Hidden quantum group symmetry in random conformal geometry

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Recent Progress in Random Conformal Geometry
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Joint work with:

- Niko Jokela (Univ. Santiago de Compostela) and Matti Järvinen (Univ. Crete) [arXiv:1311.2297]
- Eveliina Peltola (Univ. Helsinki) [arXiv:1408.1384]
- Konstantin Izyurov (Univ. Helsinki) (in progress)

Quantum group construction of boundary correlation fns:

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SLE question(s):

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$$M_{d_2} \otimes M_{d_1} \;\cong\; M_{d_1+d_2-1} \oplus M_{d_1+d_2-3} \oplus \cdots \oplus M_{|d_1-d_2|+1}$$

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Informally, $\mathcal{F}^{(x_0)}[v](\mathbf{x}) =$ "$\int_{\Gamma[v]} f(\mathbf{x};\mathbf{w})\mathrm{d}\mathbf{w}$",
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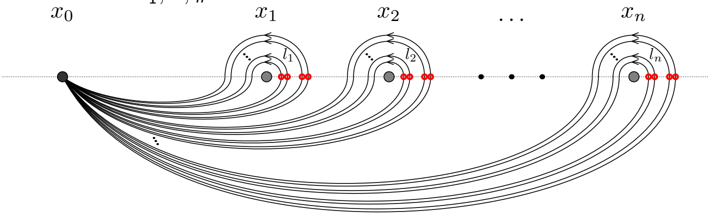

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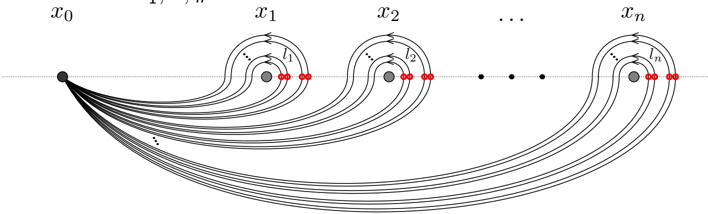
$\varphi^{(x_0)}_{l_1,\ldots,l_n}(\mathbf{x}) =$ 

$$f \propto \prod (x_j - x_i)^{\frac{2}{\kappa}(d_i-1)(d_j-1)} \times \prod (w_s - w_r)^{\frac{8}{\kappa}} \times \prod (w_r - x_i)^{-\frac{4}{\kappa}(d_i-1)}$$

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(ASY) $M_{d_{j+1}} \otimes M_{d_j} \cong \bigoplus_d M_d$ induces a decomposition of $\bigotimes_{j=1}^{n} M_{d_j}$. If $v \in \big(\bigotimes_{i>j+1} M_{d_i}\big) \otimes M_d \otimes \big(\bigotimes_{i<j} M_{d_i}\big)$, then

$$\mathcal{F}^{(x_0)}_{\ldots,d_j,d_{j+1},\ldots}[v] \sim (x_{j+1} - x_j)^{\Delta_d} \times \mathcal{F}^{(x_0)}_{\ldots,d,\ldots}[v].$$

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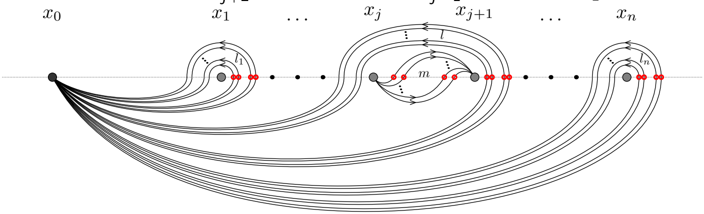

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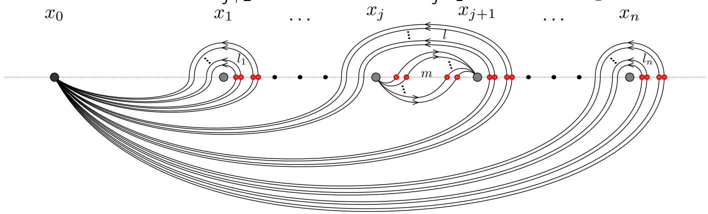

dominated convergence:

$$\frac{\mathcal{F}^{(x_0)}_{\ldots,d_j,d_{j+1},\ldots}[v](\ldots)}{(x_{j+1} - x_j)^{\Delta_d^{d_j,d_j+1}}} \underset{x_j,x_{j+1}\to\xi}{\longrightarrow} \mathcal{F}^{(x_0)}_{\ldots,d,\ldots}[v](\ldots,\xi,\ldots)$$

where $\Delta_d^{d_j,d_{j+1}} = \dfrac{2(1+d^2-d_j^2-d_{j+1}^2)+\kappa(d_j+d_{j+1}-d-1)}{2\kappa}$

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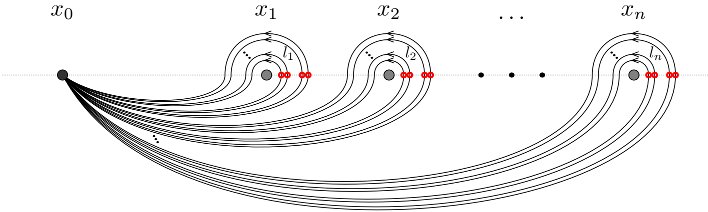

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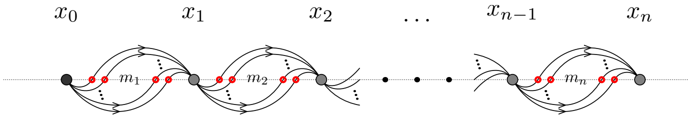

On the proof: anchor point independence

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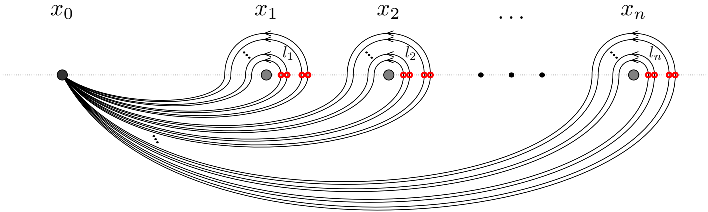

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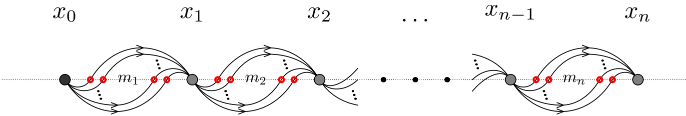

Highest weight vectors:

If $E.v = 0$, then in $\mathcal{F}^{(x_0)}[v](\mathbf{x})$, the coefficient of $\alpha^{(x_0)}_{m_1,\ldots,m_n}(\mathbf{x})$ vanishes whenever $m_1 \neq 0$.

On the proof: anchor point independence

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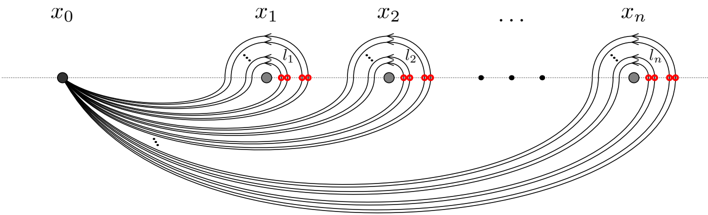

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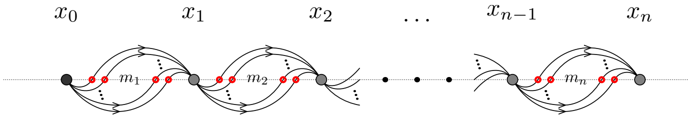

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$\rightsquigarrow$ $\mathcal{F}[v](\mathbf{x})$ well defined for $\mathbf{x} \in \mathfrak{X}_n$

On the proof: Stokes formula and highest weight vectors

- $\Game_{l_1,\ldots,l_n}$ the ℓ-dimensional integration surface of $\varphi^{(x_0)}_{l_1,\ldots,l_n}$
- $g(w_1; w_2, \ldots, w_\ell)$ single valued, symmetric in last $\ell - 1$ vars

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Stokes formula / integration by parts:

$$\int_{\Supset_{l_1,\ldots,l_n}} \sum_{r=1}^{\ell} \frac{\partial}{\partial w_r} \Big(g(w_r; w_1, \ldots, \cancel{w_r}, \ldots, w_\ell)\, f(\mathbf{x}; \mathbf{w}) \Big)\, \mathrm{d}w_1 \cdots \mathrm{d}w_\ell$$

$$= \sum_{j=1}^{n} \Big\{ (q^{-1} - q)\, [l_j]\, [d_j - l_j]\, q^{\sum_{i<j}(d_i - 1 - 2l_i)}$$

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Highest weight vect.: $v = \sum C_{l_1,\ldots,l_n} (\mu_{l_n} \otimes \cdots \otimes \mu_{l_1})$ s.t. $E.v = 0$

$$\sum C_{l_1,\ldots,l_n} \int_{\Game_{l_1,\ldots,l_n}} \sum_{r=1}^{\ell} \frac{\partial}{\partial w_r} \Big(g(w_r; \ldots)\, f(\mathbf{x}; \mathbf{w}) \Big)\, \mathrm{d}\mathbf{w} = 0.$$

On the proof: partial differential equations

Benoit & Saint-Aubin differential operators:

$$\mathcal{D}^{(j)} = \sum_{k=1}^{d_j} \sum_{\substack{n_1,\ldots,n_k \geq 1 \\ n_1+\ldots+n_k=d_j}} \frac{(\kappa/4)^{d_j-k}\,(d_j-1)!^2}{\prod_{j=1}^{k-1}(\sum_{i=1}^{j} n_i)(\sum_{i=j+1}^{k} n_i)} \times \mathcal{L}^{(j)}_{-n_1} \cdots \mathcal{L}^{(j)}_{-n_k}$$

where $\mathcal{L}^{(j)}_p$ ($j = 1, \ldots, n$ and $p \in \mathbb{Z}$) are 1[st] order diff. operators
$\mathcal{L}^{(j)}_p = -\sum_{i \neq j}(x_i - x_j)^p \left((1+p)\,\frac{(d_i-1)(2(d_i+1)-\kappa)}{2\kappa} + (x_i - x_j)\frac{\partial}{\partial x_i}\right)$

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The integrand $f(\mathbf{x};\mathbf{w})$ satisfies
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Highest weight vectors:
If $E.v = 0$, Stokes formula gives $\mathcal{D}^{(j)} \, \mathcal{F}[v](\mathbf{x}) = 0$.

On the proof: covariance under Möbius transformations

$$\varphi^{(x_0)}_{l_1,\ldots,l_n}(x_1,\ldots,x_n) \;=\; \int_{\Game_{l_1,\ldots,l_n}} f(x_1,\ldots,x_n;w_1,\ldots,w_\ell)\,\mathrm{d}w_1\cdots\mathrm{d}w_\ell$$

Möbius covariance: if $\nu(x_1) < \cdots < \nu(x_n)$ for $\nu(z) = \frac{az+b}{cz+d}$, want

$$\mathcal{F}[v]\big(\nu(x_1),\ldots,\nu(x_n)\big)\times\prod_{j=1}^{n}\nu'(x_j)^{\frac{(d_j-1)(2(d_j+1)-\kappa)}{2\kappa}} \;=\; \mathcal{F}[v](x_1,\ldots,x_n)$$

- translation invariance, $z \mapsto z+\xi$:
  $$\varphi^{(x_0+\xi)}_{l_1,\ldots,l_n}(x_1+\xi,\ldots,x_n+\xi) \;=\; \varphi^{(x_0)}_{l_1,\ldots,l_n}(x_1,\ldots,x_n)$$
  - make changes of variables $w'_r = w_r + \xi$
- homogeneity, $z \mapsto \lambda z$:
  $$\varphi^{(\lambda x_0)}_{l_1,\ldots,l_n}(\lambda x_1,\ldots,\lambda x_n) \;=\; \lambda^{\Delta}\,\varphi^{(x_0)}_{l_1,\ldots,l_n}(x_1,\ldots,x_n)$$
  - make changes of variables $w'_r = \lambda w_r$
- special conformal transformations, $z \mapsto \frac{z}{1+az}$:
  - vary a infinitesimally
  - use a property of the integrand f
  - apply Stokes formula

Summary of "spin chain - Coulomb gas correspondence"

Theorem (K. & Peltola)

$$\mathcal{F}^{(x_0)}_{d_1,\ldots,d_n} \colon \bigotimes_{j=1}^{n} M_{d_j} \longrightarrow \{\text{functions on } \mathfrak{X}_n^{(x_0)}\}$$

($\mathfrak{X}_n$) If $E.v = 0$, then $\mathcal{F}[v] \colon \mathfrak{X}_n \to \mathbb{C}$ is well-defined.

(PDE) If $E.v = 0$, then $\mathcal{D}^{(j)}\mathcal{F}[v] = 0$ for $j = 1, \ldots, n$.

(COV) $\mathcal{F}^{(\nu(x_0))}[v](\nu(\mathbf{x})) \times \prod_j \nu'(x_j)^{h_{d_j}} = \mathcal{F}^{(x_0)}[v](\mathbf{x})$

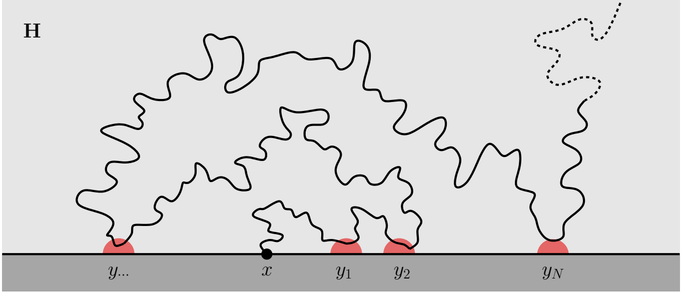
- for any translation ν
- for any affine ν, if $K.v = q^{d-1}v$
- for any Möbius transformation ν, if $K.v = v$ and $E.v = 0$

(ASY) If $v \in \left(\bigotimes_{i>j+1} M_{d_i}\right) \otimes M_d \otimes \left(\bigotimes_{i<j} M_{d_i}\right)$, then

$$\frac{\mathcal{F}^{(x_0)}_{\ldots,d_j,d_{j+1},\ldots}[v](\ldots)}{(x_{j+1}-x_j)^{\Delta_d^{d_j,d_{j+1}}}} \underset{x_j,x_{j+1}\to\xi}{\longrightarrow} \mathcal{F}^{(x_0)}_{\ldots,d,\ldots}[v](\ldots,\xi,\ldots)$$

Multi-point boundary zig-zag amplitude for chordal SLE

$$\mathsf{P}_{\mathbb{H};x,\infty}\Big[\mathrm{SLE}_\kappa \text{ visits } B_\varepsilon(y_1)\text{, then } B_\varepsilon(y_2)\text{, then } \ldots \text{then } B_\varepsilon(y_N)\Big]$$
$$\sim \text{const.} \times \varepsilon^{N\frac{8-\kappa}{\kappa}} \times \zeta_N(x; y_1, y_2, \ldots, y_N)$$



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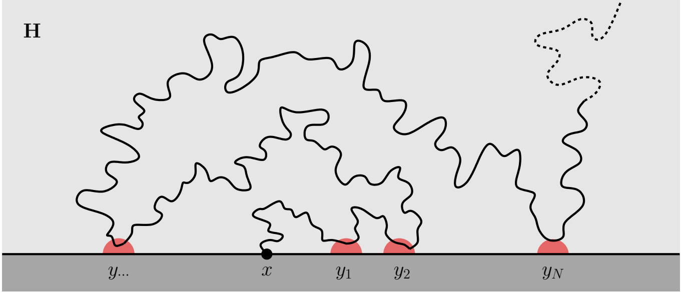

- relabel points $y_L^- < \cdots < y_2^- < y_1^- < x < y_1^+ < y_2^+ < \cdots y_R^+$

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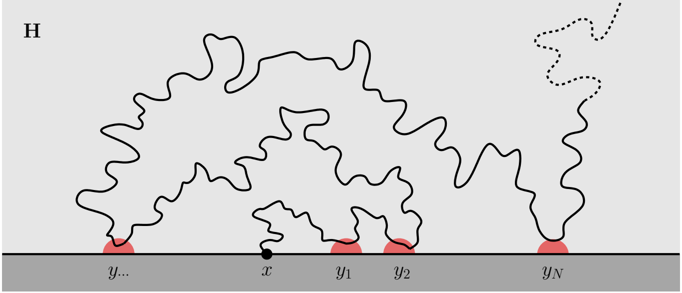

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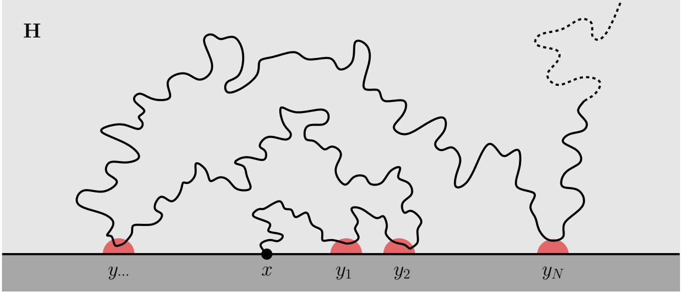

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The system for chordal SLE boundary zig-zag amplitude

(COV) ζ_ω is translation invariant

(COV) ζ_ω is homogeneous of degree $-N\frac{8-\kappa}{\kappa}$

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$\rightsquigarrow$ martingale $\prod g_t'(y_j^\pm)^{\frac{8-\kappa}{\kappa}} \times \zeta_\omega(g_t(y_L^-), \ldots, X_t, \ldots, g_t(y_R^+))$

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(ASY) As $y_1^\pm \to x$, asymptotics are

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(PDE) moreover $L+R$ third order linear homogeneous PDEs for ζ_ω

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Quantum group construction of SLE zig-zag amplitudes

- $\zeta_\omega(y_L^-, \ldots, x, \ldots, y_R^+)$ defined on $\mathfrak{X}_{L+R+1}$ will be $\zeta_\omega = \mathcal{F}[v_\omega]$, with judiciously chosen $v_\omega \in M_3^{\otimes R} \otimes M_2 \otimes M_3^{\otimes L}$

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Denote projection to M_2 by $\pi^{(2)}$, and on $M_3^{\otimes R} \otimes M_2 \otimes M_3^{\otimes L}$ define $\pi_+^{(2)} = \mathrm{id}_{M_3}^{\otimes (R-1)} \otimes \pi^{(2)} \otimes \mathrm{id}_{M_3}^{\otimes L}$.

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Decompose $M_3 \otimes M_2 \cong M_2 \oplus M_4$ and $M_2 \otimes M_3 \cong M_2 \oplus M_4$. Denote projection to M_2 by $\pi^{(2)}$, and on $M_3^{\otimes R} \otimes M_2 \otimes M_3^{\otimes L}$ define $\pi_+^{(2)} = \mathrm{id}_{M_3}^{\otimes(R-1)} \otimes \pi^{(2)} \otimes \mathrm{id}_{M_3}^{\otimes L}$ and $\pi_-^{(2)} = \mathrm{id}_{M_3}^{\otimes R} \otimes \pi^{(2)} \otimes \mathrm{id}_{M_3}^{\otimes(L-1)}$.

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Quantum group construction of SLE zig-zag amplitudes

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where $\hat\omega \in \{+,-\}^{L+R-1}$ is obtained by omitting one.

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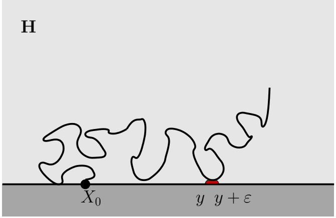
$(N=3)$ 8 explicit vectors v_ω in 54-dimensional space

$(N=4)$ 16 explicit vectors v_ω in 162-dimensional space

⋮ ⋮

Girsanov transform and the proof strategy



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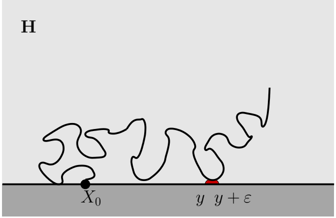

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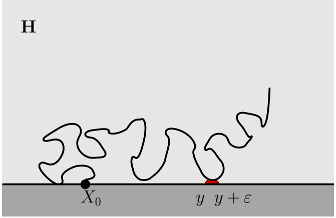

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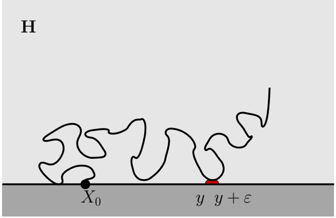

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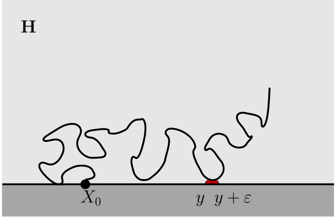

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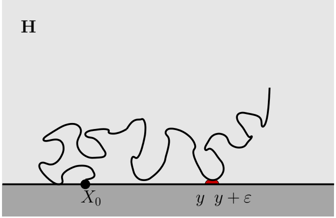

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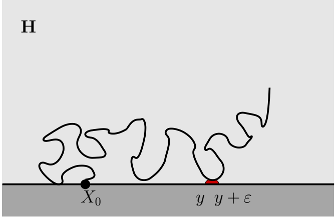

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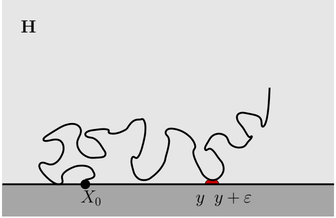

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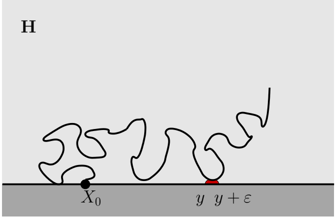

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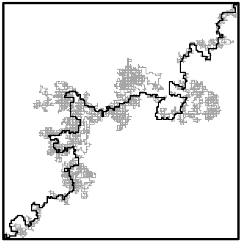
  - the solution ζ_N is used to build a martingale w.r.t. $\tilde{\mathsf{P}}$, whose end value is ζ_{N-1} — use optional stopping to conclude

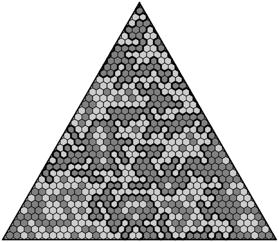
Boundary visits of interfaces in lattice models

LERW



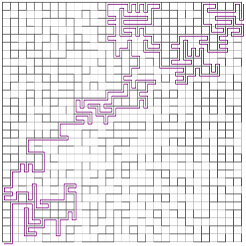
$\longrightarrow$ chordal $\mathrm{SLE}_{\kappa=2}$

Percolation



$\longrightarrow$ chordal $\mathrm{SLE}_{\kappa=6}$

Q-FK model



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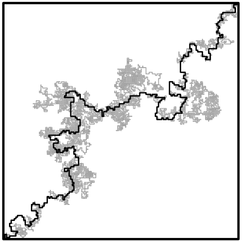
as lattice mesh $\delta \searrow 0$

Boundary visits of interfaces in lattice models

LERW



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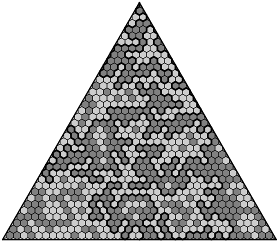

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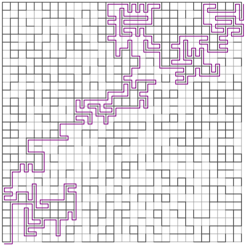

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Boundary visits of interfaces in lattice models

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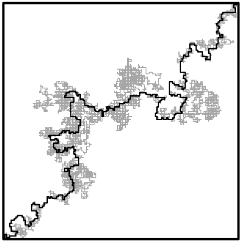

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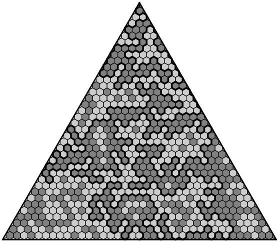

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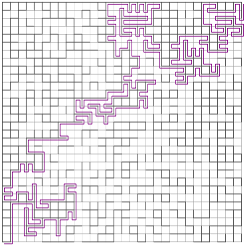

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Boundary visits of interfaces in lattice models

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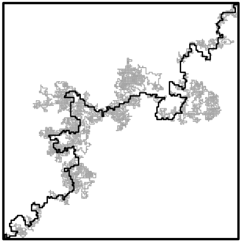

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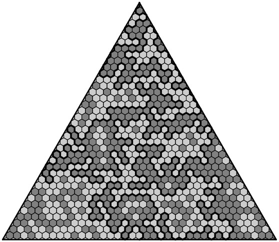

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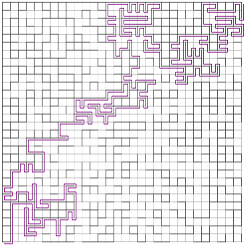

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as lattice mesh $\delta \searrow 0$

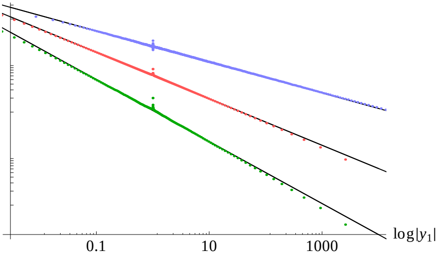
- sample configuration and find the curve (interface)
- collect frequencies of boundary visits from the samples
- $\mathsf{P}[\gamma \text{ visits } x_1, \ldots, x_N] \approx \mathrm{const.} \times \prod_j (\delta\, f'(x_j))^{\frac{8-\kappa}{\kappa}} \zeta_N(f(x_1), \ldots)$, where f = conformal map to $(\mathbb{H}; 0, \infty)$

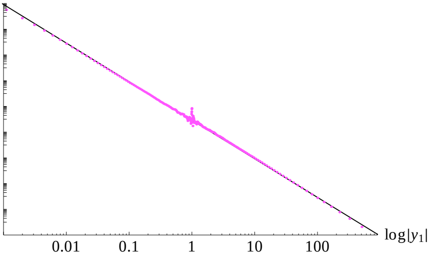
Lattice model simulation vs. evaluation of solution

$N = 1$, one-point visit frequencies, log-log-scale

$$\zeta_1(x; y_1) \;\propto\; |y_1 - x|^{\frac{\kappa-8}{\kappa}}$$

(set $x = 0$)





blue: percolation

red: $Q = 2$ FK model (exact [Smirnov])

green: $Q = 3$ FK model

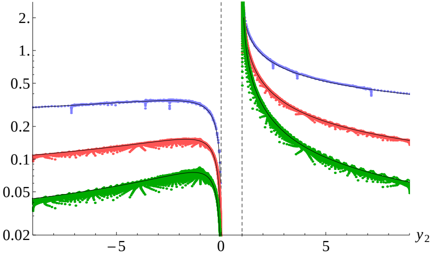
magenta: LERW

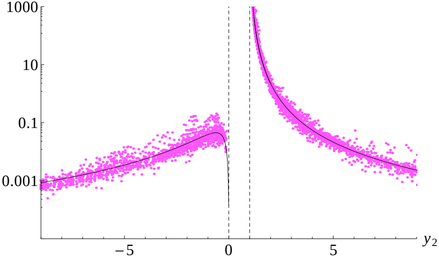
Lattice model simulation vs. evaluation of solution

$N = 2$, two-point visit frequencies, log-scale

the 4 pieces of $\zeta_2(x; y_1, y_2)$ are hypergeometric functions

(set $x = 0$, $y_1 = 1$)





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green: $Q = 3$ FK model

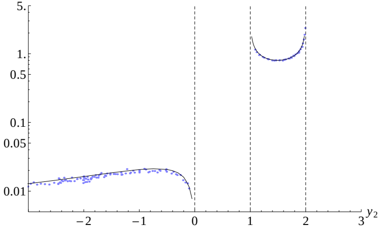
magenta: LERW

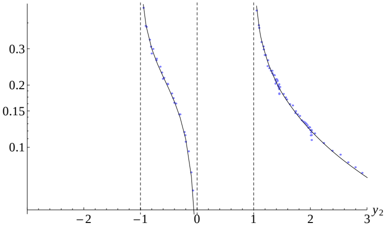
Lattice model simulation vs. evaluation of solution

$N = 3$, three-point visit frequencies, log-scale

solving for the 8 pieces of $\zeta_3(x; y_1, y_2, y_3)$ not reducible to ODE

percolation





(set $x = 0,\ y_1 = 1,\ y_3 = 2$)

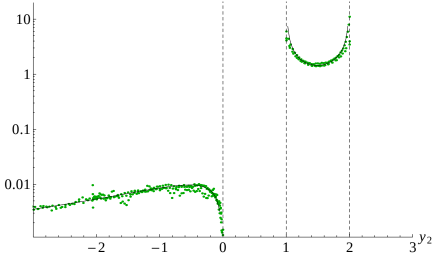
(set $x = 0,\ y_1 = 1,\ y_3 = -1$)

Lattice model simulation vs. evaluation of solution

$N = 3$, three-point visit frequencies, log-scale

solving for the 8 pieces of $\zeta_3(x; y_1, y_2, y_3)$ not reducible to ODE

$Q = 3$ FK model



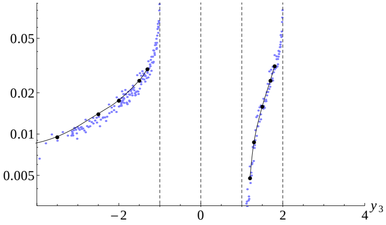
(set $x = 0$, $y_1 = 1$, $y_3 = 2$)

Lattice model simulation vs. evaluation of solution

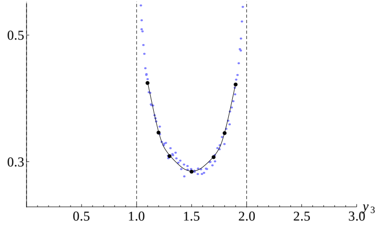
$N = 4$, four-point visit frequencies, log-scale

solving for the 16 pieces of $\zeta_4(x; y_1, y_2, y_3, y_4)$ not reducible to ODE

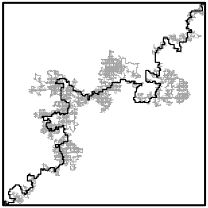
percolation

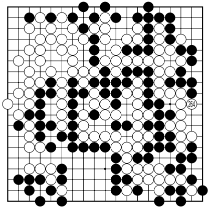


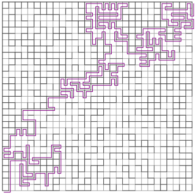
(set $x = 0$, $y_1 = 1$, $y_2 = -1$, $y_4 = 2$)



(set $x = 0$, $y_1 = -1$, $y_2 = 1$, $y_4 = 2$)







Thank you!