Percolation on uniform infinite planar maps

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based on a joint work with

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Motivation

Motivation: what is a “generic” planar geometry?

Definition: planar map =

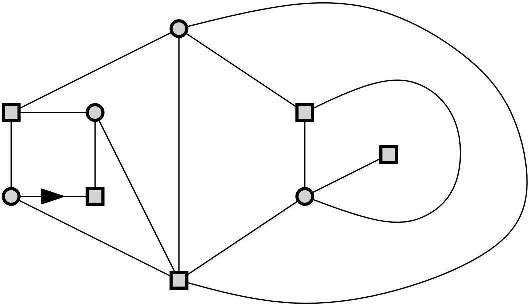
- finite connected planar graph,
- embedded on the 2-dimensional sphere,
- up to orientation-preserving deformations.

(embedding fixed $\rightsquigarrow$ some rigidity)

Goal: understand the geometry of large random planar maps.

Motivation

For instance, quadrangulation:



(rooted on a vertex + edge)

Motivation

Universality: the macroscopic object should not depend on the local combinatorics

($\leftrightarrow$ Simple Random Walk gives rise to Brownian Motion)

$\Rightarrow$ One can work with large triangulations, quadrangulations. . . or more sophisticated structures.

Motivation

Two main approaches:

(i) **scaling limit**: (pioneered by Chassaing - Schaeffer) view quadrangulation Q_n as a metric space (with the graph distance d_{gr}). Then

Theorem (Le Gall, Miermont)

We have the following convergence

$$(Q_n, n^{-1/4} d_{gr}) \xrightarrow{(d)} \mathrm{cst} \cdot (m_\infty, d^*)$$

for the Gromov-Hausdorff distance.

Motivation

(m_∞, d^*) is a random compact metric space (the "Brownian map"), which is

- a.s. homeomorphic to the 2-sphere (Le Gall - Paulin),
- a.s. of Hausdorff dimension 4 (Le Gall).

For this approach, quadrangulations work best: bipartite structure $\Rightarrow$ Cori - Vauquelin - Schaeffer bijection (can be generalized to $2p$-angulations: Bouttier - Di Francesco - Guitter)

Motivation

(ii) **local limit**: (Angel - Schramm) look at finite neighborhoods of the root

Theorem (Angel, Schramm)

For every $r \geq 0$, we have the following convergence:

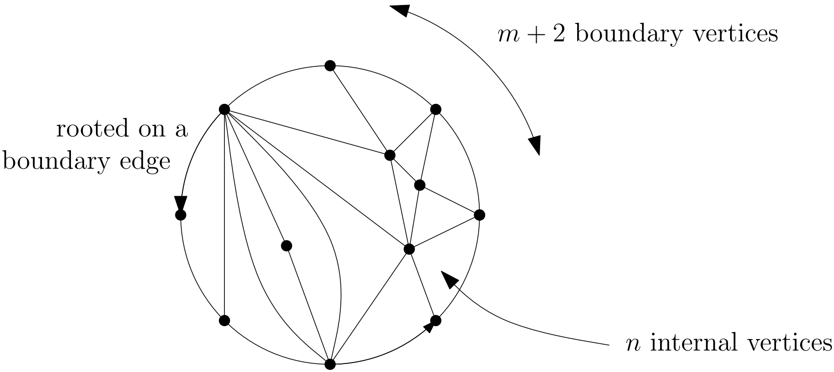
$$B_{T_n}(r) \xrightarrow[n\to\infty]{(d)} B_{T_\infty}(r)$$

where T_∞ is a random (rooted) infinite triangulation (called the Uniform Infinite Planar Triangulation, or UIPT).

Motivation

It works for any family of maps, as soon as one is able to derive explicit counting formulas. For triangulations:

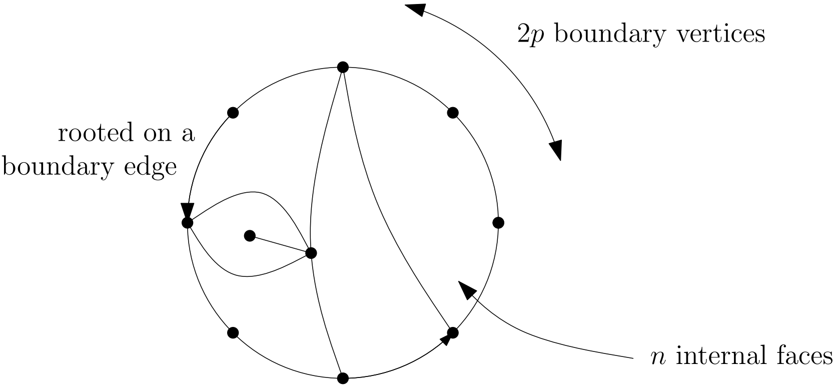
$$\phi_{n,m} = \frac{2^{n+1}(2m+1)!(2m+3n)!}{m!^2 n!(2m+2n+2)!}.$$



Counting quadrangulations

$a_{n,p}$ = number of quadrangulations of the $2p$-gon with n internal faces (rooted on the boundary face):

$$a_{n,p} = 3^n \frac{(2p)!}{p!(p-1)!} \frac{(2n+p-1)!}{n!(n+p+1)!}.$$



Counting quadrangulations

- asymptotic behavior:

$$a_{n,p} \underset{n\to\infty}{\sim} C_p 12^n n^{-5/2},$$

with $C_p = \frac{1}{2\sqrt{\pi}} \left(\frac{2}{3}\right)^p \frac{(3p)!}{p!(2p-1)!}$.

- corresponding generating function:

$$Z_p(t) := \sum_{n\geqslant 0} a_{n,p} t^n.$$

- Z_p has $1/12$ as a convergence radius, and

$$Z_p := Z_p(1/12) = 2\left(\frac{2}{3}\right)^p \frac{(3p-3)!}{p!(2p-1)!}.$$

Main properties

Main properties of the UIPQ (proved by Angel and Schramm for the UIPT – same proofs here):

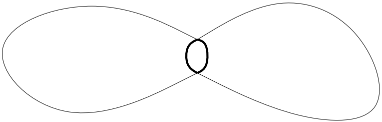
- **well-defined**: there exists

$$q_\infty = \lim_{N\to\infty} q_N$$

  in the local sense.

- degree distribution: **exponential tail**

- a.s. **one-ended**



Spatial Markov property for the UIPQ

Consider $\mathbf{q}$ a *rigid* quadrangulation with n internal faces and k boundary faces, with perimeters $2p_1, \ldots, 2p_k$.

(i) One has

$$\tau\left(\mathbf{q} \subset \mathbf{q}_\infty\right) = \frac{12^{-n}}{C_1}\left(\prod_{i=1}^{k} Z_{p_i}\right)\sum_{i=1}^{k}\frac{C_{p_i}}{Z_{p_i}}. \tag{1}$$

Spatial Markov property for the UIPQ

When $\mathbf{q} \subset \mathbf{q}_\infty$, denote $\mathbf{q}_i$ = component of the UIPQ in the ith face. Then,

(ii) A.s., only one of these components is infinite: it is $\mathbf{q}_j$ with probability

$$\tau\left(\mathbf{q} \subset \mathbf{q}_\infty, \mathbf{q}_j \text{ is infinite}\right) = \frac{12^{-n}}{C_1} C_{p_j} \left(\prod_{\substack{i=1 \\ i \neq j}}^{k} Z_{p_i} \right)$$

(jth term in the previous sum).

Spatial Markov property for the UIPQ

(iii) If we condition on $\{\mathbf{q} \subset \mathbf{q}_\infty\}$, and that the external faces of $\mathbf{q}$ all contain finitely many vertices of $\mathbf{q}_\infty$ except the jth one,

- the quadrangulations $(\mathbf{q}_i)_{1 \le i \le k}$ are independent,
- $\mathbf{q}_j$ has the same distribution as the UIPQ of the $2p_j$-gon,
- and for $i \neq j$, $\mathbf{q}_i$ is distributed as the free quadrangulation of a $2p_i$-gon.

free quadrangulation of a $2p$-gon = probability measure μ^p s.t.

$$\mu^p(\mathbf{q}) = \frac{12^{-n}}{Z_p(1/12)}$$

for each quadrangulation $\mathbf{q}$ of the $2p$-gon with n internal faces.

Peeling process for the UIPQ

peeling process (Angel) = sequence $(\mathbf{q}_n)_{n\geq 0}$ of (finite) random quadrangulations with simple boundary, such that:

- $\mathbf{q}_0$ is the root edge of $\mathbf{q}_\infty$,
- $\mathbf{q}_0 \subset \mathbf{q}_1 \subset \cdots \subset \mathbf{q}_n \subset \cdots \subset \mathbf{q}_\infty$,
- Conditionally on $\mathcal{F}_n$ (fitration generated by $\mathbf{q}_0, \mathbf{q}_1, \ldots, \mathbf{q}_n$), the part of $\mathbf{q}_\infty$ that has not been discovered yet, that is $\mathbf{q}_\infty \setminus \mathbf{q}_n$, is the UIPQ of the $|\partial \mathbf{q}_n|$-gon.

(one adds quadrangles "one by one")

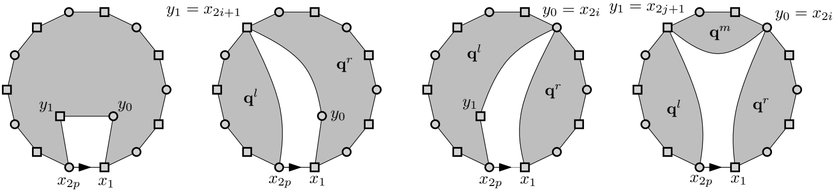
Peeling process for the UIPQ

Conditional distribution of $\mathbf{q}_{n+1}$ knowing $\mathcal{F}_n$?

- choose an oriented edge e on $\partial \mathbf{q}_n$ such that $\mathbf{q}_n$ lies on the right-hand side of e (**any** choice, deterministic or random, is acceptable as long as it depends only on $\mathcal{F}_n$),
- $\mathbf{q}_\infty \setminus \mathbf{q}_n$ rooted at e is a UIPQ of the $|\partial \mathbf{q}_n|$-gon,
- reveal the face of $\mathbf{q}_\infty \setminus \mathbf{q}_n$ containing e.

Peeling process for the UIPQ

Let $p = |\partial \mathbf{q}_n|/2$. Four cases may occur for the new face (x_{2p}, x_1, y_0, y_1), depending on whether y_0 and / or y_1 belong to $\partial \mathbf{q}_n$:



(note that y_1 can coincide with x_1, and y_0 can coincide with x_{2p})

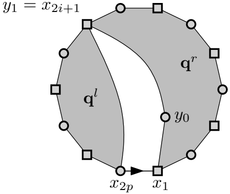
Peeling process for the UIPQ

Case 2: $y_0 \notin \partial \mathbf{q}_n$ and $y_1 = x_{2i+1} \rightsquigarrow$ two separate quadrangulations: $\mathbf{q}_n^r$ with perimeter $2(i+1)$ and $\mathbf{q}_n^l$ with perimeter $2(p-i)$ (**exactly one is infinite**)

If $\mathbf{q}_n^r$ infinite, it is a UIPQ of the $2(i+1)$-gon, and $\mathbf{q}_n^l$ is independent of $\mathbf{q}_n^r$ and is a free quadrangulation of the $2(p-i)$-gon $\rightsquigarrow$ set $\mathbf{q}_{n+1} = \mathbf{q}_n$ + face discovered + $\mathbf{q}_n^l$

This has conditional probability

$$\tau\left(y_0 \notin \partial \mathbf{q}_n, y_1 = x_{2i+1}, \mathbf{q}_n^r \text{ infinite } | \mathcal{F}_n\right) = \frac{Z_{p-i} C_{i+1}}{12 C_p}.$$



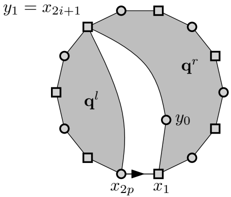
Peeling process for the UIPQ

Case 2: $y_0 \notin \partial \mathbf{q}_n$ and $y_1 = x_{2i+1}$

If $\mathbf{q}_n^l$ is infinite, set $\mathbf{q}_{n+1} = \mathbf{q}_n$ + face discovered + $\mathbf{q}_n^r$

The corresponding probability is

$$\tau\left(y_0 \notin \partial \mathbf{q}_n, y_1 = x_{2i+1}, \mathbf{q}_n^l \text{ infinite } \middle| \mathcal{F}_n\right) = \frac{C_{p-i} Z_{i+1}}{12 C_p}.$$



Peeling process for the UIPQ

If we write $|\partial \mathbf{q}_{n+1}| = |\partial \mathbf{q}_n| + 2X_n$:

$$P\left(X_n = 1 \middle| |\partial \mathbf{q}_n| = 2p\right) = \frac{C_{p+1}}{12C_p}$$

(corresponding to case (1)), and for every $k = 0, \ldots, p-1$,

$$P\left(X_n = -k \middle| |\partial \mathbf{q}_n| = 2p\right) = 4\frac{C_{p-k}Z_{k+1}}{12C_p} + 3\frac{C_{p-k}}{12C_p}\sum_{i=1}^{k} Z_i Z_{k+1-i}$$

(combining cases (2) and (3) for the first term, and (4) for the second term).

Peeling process for the UIPQ

Lemma (Angel, Benjamini - Curien)

If $\mathbf{q}_0, \mathbf{q}_1, \ldots, \mathbf{q}_n, \ldots$ is generated by a peeling procedure of the UIPQ, then one has

$$|\partial \mathbf{q}_n| \approx n^{2/3},$$
$$|\mathbf{q}_n| \approx n^{4/3}.$$

We use only $|\partial \mathbf{q}_n| \to \infty$ a.s., and prove

$$E\left[X_n | \mathcal{F}_n\right] \underset{n \to \infty}{\longrightarrow} 0.$$

Bernoulli percolation

The peeling process can be used to study bond percolation on the UIPQ:

Theorem (Ménard, N.)

For bond percolation on the UIPQ, one has $p_c^{bond} = 1/3$ almost surely.

Easier to apply it for the Uniform Infinite Planar Map (UIPM):

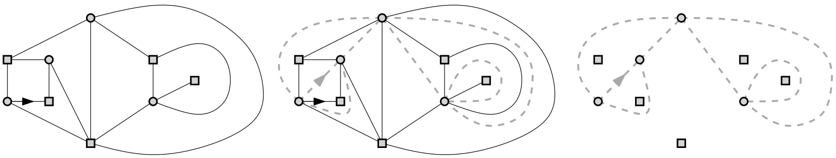
Theorem (Ménard, N.)

For site and bond percolation on the UIPM, one has, respectively, $p_c^{site} = 2/3$ and $p_c^{bond} = 1/2$ almost surely.

Uniform Infinite Planar Map

Uniform Infinite Planar Map (UIPM): $n \to \infty$ limit of uniform planar map with n edges (no constraint on degree).

Bijection with quadrangulations ($\Rightarrow$ UIPM can be obtained from UIPQ)



(circles = primal vertices / squares = dual vertices)

Site percolation on the UIPM

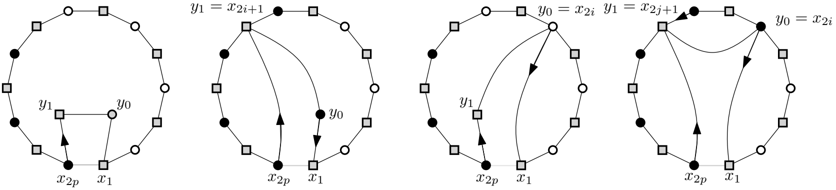
Now: Bernoulli site percolation on the UIPM: the vertices are colored, independently of each other, *black* with probability q, and *white* with probability $(1 - q)$.

$\rightsquigarrow$ exploration process (Angel for triangulations): at each step, choose the quadrangle revealed so that $\partial \mathbf{q}_n$ remains divided in two arcs: one arc of black sites and one arc of white sites.

$\Rightarrow$ all black vertices on $\partial \mathbf{q}_n$ belong to the percolation cluster containing the root vertex of $\mathbf{m}_\infty$, as long as the boundary does not become totally white (corresponds to detecting a white circuit).

Site percolation on the UIPM

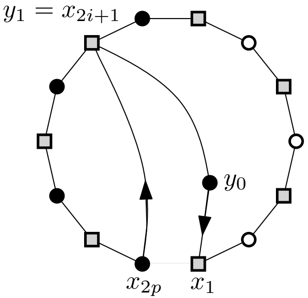
Denote B_n = number of black vertices on $\partial \mathbf{q}_n$, W_n = number of white vertices. Also, write $|\partial \mathbf{q}_n| = 2p$.



Site percolation on the UIPM

Case 2: $y_0 \notin \partial \mathbf{q}_n$ and $y_1 \in \partial \mathbf{q}_n$, $X_n = -k$:

- if $\mathbf{q}_n^l$ infinite, $B_{n+1} = \min(B_n, p-k)$.
- if $\mathbf{q}_n^r$ infinite, $B_{n+1} = \max(B_n - k - 1, 0) + 1$ with probability q and $B_{n+1} = \max(B_n - k - 1, 0)$ with probability $(1-q)$.



Site percolation on the UIPM

$\Rightarrow$ if $|\partial \mathbf{q}_n| = 2p$, when $X_n = -k$:

$$B_{n+1} = \begin{cases} \min(B_n, p-k) & \text{w. p. } 2\frac{C_{p-k}Z_{k+1}}{12C_p} + \frac{C_{p-k}}{12C_p}\sum_{i=1}^k Z_i Z_{k+1-i}, \\ \max(B_n - k, 0) & \text{w. p. } \frac{C_{p-k}Z_{k+1}}{12C_p} + \frac{C_{p-k}}{12C_p}\sum_{i=1}^k Z_i Z_{k+1-i}, \\ \max(B_n - k - 1, 0) + 1 & \text{w. p. } q\frac{C_{p-k}Z_{k+1}}{12C_p}, \\ \max(B_n - k - 1, 0) & \text{w. p. } (1-q)\frac{C_{p-k}Z_{k+1}}{12C_p}, \\ \max(B_n - i, 0) & \text{w. p. } \frac{C_{p-k}}{12C_p} Z_i Z_{k+1-i} \text{ for } 1 \leqslant i \leqslant k. \end{cases}$$

By analyzing carefully this chain, we prove: if $q < 2/3$, B_n comes back to 0 i.o., and $= O(\log n)$.

For $q > 2/3$, use W_n instead to prove that $B_n \to \infty$.

Site percolation on the UIPM

$\rightsquigarrow$ modified Markov chain (B'_n) obtained by "simplifying" (B_n):

$$B'_{n+1} = \begin{cases} B'_n + 1 & \text{with probability } q\frac{C_{p+1}}{12C_p}, \\ B'_n & \text{with probability } (1-q)\frac{C_{p+1}}{12C_p} \end{cases}$$

(corresponding to $X_n = 1$), and

$$B'_{n+1} = \begin{cases} B'_n & \text{w. p. } 2\frac{C_{p-k}Z_{k+1}}{12C_p} + \frac{C_{p-k}}{12C_p}\sum_{i=1}^{k} Z_i Z_{k+1-i}, \\ B'_n - k & \text{w. p. } (1+q)\frac{C_{p-k}Z_{k+1}}{12C_p} + \frac{C_{p-k}}{12C_p}\sum_{i=1}^{k} Z_i Z_{k+1-i}, \\ B'_n - k - 1 & \text{w. p. } (1-q)\frac{C_{p-k}Z_{k+1}}{12C_p}, \\ B'_n - i & \text{w. p. } \frac{C_{p-k}}{12C_p} Z_i Z_{k+1-i} \text{ for } 1 \leqslant i \leqslant k \end{cases}$$

for every $k = 0, \ldots, p-1$ (corresponding to $X_n = -k$).

Site percolation on the UIPM

We find:

$$
\begin{aligned}
& E\left[B'_{n+1} - B'_n \middle| |\partial \mathbf{q}_n| = 2p\right] \\
& = qP\left(X_n = 1 \middle| |\partial \mathbf{q}_n| = 2p\right) - \sum_{k=0}^{p-1} k\left(2\frac{C_{p-k}Z_{k+1}}{12C_p} + \frac{C_{p-k}}{12C_p}\sum_{i=1}^{k} Z_i Z_{k+1-i}\right) \\
& \qquad - (1-q)\sum_{k=0}^{p-1}\frac{C_{p-k}Z_{k+1}}{12C_p} - \sum_{k=1}^{p-1}\sum_{i=1}^{k} i\frac{C_{p-k}}{12C_p}Z_i Z_{k+1-i} \\
& = \left(q - \frac{1}{2}\right)P\left(X_n = 1 \middle| |\partial \mathbf{q}_n| = 2p\right) + \frac{1}{2}E\left[X_n \middle| |\partial \mathbf{q}_n| = 2p\right] \\
& \qquad - (1-q)\sum_{k=0}^{p-1}\frac{C_{p-k}Z_{k+1}}{12C_p} + \sum_{k=1}^{p-1}\sum_{i=1}^{k}\left(\frac{k}{2} - i\right)\frac{C_{p-k}}{12C_p}Z_i Z_{k+1-i} \\
& = \left(q - \frac{1}{2}\right)\frac{C_{p+1}}{12C_p} + \frac{1}{2}E\left[X_n \middle| |\partial \mathbf{q}_n| = 2p\right] \\
& \qquad - (1-q)\sum_{k=0}^{p-1}\frac{C_{p-k}Z_{k+1}}{12C_p} - \frac{1}{2}\sum_{k=1}^{p-1}\sum_{i=1}^{k}\frac{C_{p-k}}{12C_p}Z_i Z_{k+1-i}.
\end{aligned}
$$

Site percolation on the UIPM

Hence,

$$E\left[B'_{n+1} - B'_n \middle| \mathcal{F}_n\right] \longrightarrow \left(q - \frac{1}{2}\right)\frac{3}{8} - (1-q)\frac{1}{8} - \frac{1}{2}\frac{1}{24} = \frac{q}{2} - \frac{1}{3}$$

as $n \to \infty$.

In particular, negative and bounded away from 0 for $q < 2/3$.

Conclusion

- Explicit counting formulas $\rightsquigarrow$ derivation of percolation thresholds
- Recent (independent) work of Angel-Curien: p_c^{bond} and p_c^{face} for several random maps in the *half-plane* (and some critical exponents) – in particular uniform quadrangulations
- Curien-Kortchemski: on the UIPT, scaling limit of the boundary for large critical percolation clusters, and some critical exponents

End

Thank you!