Large deviation bounds for the size of the largest critical percolation cluster in two dimensions

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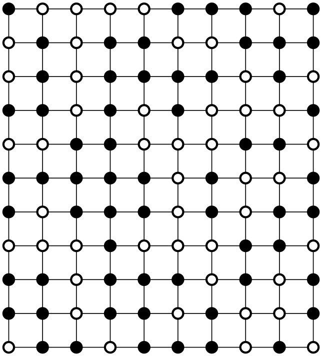
Site percolation on the square lattice

Let $p \in [0, 1]$. Each vertex (site) is

- open with probability p (black),
- closed with probability $1 - p$ (white),

independently from each other.



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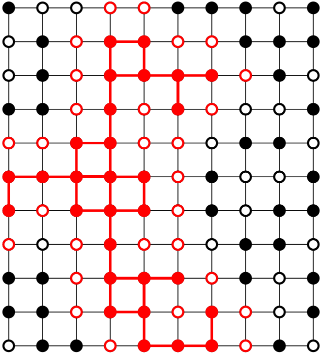

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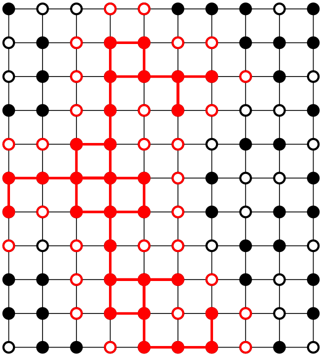

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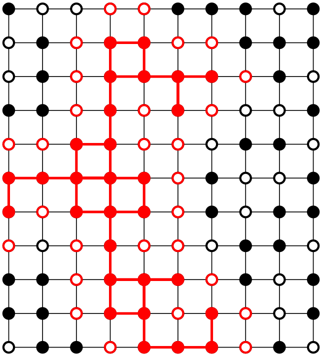

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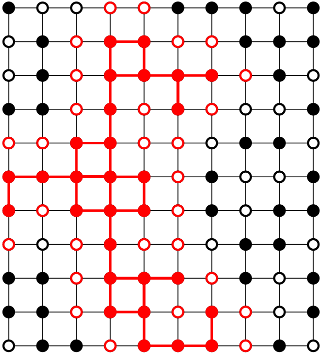

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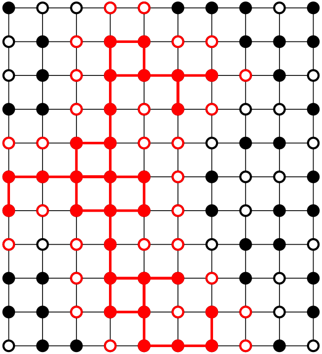

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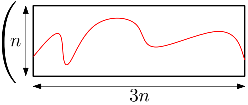
Lemma (RSW)

$\mathbb{P}_{p_c}\Big($ 

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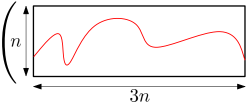
$\mathbb{P}_{p_c}\Bigg($ 

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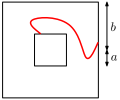
$\pi_1(a,b) := \mathbb{P}_{p_c}\Bigg($ 

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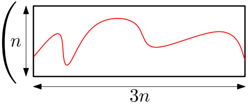
$$\mathbb{P}_{p_c}\Bigg($$  $$\Bigg) \geq c > 0.$$

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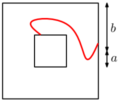
$$\pi_1(a,b) := \mathbb{P}_{p_c}\Bigg($$  $$\Bigg) \begin{cases} \geq \left(\frac{a}{b}\right)^{\alpha}, \\ \leq \left(\frac{a}{b}\right)^{\alpha'}. \end{cases}$$

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$$\pi_1(a_1,a_2)\pi_1(a_2,a_3) \asymp \pi_1(a_1,a_3).$$

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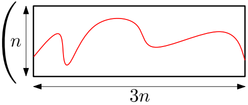
$$\mathbb{P}_{p_c}\Bigg($$ 

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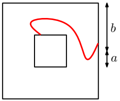
$$\pi_1(a,b) := \mathbb{P}_{p_c}\Bigg($$ 

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We set $\pi_1(n) := \pi_1(1,n)$.

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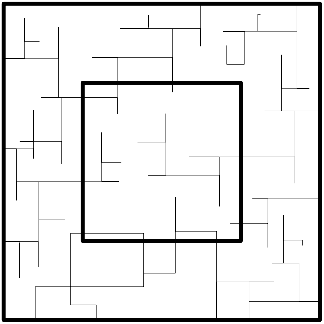
Let $\mathcal{V}_n = \{v \in \Lambda_n \,|\, v \leftrightarrow \partial\Lambda_{2n}\}$.



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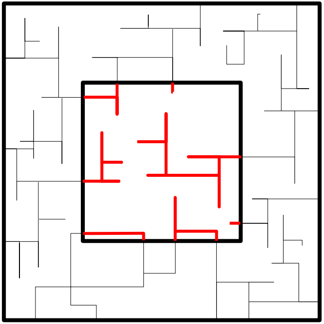

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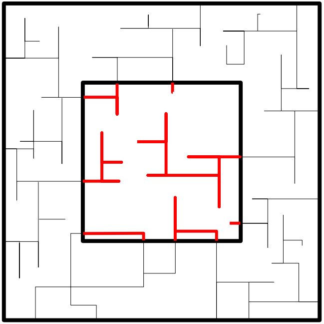

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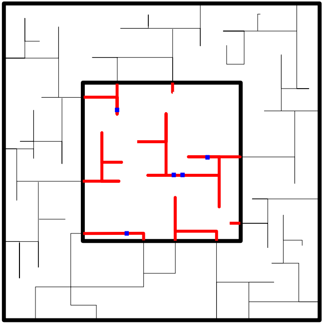

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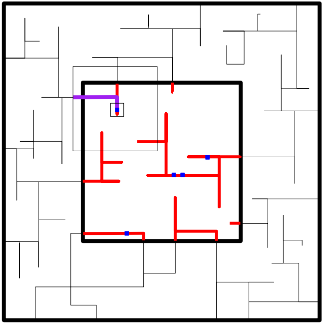

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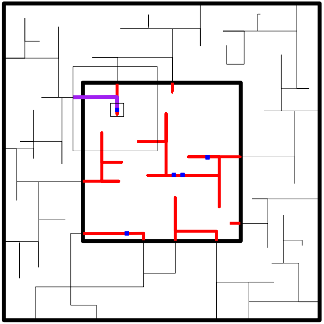

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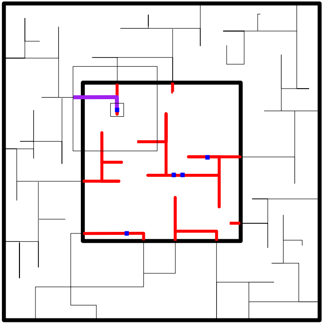

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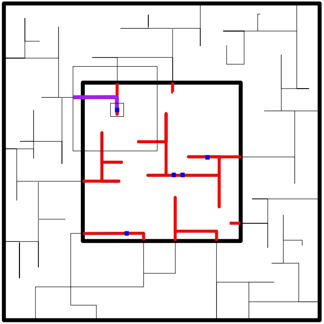

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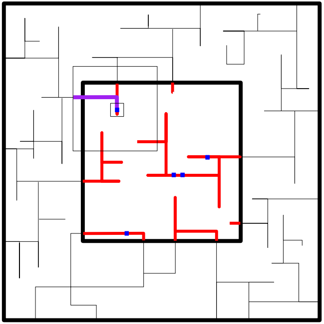

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$$\leq c^k \left(\frac{n^2}{k}\pi_1(n/\sqrt{k})\right)^k. \;\square$$

What is self-destructive percolation?

Let $p, \delta \in [0,1]$.
Two percolation configurations:

- ω - intensity p (measure $\mathbb{P}_p$).
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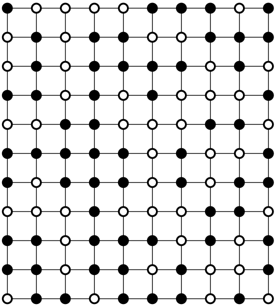

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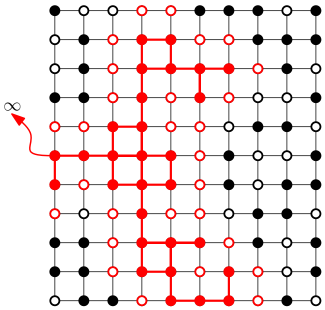

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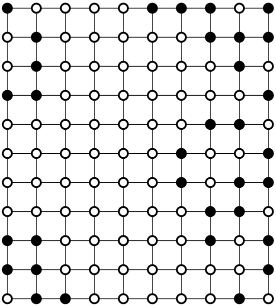

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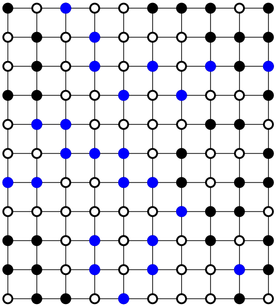

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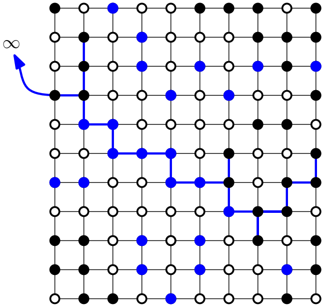
$\delta_c(p) = \sup\{\delta : \mathbb{P}_{p,\delta}(0 \overset{\overline{\omega}^{\delta}}{\longleftrightarrow} \infty) = 0\}$.



Question: $\delta_c(p) \to 0$ as $p \searrow p_c$?

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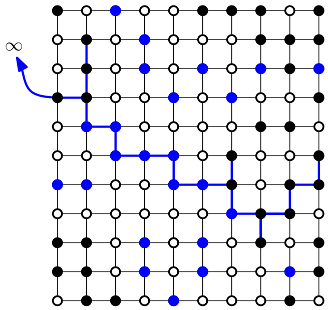

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$\delta_c(p) = \sup\{\delta : \mathbb{P}_{p,\delta}(0 \stackrel{\overline{\omega}^\delta}{\longleftrightarrow} \infty) = 0\}$.

Theorem (K., Manolescu, Sidoravicius '13)

There exists $\delta > 0$ such that, for all $p > p_c$,

$$\delta_c(p) > \delta.$$

Thank you!

Proof of the lower bound

$$\left.\begin{aligned} \mathbb{P}_p(\#\mathcal{V}_n \geq n^2\pi_1(n/u)) \leq Ce^{-cu^2} \\ \mathbb{E}_{p_c}\#\mathcal{V}_n \geq cn^2\pi_1(n) \end{aligned}\right\}$$

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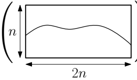
$\mathbb{P}_{p_c}\Big($  $\Big) \geq c > 0.$

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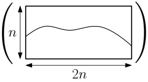
$\mathbb{P}_{p_c}$ 

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$$\mathbb{P}_{p_c}(A \cap B) \geq \mathbb{P}_p(A)\mathbb{P}_p(B)$$

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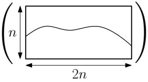
$\mathbb{P}_{p_c}\Bigg($ 

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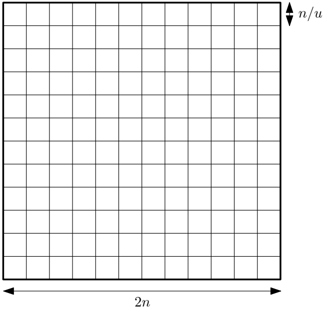

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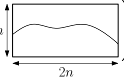
$\mathbb{P}_{p_c}\Bigg($  $\Bigg) \geq c > 0.$

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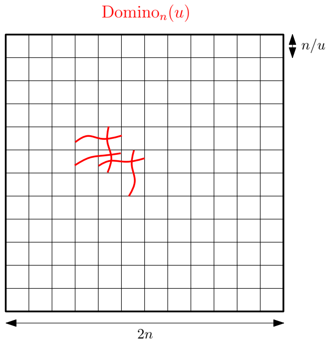

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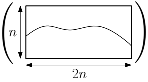
$\mathbb{P}_{p_c}$  $\geq c > 0.$

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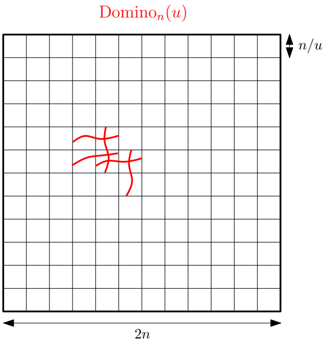

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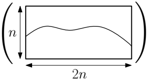
$\mathbb{P}_{p_c}$ 

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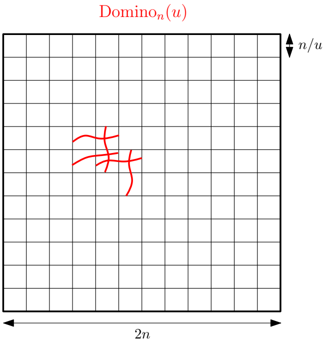

On $Domino_n(u)$, $\mathcal{C}_n^{(1)} \geq \#\{v \in \Lambda_{\frac{u-1}{u}n} \,|\, v \leftrightarrow \partial\Lambda_{2\frac{n}{u}}(v)\} := \mathcal{A}_n(u).$

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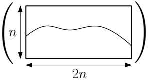
$\mathbb{P}_{p_c}$ 

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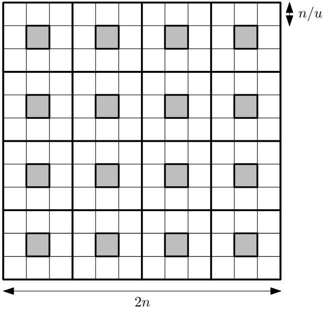

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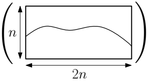
$\mathbb{P}_{p_c}$  $\geq c > 0.$

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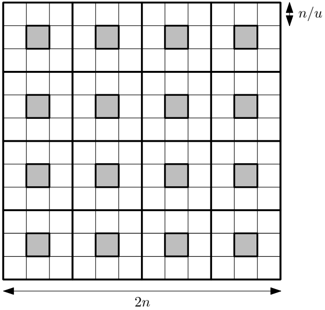

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