

Large deviation bounds for the size of the largest critical percolation cluster in two dimensions

Demeter Kiss

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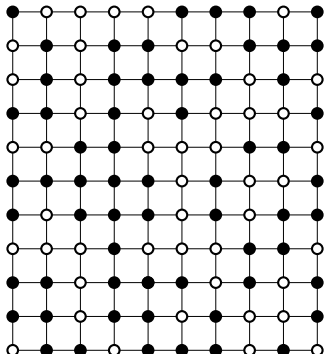
11 August 2014

Site percolation on the square lattice

Let $p \in [0, 1]$. Each vertex (site) is

- open with probability p (black),
- closed with probability $1 - p$ (white),

independently from each other.

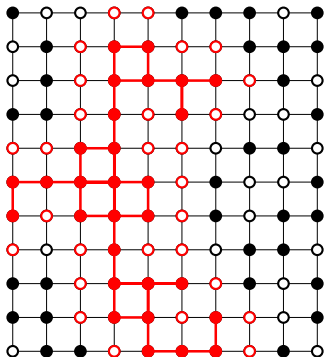


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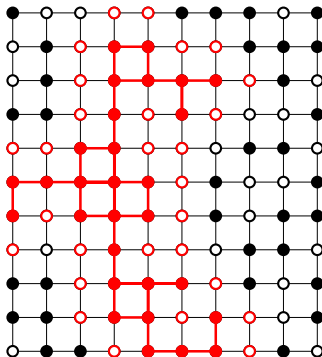
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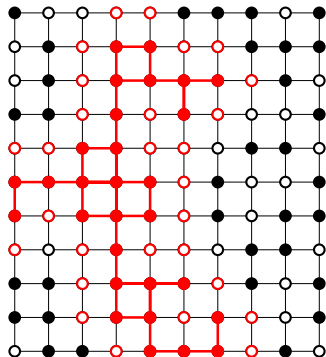
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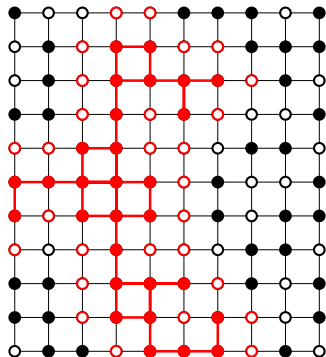
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Cluster = open connected component.

From now on, $p = p_c$.



Scale invariance and RSW

Lemma (RSW)

$$\mathbb{P}_{p_c} \left(\begin{array}{c} \updownarrow n \\ \boxed{\text{red path}} \\ \leftarrow 3n \rightarrow \end{array} \right) \geq c > 0.$$

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Consequences:

$$\pi_1(a, b) := \mathbb{P}_{p_c} \left(\begin{array}{c} \boxed{\text{red path}} \\ \updownarrow b \\ \updownarrow a \end{array} \right) \begin{cases} \geq \left(\frac{a}{b}\right)^\alpha, \\ \leq \left(\frac{a}{b}\right)^{\alpha'}. \end{cases}$$

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Quasi-multiplicativity

$$\pi_1(a_1, a_2)\pi_1(a_2, a_3) \asymp \pi_1(a_1, a_3).$$

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We set $\pi_1(n) := \pi_1(1, n)$.

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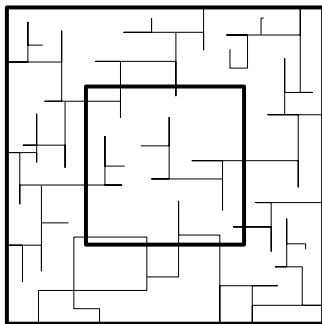
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On the triangular lattice

$$\mathbb{P}_{p_c}(\#\mathcal{C}_n^{(1)} \geq xn^2\pi_1(n)) \begin{cases} \leq Ce^{-cx^{96/5}} \\ \geq ce^{-Cx^{96/5}}. \end{cases}$$

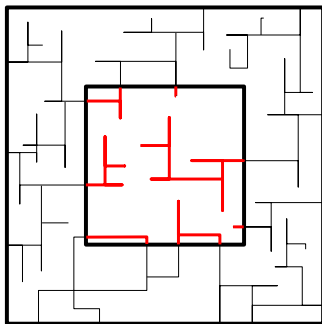
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Let $\mathcal{V}_n = \{v \in \Lambda_n \mid v \leftrightarrow \partial\Lambda_{2n}\}$.



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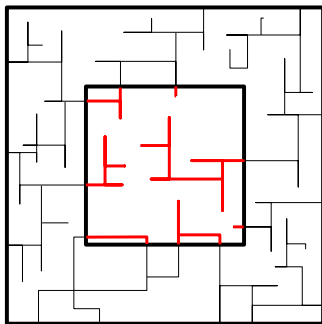
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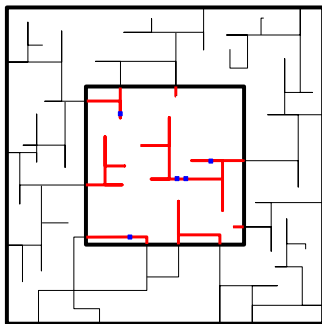


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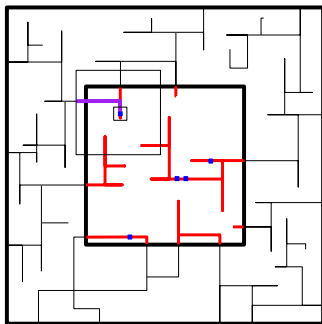


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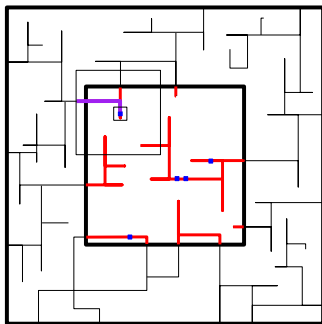
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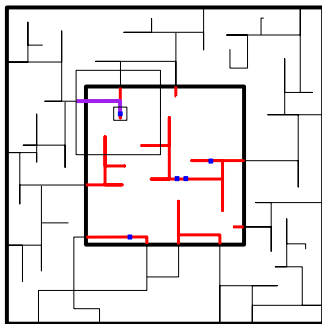
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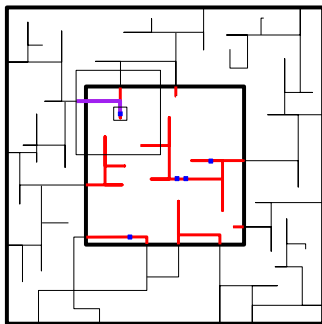
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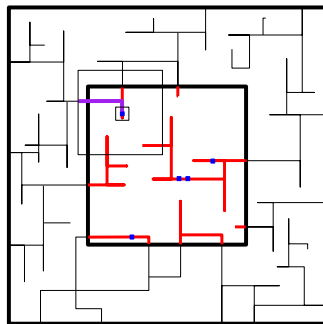
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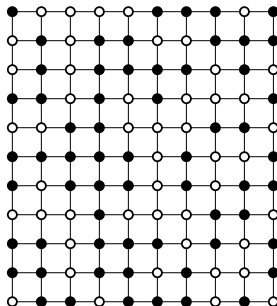
$$\leq c^k \left(\frac{n^2}{k} \pi_1(n/\sqrt{k}) \right)^k . \square$$

What is self-destructive percolation?

Let $p, \delta \in [0, 1]$.

Two percolation configurations:

- ω - intensity p (measure \mathbb{P}_p).
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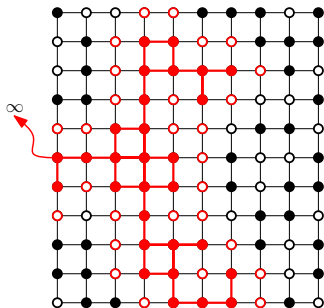


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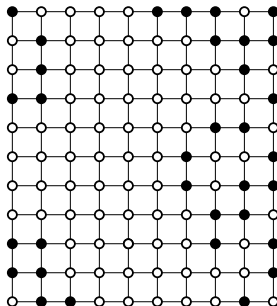
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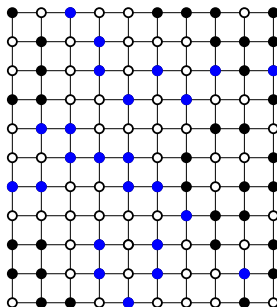
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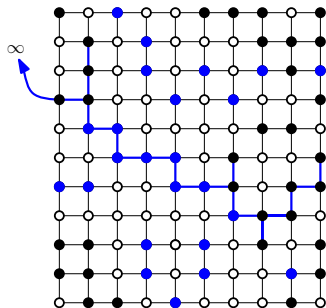
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$\delta_c(p) = \sup\{\delta : \mathbb{P}_{p,\delta}(0 \overset{\bar{\omega}^\delta}{\longleftrightarrow} \infty) = 0\}$.

Question: $\delta_c(p) \rightarrow 0$ as $p \searrow p_c$?



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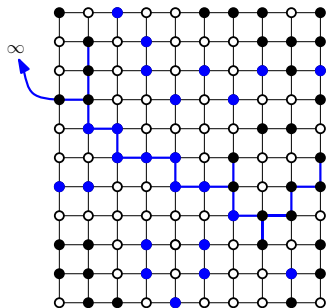
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Theorem (K., Manolescu, Sidoravicius '13)

There exists $\delta > 0$ such that, for all $p > p_c$,

$$\delta_c(p) > \delta.$$



Thank you!

Proof of the lower bound

$$\left. \begin{aligned} \mathbb{P}_p(\#\mathcal{V}_n \geq n^2\pi_1(n/u)) &\leq Ce^{-cu^2} \\ \mathbb{E}_{p_c}\#\mathcal{V}_n &\geq cn^2\pi_1(n) \end{aligned} \right\}$$

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The diagram shows a rectangle with a wavy line inside. A vertical double-headed arrow on the left side of the rectangle is labeled 'n', indicating its height. A horizontal double-headed arrow at the bottom of the rectangle is labeled '2n', indicating its width. The wavy line starts at the bottom-left corner and ends at the bottom-right corner, staying within the top and bottom boundaries of the rectangle.

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For all increasing events A, B

$$\mathbb{P}_{p_c}(A \cap B) \geq \mathbb{P}_p(A)\mathbb{P}_p(B)$$

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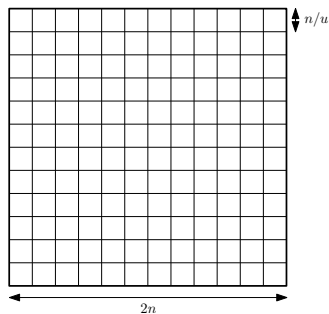
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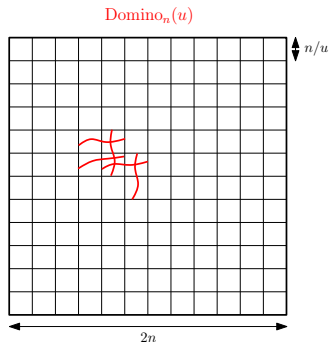
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$$\mathbb{P}_{p_c}(A \cap B) \geq \mathbb{P}_p(A)\mathbb{P}_p(B)$$



Proof of the lower bound

$$(1) \mathbb{P}_p(\#\mathcal{V}_n \geq cn^2\pi_1(n)) \geq c' > 0$$

Lemma (RSW)

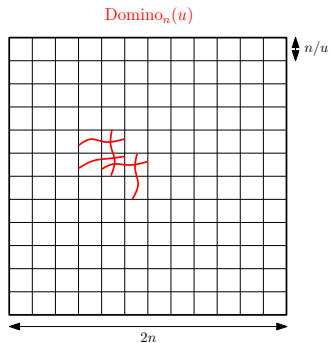
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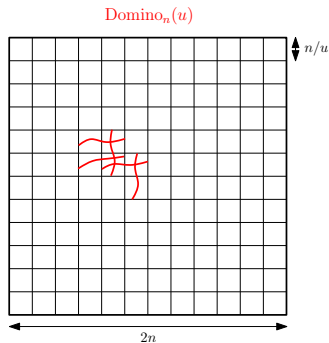
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$$(1) \Rightarrow \mathbb{P}_{p_c}(\mathcal{A}_n(u) \geq cu^2(n/u)^2\pi_1(n/u)) \geq c'u^2 > 0.$$

