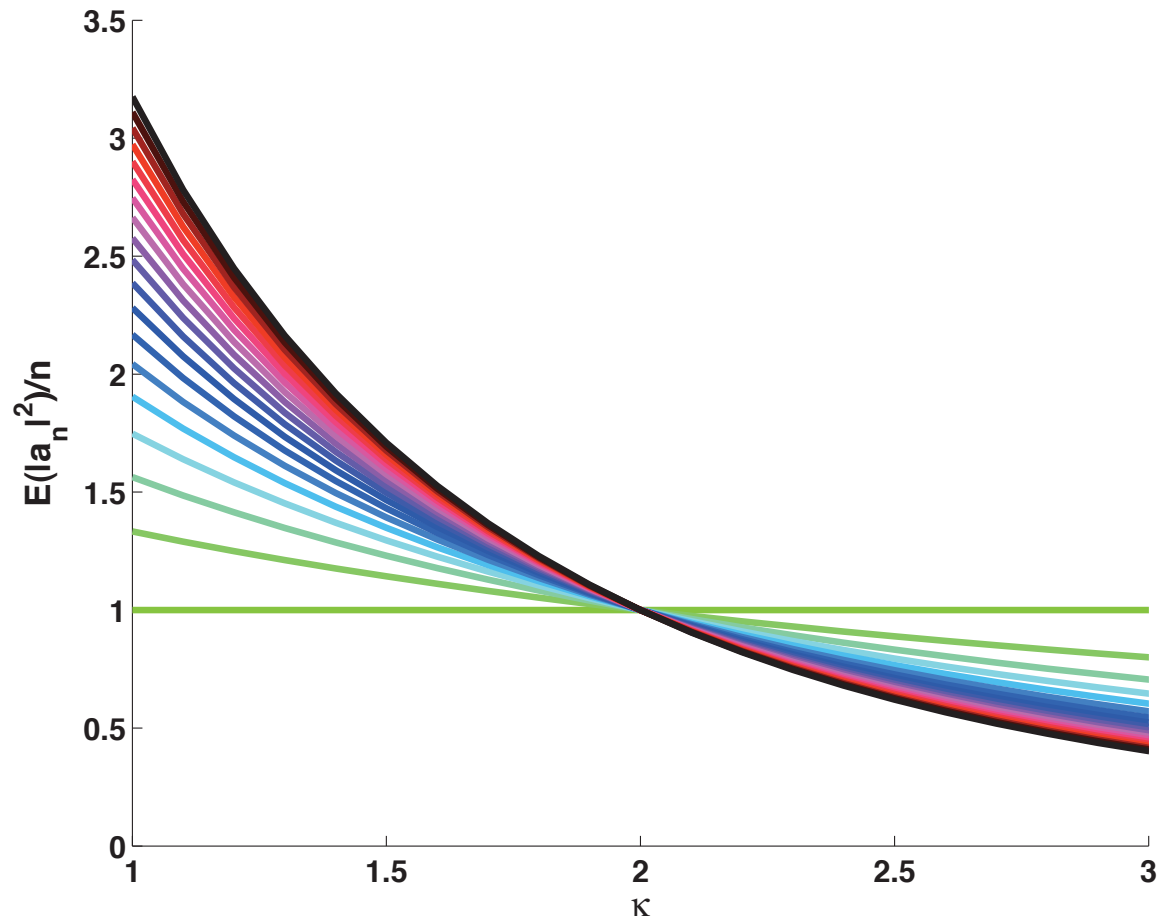


# Expected Square Coefficients $\mathbb{E}(|a_n|^2)/n$

For  $SLE_2$



Expected Square Coefficients  $\mathbb{E}(|a_n|^2)/n$   
For  $SLE_2$



## Expected Square Coefficients

*Examples:* For  $SLE_6$  and  $SLE_2$

$$\mathbb{E}(|a_n|^2) = 1, \kappa = 6, \forall n \geq 1; \quad \mathbb{E}(|a_n|^2) = n, \kappa = 2, \forall n \geq 1$$

$$\mathbb{E}(|a_4|^2) = \frac{8\kappa^5 + 104\kappa^4 + 4576\kappa^3 + 18288\kappa^2 + 22896\kappa + 8640}{9(\kappa + 10)(3\kappa + 2)(\kappa + 6)(\kappa + 1)(\kappa + 2)^2}.$$

*[Recursion:  $n \leq 4$ ; Computer assisted:  $n \leq 8$  (formal),  $n \leq 19$  (num.)]*

### Theorem 2.

- (i) if  $\eta_1 = 3$ ,  $\mathbb{E}(|a_n|^2) = 1, \forall n \geq 1$  ( $SLE_6$ );
- (ii) if  $\eta_1 = 1, \eta_2 = 4$ ,  $\mathbb{E}(|a_n|^2) = n, \forall n \geq 1$  ( $SLE_2$ );
- (iii) if  $\eta_1 = 2$ ,  $\mathbb{E}(|b_{2n+1}|^2) = 1/(2n+1), \forall n \geq 1$  ( $SLE_4$ ).

## Derivative Moments

**Theorem 3.** *The whole-plane  $SLE_{\kappa}$  map  $f_0(z)$  has derivative moments*

$$\begin{aligned}\mathbb{E}[(f'_0(z))^{p/2}] &= (1-z)^\alpha, \\ \mathbb{E}[|f'_0(z)|^p] &= \frac{(1-z)^\alpha(1-\bar{z})^\alpha}{(1-z\bar{z})^\beta},\end{aligned}$$

*for the special set of exponents  $p = p(\kappa) := (6 + \kappa)(2 + \kappa)/8\kappa$ , with  $\alpha = (6 + \kappa)/2\kappa$  and  $\beta = (6 + \kappa)^2/8\kappa$ . [See also *Loutsenko & Yermolayeva '12*]*

**Corollary 4.**  *$p = 2$  case: for  $\kappa = 6$ :*

$$\mathbb{E}(f'_0(z)) = 1 - z, \quad \mathbb{E}(|f'_0(z)|^2) = \frac{(1-z)(1-\bar{z})}{(1-z\bar{z})^3};$$

*for  $\kappa = 2$ :*

$$\mathbb{E}(f'_0(z)) = (1-z)^2, \quad \mathbb{E}(|f'_0(z)|^2) = \frac{(1-z)^2(1-\bar{z})^2}{(1-z\bar{z})^4}.$$

## Logarithmic Moments [D., Ho, Le, Zinsmeister '14]

**Theorem 5.** *The whole-plane  $SLE_{\kappa}$  map  $f_0(z)$  has logarithmic moments*

$$\mathbb{E} \left( \left| \frac{f'_0(z)}{f_0(z)} \right|^{\frac{\kappa+2}{\kappa}} \right) = \frac{(1-z)^{\frac{2}{\kappa}} (1-\bar{z})^{\frac{2}{\kappa}}}{(1-z\bar{z})^{\frac{2}{\kappa}} (z\bar{z})^{\frac{\kappa+2}{2\kappa}}}.$$

**Corollary 6.**  *$p = 2$  case, for  $\kappa = 2$ :*

$$\mathbb{E} \left( \left| \frac{f'_0(z)}{f_0(z)} \right|^2 \right) = \frac{(1-z)(1-\bar{z})}{(1-z\bar{z})z\bar{z}}.$$

For the whole-plane  $SLE_{\kappa}$  map, let  $\log \frac{f_0(z)}{z} = \sum_{n \geq 1} \gamma_n z^n$ ;  
then, for  $\kappa = 2$ ,  $\mathbb{E}(|\gamma_n|^2) = 2/n^2$ ,  $\forall n \geq 1$ .

## Mixed Moments [D., Ho, Le, Zinsmeister '14]

General case

$$\mathbb{E} \left( \frac{|f_0'(z)|^p}{|f_0(z)|^q} \right).$$

**Theorem 6.** *For  $p = 0$ , the whole-plane  $SLE_\kappa$  map  $f_0(z)$  has a special moment*

$$\mathbb{E}(|f_0(z)|^{\frac{(\kappa+4)(\kappa+2)}{2\kappa}}) = \frac{(1-z)^{\frac{\kappa+4}{\kappa}} (1-\bar{z})^{\frac{\kappa+4}{\kappa}} (z\bar{z})^{\frac{(\kappa+4)(\kappa+2)}{4\kappa}}}{(1-z\bar{z})^{\frac{(\kappa+4)^2}{2\kappa}}}.$$

## *m*-fold version

**Theorem 7.** *The  $m$ -fold whole-plane  $SLE_{\kappa}$  map*

*$h_0^{(m)}(z) := [f_0(z^m)]^{1/m}$  has derivative moments*

$$\begin{aligned}\mathbb{E} \left[ \left( (h_0^{(m)})'(z) \right)^{p/2} \right] &= (1 - z^m)^{\alpha}, \\ \mathbb{E} \left[ \left| (h_0^{(m)})'(z) \right|^p \right] &= \frac{(1 - z^m)^{\alpha} (1 - \bar{z}^m)^{\alpha}}{(1 - z^m \bar{z}^m)^{\beta}},\end{aligned}$$

*for the special set of exponents*

*$p = p_m(\kappa) = m(2m + 4 + \kappa)(2 + \kappa) / 2(m + 1)^2 \kappa$ , with*

*$\alpha = (2m + 4 + \kappa) / (m + 1) \kappa$  and*

*$\beta = (2m + 4 + \kappa)^2 / 2(m + 1)^2 \kappa$ .*

For  $m = 2$

$$p_2(\kappa = 4) = 2, \alpha = 1, \beta = 2.$$



## The B–S Equation

Beliaev and Smirnov (2005) obtained by martingale arguments the following equation for the *exterior whole-plane* case

$(F(z) = F(re^{i\theta}), r \geq 1, \sigma = +1)$

$$p \left( \frac{r^4 + 4r^2(1 - r \cos \theta) - 1}{(r^2 - 2r \cos \theta + 1)^2} - \sigma \right) F + \frac{r(r^2 - 1)}{r^2 - 2r \cos \theta + 1} F_r - \frac{2r \sin \theta}{r^2 - 2r \cos \theta + 1} F_\theta + \Lambda F = 0.$$

**Proposition 1.** *For the interior whole-plane Schramm (or Lévy)-Loewner evolution, the moments of the derivative modulus,  $F(z) := \mathbb{E}(|f'_0(z)|^p)$ , satisfy the same B–S equation, but with  $\sigma = -1$ , and  $\Lambda = (\kappa/2)\partial^2/\partial\theta^2$  the generator of the driving Brownian process (or of the Lévy process).*

## Holomorphic Coordinates

Switch to  $z, \bar{z}$  variables, instead of polar coordinates, and write  $F(z)$  above as

$$F(z, \bar{z}) := \mathbb{E}(|f'_0(z)|^p) = \mathbb{E}[(f'_0(z))^{p/2}(\bar{f}'_0(\bar{z}))^{p/2}].$$

Using  $\partial := \partial_z$ ,  $\bar{\partial} := \partial_{\bar{z}}$ , the equation then becomes

$$-\frac{\kappa}{2}(z\partial - \bar{z}\bar{\partial})^2 F + \frac{z+1}{z-1}z\partial F + \frac{\bar{z}+1}{\bar{z}-1}\bar{z}\bar{\partial}F - p \left[ \frac{1}{(z-1)^2} + \frac{1}{(\bar{z}-1)^2} + (\sigma - 1) \right] F = 0.$$

*Exterior/Interior whole-plane:  $\sigma = \pm 1$ .*

- *Consider the action of the differential operator  $\mathcal{P}(D)$  on a function of the form  $F(z, \bar{z}) = \varphi(z)\bar{\varphi}(\bar{z})P(z, \bar{z})$ .*

- *Simplify it for the particular choice of a rotationally invariant  $P(z, \bar{z}) := P(z\bar{z})$ .*

- *Study the algebra generated by the action of  $\mathcal{P}(D)$  on*

*$\varphi(z) = \varphi_\alpha(z) := (1-z)^\alpha$ , and  $P(z\bar{z}) := (1-z\bar{z})^{-\beta}$ ,  $\forall \alpha, \beta$ .*

## Integral means spectrum

**Definition 1.** *The integral means spectrum of a conformal mapping  $f$  is the function defined on  $\mathbb{R}$  by*

$$\beta(p) := \overline{\lim}_{r \rightarrow 1} \frac{\log(\int_{\partial D} |f'(rz)|^p |dz|)}{\log(\frac{1}{1-r})}.$$

In the *stochastic* setting, one defines the *average* integral means spectrum

**Definition 2.**

$$\beta(p) := \overline{\lim}_{r \rightarrow 1} \frac{\log(\int_{\partial D} \mathbb{E} |f'(rz)|^p |dz|)}{\log(\frac{1}{1-r})}.$$

**Corollary 7.** *For a Lévy-Loewner evolution with  $\eta_1 = 1, \eta_2 = 4$ , or  $\eta_1 = 3$  (thus including SLE for  $\kappa = 2, 6$ ), and for an oddified LLE with  $\eta_1 = 2$  (thus including SLE for  $\kappa = 4$ ), one has, respectively:*

$$\mathbb{E} \left( \frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta \right) = \frac{1 + 4r^2 + r^4}{(1 - r^2)^4}; \frac{1 + r^2}{(1 - r^2)^3}; \frac{1 + r^4}{(1 - r^4)^2}.$$

This gives the values of the average integral means spectrum  $\beta(2) = 4, 3$  for whole-plane LLE with  $\eta_1 = 1, \eta_2 = 4$  or  $\eta_1 = 3$  (thus whole-plane SLE with  $\kappa = 2, 6$ ) respectively. For the oddified LLE with  $\eta_1 = 2$  (thus the oddified whole-plane SLE<sub>4</sub>),  $\beta_2(2) = 2$ .

- They differ from the corresponding values at  $p = 2$  of the SLE integral means spectrum of *Beliaev & Smirnov '05*, which corresponds to the  $f(\alpha)$  bulk multifractal spectrum for the SLE harmonic measure [*D. '00*].

Define

$$\beta_0(p, \kappa) := -p + \frac{4 + \kappa}{4\kappa} \left( 4 + \kappa - \sqrt{(4 + \kappa)^2 - 8\kappa p} \right),$$

$$\hat{\beta}_0(p, \kappa) := p - \frac{(4 + \kappa)^2}{16\kappa}.$$

This is the *average* integral means spectrum  $\bar{\beta}_0(p, \kappa)$  of the bulk of  $SLE_\kappa$ , as obtained in *Beliaev & Smirnov '05*:

$$\bar{\beta}_0(p, \kappa) = \beta_0(p, \kappa), \quad 0 \leq p \leq p_0^*(\kappa),$$

$$= \hat{\beta}_0(p, \kappa), \quad p \geq p_0^*(\kappa),$$

$$p_0^*(\kappa) := \frac{3(4 + \kappa)^2}{32\kappa}.$$

## Integral means spectra

The whole-plane SLE $_{\kappa}$ ,  $f_{t=0}(z)$ ,  $z \in \mathbb{D}$ , and its  $m$ -fold transforms,  $h_0^{(m)}(z) := z[f_0(z^m)/z^m]^{1/m}$ ,  $m \geq 1$ , have average integral means spectra  $\beta_m(p, \kappa)$  that exhibit a *phase transition* and are given, for  $p \geq 0$ , by

$$\beta_1(p, \kappa) = \max \left\{ \beta_0(p, \kappa), 3p - \frac{1}{2} - \frac{1}{2} \sqrt{1 + 2\kappa p} \right\},$$

$$\beta_2(p, \kappa) = \max \left\{ \beta_0(p, \kappa), 2p - \frac{1}{2} - \frac{1}{2} \sqrt{1 + \kappa p} \right\},$$

$$\beta_m(p, \kappa) = \max \left\{ \bar{\beta}_0(p, \kappa), (1 + 2/m)p - \frac{1}{2} - \frac{1}{2} \sqrt{1 + 2\kappa p/m} \right\}.$$

The first spectrum  $\beta_1$  has its transition point at

$$p^*(\kappa) := \frac{1}{16\kappa} \left( (4 + \kappa)^2 - 4 - 2\sqrt{4 + 2(4 + \kappa)^2} \right) < p_0^*(\kappa).$$

[See also *Loutsenko & Yermolayeva '12, '13*]

## Koebe function

$\lim_{\kappa \rightarrow 0} \beta_0(p, \kappa) = 0$ , and the spectra (for  $p \geq 0$ ):

$$\beta(p, \kappa = 0) = \max\{0, 3p - 1\},$$

$$\beta_2(p, \kappa = 0) = \max\{0, 2p - 1\},$$

$$\beta_m(p, \kappa = 0) = \max\{0, (1 + 2/m)p - 1\},$$

coincide with those directly derived for the **Koebe function**. Also

$$\beta_1(p, \kappa) \leq 3p - 1,$$

$$\beta_2(p, \kappa) \leq 2p - 1,$$

$$\beta_m(p, \kappa) \leq (1 + 2/m)p - 1,$$

in agreement with *Feng and McGregor (1976)* and *Makarov (1998)* for  $f$  holomorphic and injective in the unit disk, and its  $m$ -fold transforms.

**Theorem 8.** *The average integral means spectrum  $\beta(p, \kappa)$  of the unbounded whole-plane  $SLE_\kappa$  has a phase transition at  $p^*(\kappa)$  and a special point at  $p(\kappa) := (6 + \kappa)(2 + \kappa)/8\kappa$ , such that*

$$\beta(p, \kappa) = \beta_0(p, \kappa), \quad 0 \leq p \leq p^*(\kappa);$$

$$\beta(p, \kappa) = 3p - \frac{1}{2} - \frac{1}{2} \sqrt{1 + 2\kappa p} \geq \beta_0(p, \kappa), \quad p^*(\kappa) \leq p \leq \min\{1 + \kappa/2, p(\kappa)\};$$

$$\beta(p, \kappa) \geq 3p - 1/2 - (1/2) \sqrt{1 + 2\kappa p}, \quad \min\{1 + \kappa/2, p(\kappa)\} \leq p \leq p(\kappa);$$

$$\beta(p, \kappa) \leq 3p - 1/2 - (1/2) \sqrt{1 + 2\kappa p}, \quad p(\kappa) < p.$$

- For  $p > p^*(\kappa)$  the  $B$ – $S$  solution ceases to be uniformly positive on  $\partial\mathbb{D}$ .
- Existence of a **subsolution/supersolution** for the parabolic operator  $\mathcal{P}(D)[\psi(z, \bar{z})\ell_\delta(z\bar{z})] \stackrel{\leq}{\geq} 0$  in some annulus of  $\mathbb{D}$  whose boundary includes  $\partial\mathbb{D}$ , corresponding respectively to  $p \stackrel{\leq}{\geq} p^*(\kappa)$ . Trial functions:  
 $\psi(z, \bar{z}) := (1 - z\bar{z})^{-\beta} |1 - z|^2 \gamma$ ,  $\ell_\delta(z\bar{z}) := [-\log(1 - z\bar{z})]^\delta$ .
- Hidden  **$\gamma$ -duality** in the space of hypergeometric  $B$ – $S$  solutions on  $\partial\mathbb{D}$ .



## Integral means spectrum: *Inner whole-plane SLE*

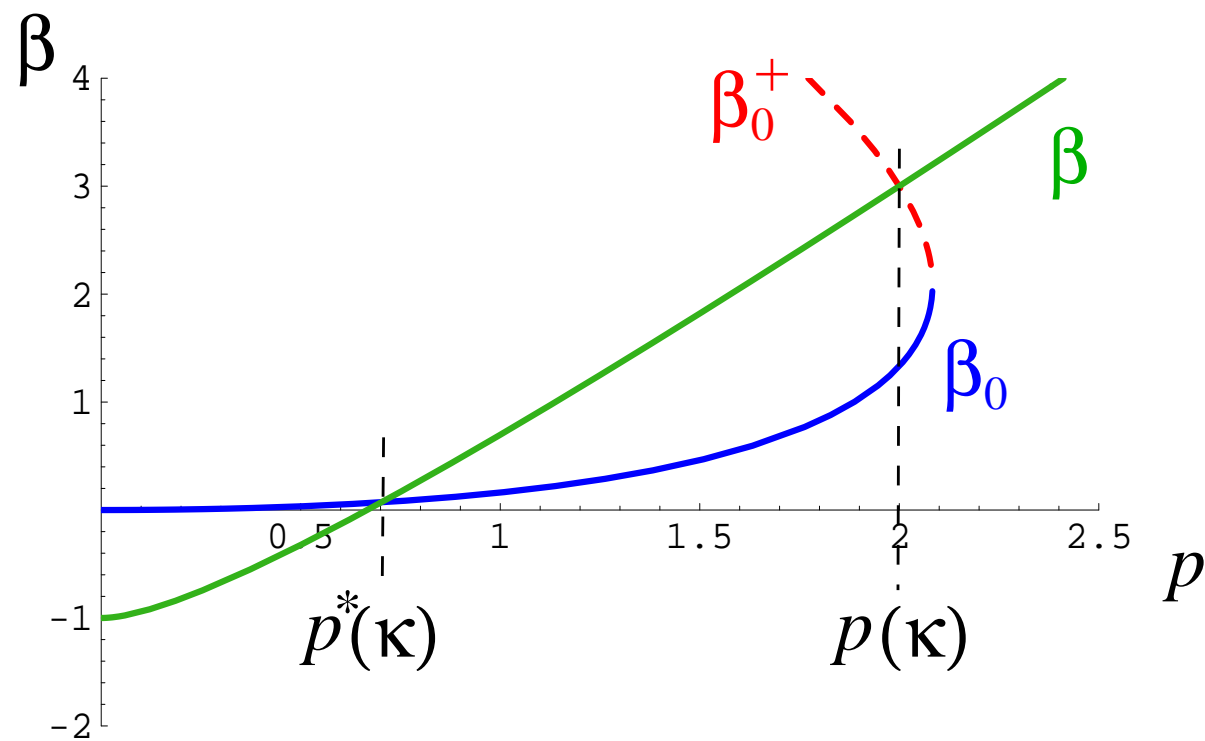


Figure 1:  $\beta(p) = 3p - \frac{1}{2} - \frac{1}{2}\sqrt{1 + 2\kappa p}$

## Integral means spectrum: *Outer whole-plane SLE*

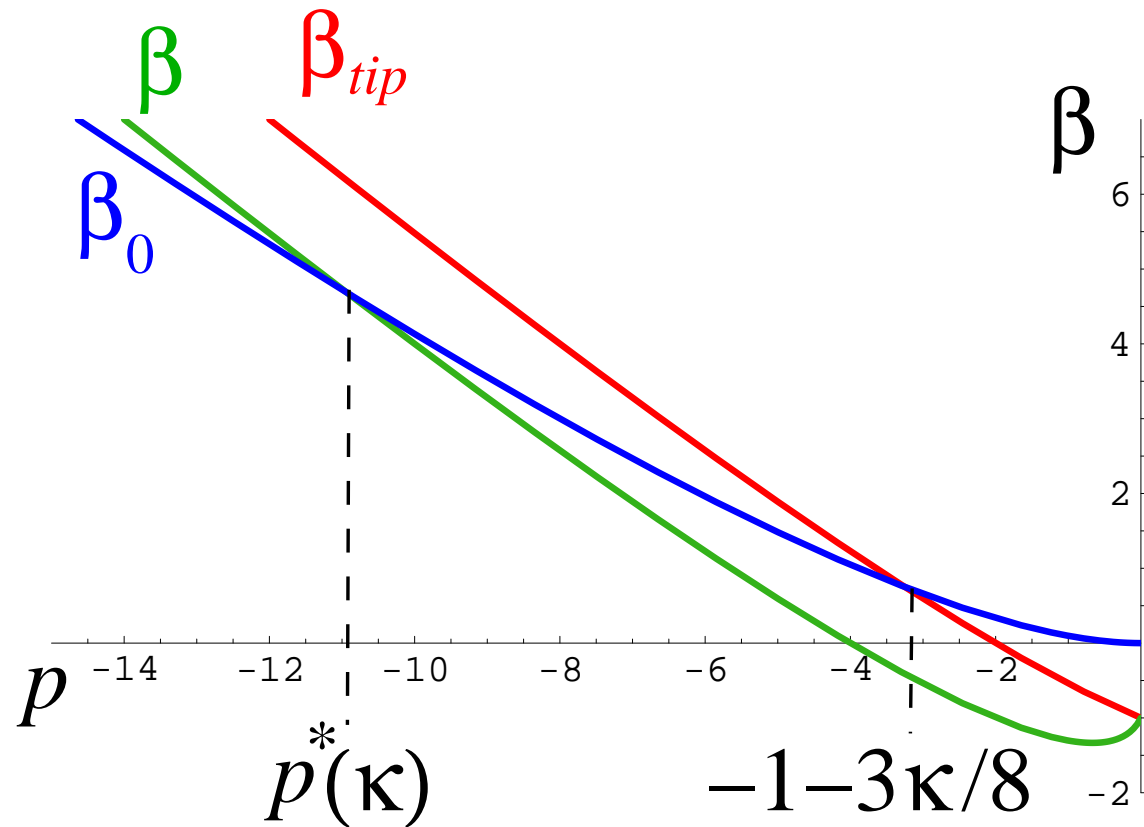


Figure 2:  $\beta(p) = -p - \frac{1}{2} - \frac{1}{2}\sqrt{1 - 2\kappa p}$ ,  $p^*(\kappa) = -(4 + \kappa)^2(8 + \kappa)/128$  (Beliaev, D., Zinsmeister '14)

# Packing Spectrum

The *packing spectrum* [Makarov] is defined as

$$s(p) := \beta(p) - p + 1.$$

For the unbounded whole-plane  $SLE_{\kappa}$ , we have for  $p \geq p^*(\kappa)$

$$\begin{aligned} s(p, \kappa) &= \beta(p, \kappa) - p + 1 \\ &= 2p + \frac{1}{2} - \frac{1}{2} \sqrt{1 + 2\kappa p}. \end{aligned}$$

Consider its *inverse function*

$$\begin{aligned} p = p(s, \kappa) &:= \frac{s}{2} + \frac{\kappa}{8} u_{\kappa}^{-1}(s), \\ u_{\kappa}^{-1}(s) &:= \frac{1}{2\kappa} \left( \kappa - 4 + \sqrt{(4 - \kappa)^2 + 16\kappa s} \right) \end{aligned}$$

(KPZ formula)

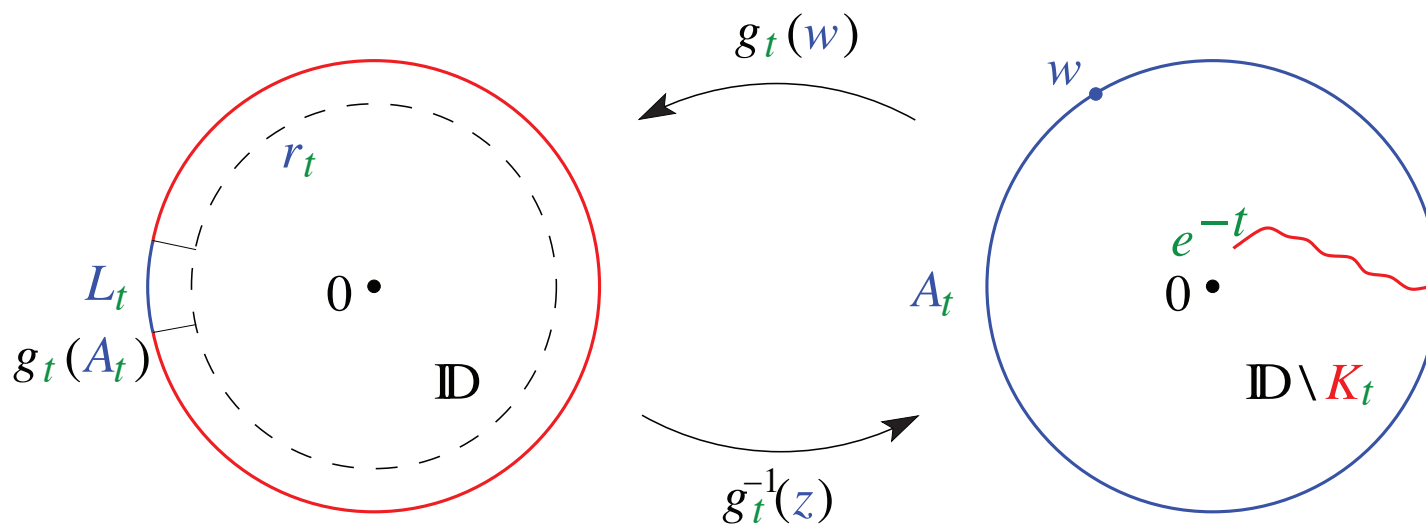
## Relation to Tip & Derivative Exponents

(Non-standard) tip multifractal exponents obtained by quantum gravity [D. '00], corresponding geometrically to the extremity of an  $SLE_{\kappa}$  path avoiding a packet of  $s$  independent Brownian motions.

Differ from the ones associated to the *standard SLE tip multifractal spectrum* [Hastings '02, Beliaev & Smirnov '05, Johansson & Lawler '09].

Identical to the *derivative exponents* obtained for radial  $SLE_{\kappa}$  [Lawler, Schramm & Werner '01].

# (Inverse) Radial SLE Map



$$f_0(z) \stackrel{(\text{law})}{=} \lim_{t \rightarrow +\infty} [e^t g_t^{-1}(z)].$$

# Derivative exponents

**Lemma 1.** (Lawler, Schramm, Werner '01) Let

$$A_t := \partial\mathbb{D} \setminus \overline{K}_t,$$

which is either an arc on  $\partial\mathbb{D}$  or  $A_t = \emptyset$ . Let  $s \geq 0$ , and set

$$p = p(s, \kappa) := \frac{s}{2} + \frac{1}{16} \left( \kappa - 4 + \sqrt{(4 - \kappa)^2 + 16\kappa s} \right).$$

Let  $\mathcal{H}(\theta, t)$  denote the event  $\{w = \exp(i\theta) \in A_t\}$ , and set

$$\mathcal{F}(\theta, t) := \mathbb{E} \left[ \left| g'_t(\exp(i\theta)) \right|^s 1_{\mathcal{H}(\theta, t)} \right],$$

$$q = q(s, \kappa) := \mathcal{U}_\kappa^{-1}(s) = \frac{\kappa - 4 + \sqrt{(4 - \kappa)^2 + 16\kappa s}}{2\kappa},$$

$$\mathcal{F}(\theta, t) \asymp \exp(-pt) (\sin(\theta/2))^q, \quad \forall t \geq 1, \quad \forall \theta \in (0, 2\pi).$$

# Harmonic measure

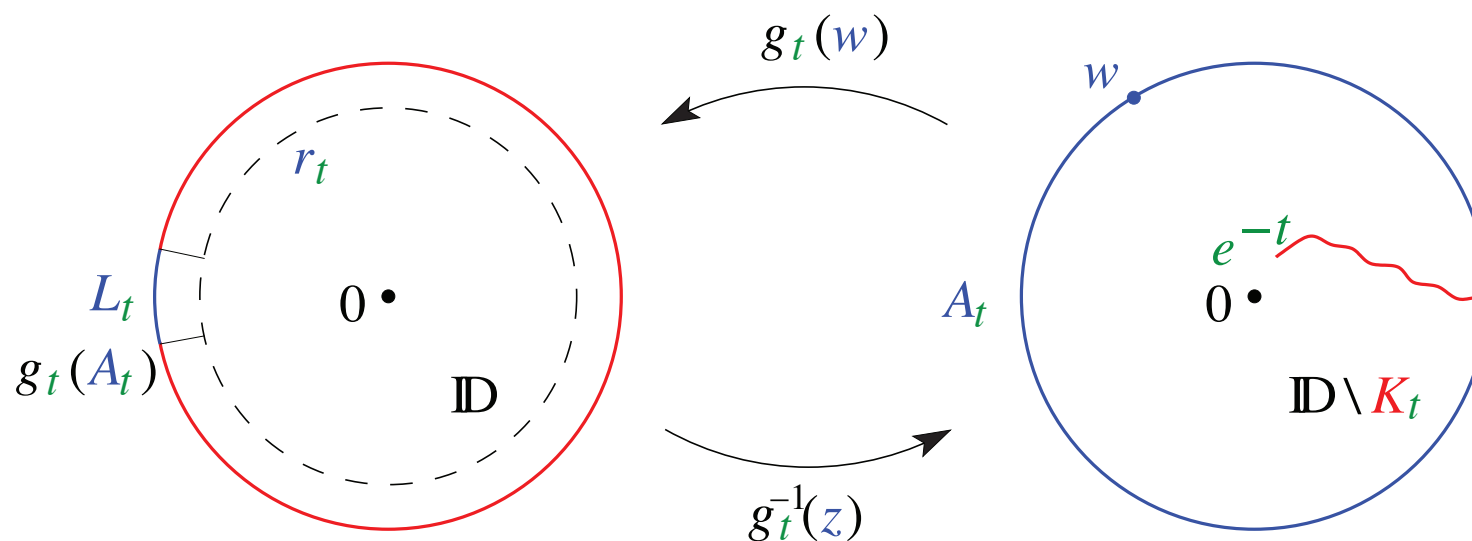


Figure 3:  $f_0(z) \stackrel{(\text{law})}{=} \lim_{t \rightarrow +\infty} e^t g_t^{-1}(z)$ , where  $z \mapsto g_t^{-1}(z)$  maps  $\mathbb{D}$  to the slit domain  $\mathbb{D} \setminus K_t$  ( $K_t$  SLE hull). The length  $L_t := |g_t(A_t)|$  of the image of the boundary set  $A_t := \partial\mathbb{D} \setminus \overline{K_t}$  is the  $(2\pi) \times$  the **harmonic measure** of  $A_t$  as seen from  $0$  in  $\mathbb{D} \setminus K_t$ , with  $\mathbb{E}[L_t^s] \asymp e^{-p(s, \kappa)t}$  for  $t \rightarrow +\infty$  [LSW '01].

## Packing spectrum & derivative exponents

The average integral means spectrum involves evaluating, for the whole-plane SLE map  $f_0(z)$ , the integral

$$\mathbb{I}_p(r) := \int_{\partial D} \mathbb{E} [ |f_0'(rz)|^p ] |dz|,$$

on a circle of radius  $r < 1$  concentric to  $\partial\mathbb{D}$ , and looking for the smallest  $\beta(p)$  such that

$$(1-r)^{\beta(p)} \mathbb{I}_p(r) \stackrel{r \rightarrow 1}{<} +\infty.$$

*For  $p \geq p^*(\kappa)$ , the integrand behaves like a distribution and the circle integral concentrates in the vicinity of the pre-image point of infinity by the whole-plane map,  $z_0 := f_0^{-1}(\infty) \in \partial\mathbb{D}$ . In the large- $t$  approximation to  $f_0$ , this is the neighborhood of  $g_t(A_t)$ .*



## Condensation

The circle integral there is the *restricted* integral in the image  $w$ -unit circle

$$I_p(t) := \int_{A_t} e^{pt} |g'_t(w)|^s |dw|; \quad s = s(p) = \beta(p) + 1 - p,$$

From **LSW**'s Lemma above

$$\mathbb{E}[I_p(t)] \asymp \int_0^{2\pi} \sin^q(\theta/2) d\theta < +\infty.$$

By defining the *stochastic radius*  $r_t := 1 - L_t \rightarrow 0$ , this can be recast as

$$\mathbb{E} \left[ (1 - r_t)^{\beta(p)} \int_{\partial\mathbb{D}} |\tilde{f}'_t(r_t z)|^p |dz| \right] \asymp 1, \quad t \rightarrow +\infty,$$

where  $f_0(z) \stackrel{(\text{law})}{=} \lim_{t \rightarrow +\infty} [\tilde{f}_t(z) := e^t g_t^{-1}(z)]$ . This is (*formally*) reminiscent of the definition of the average integral means spectrum, hinting at why *the derivative exponent*  $p = p(s, \kappa)$  *is the inverse function of the unbounded whole-plane packing spectrum*  $s(p, \kappa)$ . □

## Complex exponents (i.e., windings)

The *packing spectrum* [Makarov] is defined as

$$s(p) := \beta(p) - p + 1.$$

For the unbounded whole-plane  $SLE_{\kappa}$ , we have for  $p \geq p^*(\kappa)$

$$\begin{aligned} s(p, \kappa) &= \beta(p, \kappa) - p + 1 \\ &= 2p + \frac{1}{2} - \frac{1}{2} \sqrt{1 + 2\kappa p}. \end{aligned}$$

For complex exponent  $p + i\tilde{p}$ , there exists the *scaling law*

$$\begin{aligned} s(p + i\tilde{p}, \kappa) &= s(p', \kappa) = \beta(p', \kappa) - p' + 1 \\ &= 2p' + \frac{1}{2} - \frac{1}{2} \sqrt{1 + 2\kappa p'} \\ 1 + 2\kappa p' &:= \frac{1}{2} \left[ 1 + 2\kappa p + \sqrt{(1 + 2\kappa p)^2 + (4\kappa\tilde{p})^2} \right] \end{aligned}$$