Slit holomorphic stochastic flows

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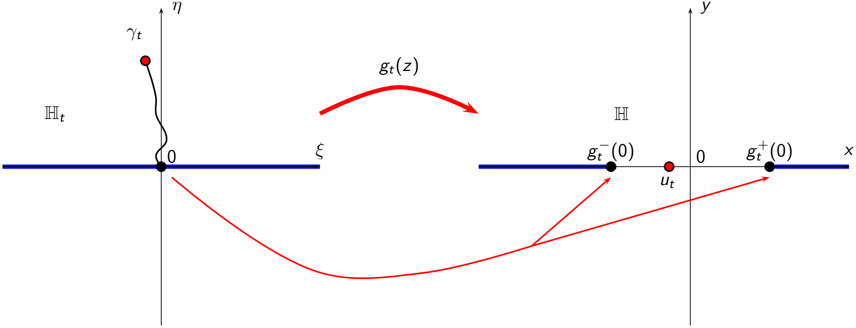
SLE

- Introduced by Oded Schramm in 1999 as a candidate for the scaling limit of loop erased random walk and the interfaces in critical percolation;
- It been shown to be the scaling limit of a number of models.

Half-plane Löwner equation

Löwner 1923 (radial version). From 1946 Kufarev and his students (chordal version)

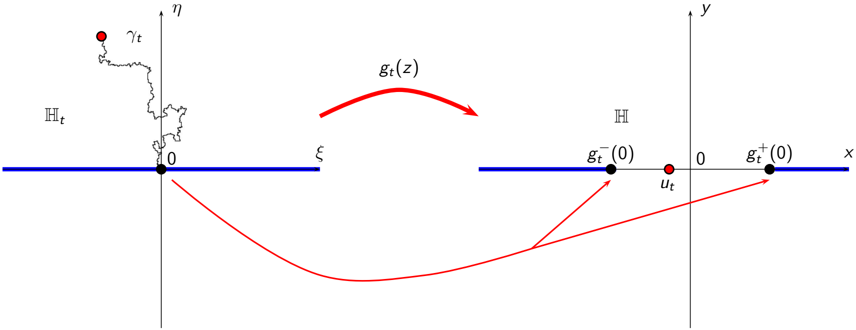
$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - u_t}, \quad g_0(z) = z, \quad g_t(z) = z + \frac{2t}{z} + O\left(\frac{1}{z^2}\right),$$



SLE

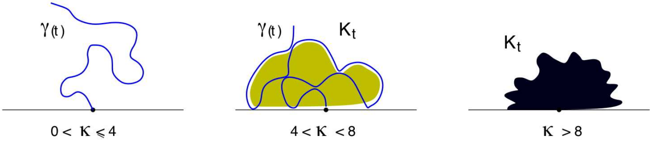
The driving term $u_t = \sqrt{k} B_t$, 1-dimensional standard Brownian motion:

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - u_t}, \quad g_0(z) = z, \quad g_t(z) = z + \frac{2t}{z} + O\left(\frac{1}{z^2}\right),$$



SLE

- The driving term $u_t = \sqrt{k} B_t$;
- Depending on k we can have:

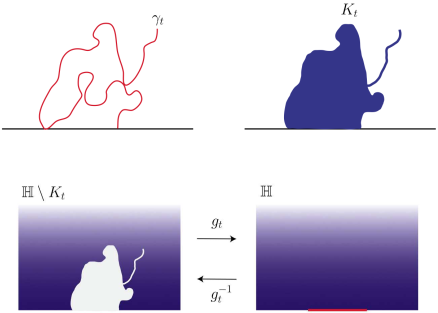


(courtesy of S. Rohde)

- Scale invariance: given $g_t(z)$ SLE$_\kappa$ for any $\lambda > 0 \Rightarrow \frac{1}{\lambda} g_{\lambda^2 t}(\lambda z)$ is also SLE$_\kappa$.

SLE-hull

- The solution to SLE$_k$ equation exists as long as $g_t(z) - u_t$ remains away from zero. Denote by T_z the first time such that $\lim_{t \to T_z - 0}(g_t(z) - u_t) = 0$;
- $K_t = \{z \in \mathbb{H} \colon T_z \le t\}$ SLE-hull;
- $\mathbb{H}_t = \mathbb{H} \setminus K_t = \{z \in \mathbb{H} \colon T_z > t\}$.



(courtesy of Kazumitsu Sakai)

Invariant approach to SLE

Certain probability measures on non-self-crossing random curves in a domain Ω connecting two given points $a, b \in \partial\Omega$ and satisfying

- Conformal invariance;
- Domain Markov property.

Conformal invariance

Family of curves

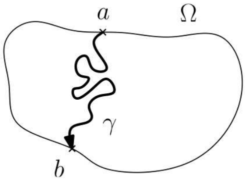
$$\mathcal{F}_{(\Omega,a,b)} = \{\gamma : \gamma \text{ is a non-self-traversing curve connecting } a \text{ and } b\},$$

$\mathcal{T}$ is the set of all possible triples

$$\mathcal{T} := \{(\Omega, a, b) \colon \Omega \text{ is a hyperbolic simply connected domain, } a, b \in \partial\Omega\}.$$

M is a family of measures indexed by $\mathcal{T}$, that is,

$$M = \{\mu_{(\Omega,a,b)} : (\Omega, a, b) \in \mathcal{T},\ \mu_{(\Omega,a,b)} \text{ is a measure on } \mathcal{F}_{\Omega,a,b}\}.$$



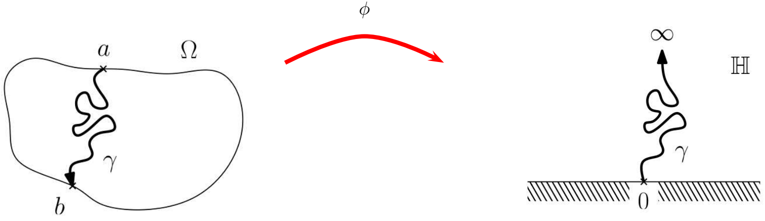
Definition

The family of measures M indexed by $\mathcal{T}$ is conformally invariant if for each pair of triples (Ω, a, b) and (Ω', a', b') from $\mathcal{T}$

$$\mu_{(\Omega', a', b')} = \phi_* \mu_{(\Omega, a, b)},$$

where $\phi\colon \Omega \to \Omega'$ is the unique conformal isomorphism $a \mapsto a'$, $b \mapsto b'$, $\phi_* \mu$ is pushforward of μ by ϕ.

For example,



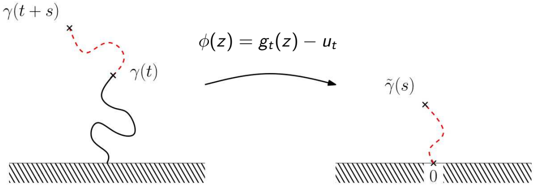
Domain Markov property

Definition

The family of measures M indexed by $\mathcal{T}$ satisfies the domain Markov property if for any Borel set B the conditional law $\mu_{(\Omega,a,b)}(\cdot \mid \gamma[0,t] = \gamma_0)$ is such that

$$\mu_{(\Omega,a,b)}(B' \mid \gamma[0,t] = \gamma_0) = \mu_{(\Omega_t,\gamma_0(t),b)}(B).$$

$B' = \{\gamma \colon \gamma[t,\infty) \in B\}$



+conformal invariance$\Rightarrow$ $\mu_{(\mathbb{H},0,\infty)}(B \mid \gamma[0,t] = \gamma_0) = \phi_* \mu_{(\mathbb{H},0,\infty)}$.

Infinitesimal generator

- SLE: $z \in \mathbb{H}$, $u_t = \sqrt{\kappa} dB_t$

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - u_t}, \quad g_0(z) = z, \quad g_t(z) = z + \frac{2t}{z} + O\left(\frac{1}{z^2}\right),$$

- SDE for $h_t(z) = g_t(z) - u_t$ is

$$dh_t(z) = \frac{2}{h_t(z)} dt - \sqrt{\kappa} dB_t.$$

- For any holomorphic function $M(z)$ we have the Itô formula

$$(dM)(h_t) = -du_t \ell_{-1} M(h_t) + dt(\frac{\kappa}{2}\ell_{-1}^2 - 2\ell_{-2})M(h_t),$$

where $\ell_n = -z^{n+1}\partial$. Infinitesimal generator $(\frac{\kappa}{2}\ell_{-1}^2 - 2\ell_{-2})$.

- $M(z)$ is a local martingale if $(\frac{\kappa}{2}\ell_{-1}^2 - 2\ell_{-2})M(z) = 0$.

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Question?

Analogously to

$$dh_t(z) = \frac{2}{h_t(z)}dt - \sqrt{\kappa}dB_t.$$

Define the process $G_t(z)$ solving the Stratonovich SDE

$$\begin{cases} dG_t(z) = -b(G_t(z))\,dt + \sqrt{\kappa}\,\sigma(G_t(z)) \circ dB_t, \\ G_0(z) = z, \end{cases} \quad z \in \mathbb{D}.$$

For which vector fields b and σ is the process G_t a slit holomorphic flow?

In the chordal case $b(z) = -\frac{2}{z}$ and $\sigma(z) = -1$.

General (deterministic) Löwner theory

Filippo Bracci, Manuel Contreras, Santiago Díaz-Madrigal, Pavel Gumenyuk, Christian Pommerenke (2000-2012).

Herglotz vector fields

Definition

A (generalized) Herglotz vector field of order d is $V : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ satisfying the following conditions:

- the function $[0, +\infty) \ni t \mapsto V(t, z)$ is measurable for every $z \in \mathbb{D}$;
- the function $z \mapsto V(t, z)$ is holomorphic in the unit disk for $t \in [0, +\infty)$;
- for any compact set $K \subset \mathbb{D}$ and for every $T > 0$, there exists a non-negative function $k_{K,T} \in L^d([0, T], \mathbb{R})$, such that

$$|V(t, z)| \leq k_{K,T}(t)$$

  for all $z \in K$, and for almost every $t \in [0, T]$;
- for almost every $t \in [0, +\infty)$, the function $V(\cdot, t)$ is a **semicomplete** vector field.

General Löwner equation

Theorem (Contreras, Díaz-Madrigal, Gumenyuk)

Let V be a Herglotz vector field of order $d \in [1, +\infty]$. Then,

- *For every $z \in \mathbb{D}$, there exists a unique maximal solution $g_t(z) \in \mathbb{D}$ to the following initial value problem*

$$\begin{cases} \dfrac{\partial g_t(z)}{\partial t} = -V(t, g_t(z)), \\ g_0(z) = z. \end{cases}$$

- *For every $t \geq 0$, the set D_t of all $z \in \mathbb{D}$, for which $g_t(z)$ is defined at the moment t, is a simply connected domain, and the function $g_t(z)$ defined for all $z \in D_t$ maps D_t conformally onto $\mathbb{D}$.*

Semicomplete vector fields

Laurent polynomial vector fields in the upper half-plane $\mathbb{H}$

$$\ell_n^{\mathbb{H}}(z) := -z^{n+1}, \quad n \in \mathbb{Z}.$$

The push-forward of the vector fields $\phi \colon \mathbb{H} \to D$:

$$\ell_n := \phi_* \ell_n^{\mathbb{H}}.$$

Proposition

- σ is a complete holomorphic vector field in D if and only if it admits a decomposition
$$\sigma = \sigma_{-1}\,\ell_{-1} + \sigma_0\,\ell_0 + \sigma_1\,\ell_1,$$
where $\sigma_{-1}, \sigma_0, \sigma_1 \in \mathbb{R}$.
- Vector fields ℓ_{-2} and $-\ell_2$ are semicomplete in D.

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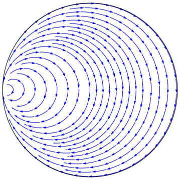
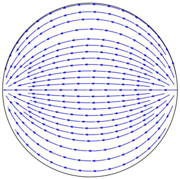
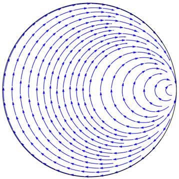


Vector field ℓ_{-1}



Vector field ℓ_0



Vector field ℓ_1

Slit holomorphic vector fields

In the unit disk $\mathbb{D}$:

PROPOSITION

The following statements are equivalent.

- $b(z)$ is a semicomplete vector field in $\mathbb{D}$ satisfying

$$\lim_{r\to 1} \operatorname{Re} b(re^{i\theta})\, re^{-i\theta} = 0$$

  for all $e^{i\theta} \in \partial\mathbb{D}$, except perhaps for $e^{i\theta} = 1$.

- $b(z)$ can be written as

$$b(z) = b_{-2}\ell^{\mathbb{D}}_{-2}(z) + b_{-1}\ell^{\mathbb{D}}_{-1}(z) + b_0\ell^{\mathbb{D}}_0(z) + b_1\ell^{\mathbb{D}}_1(z),$$

  for some $b_{-2} \geq 0$, $b_{-1}, b_0, b_1 \in \mathbb{R}$.

- The flow of b is generically a slit flow in $\mathbb{D}$ starting at 1.

Idea of time substitution

In the unit disk $\mathbb{D}$:

- Take
$$b(z) = -z\frac{1+z}{1-z} = 2\ell^{\mathbb{D}}_{-2}(z) + \frac{1}{2}\ell^{\mathbb{D}}_0(z), \quad \sigma(z) = -iz = \ell^{\mathbb{D}}_{-1}(z) + \frac{1}{4}\ell^{\mathbb{D}}_1(z);$$

- The flow of σ is $h_t(z) = ze^{-it}$;
- For a continuous function $u_t : [0, +\infty) \to \mathbb{R}$, $u_0 = 0$, define a Herglotz vector field $V(z,t) := (h^{-1}_{u_t}{}_* b)(z) = \frac{1}{h'_{u_t}(z)} b(h_{u_t}(z)) =$
$$= -z\frac{e^{iu_t} + z}{e^{iu_t} - z};$$
- The family of maps $\{g_t\}_{t\geq 0}$
$$\begin{cases} \frac{\partial}{\partial t} g_t(z) = -V(t, g_t(z)) = g_t \dfrac{e^{iu_t} + g_t}{e^{iu_t} - g_t}, \quad t \geq 0, \\ g_0(z) = z, \quad z \in \mathbb{D}, \end{cases}$$
is a radial decreasing Löwner chain.

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General slit Löwner chains

- Take $b(z)$ a slit vector field in D,

$$b = b_{-2}\ell_{-2} + b_{-1}\ell_{-1} + b_0\ell_0 + b_1\ell_1, \quad b_{-2} > 0,\ b_{-1},\ b_0,\ b_1 \in \mathbb{R};$$

- Take $\sigma(z)$ a complete vector field in D,

$$\sigma = \sigma_{-1}\ell_1 + \sigma_0\ell_0 + \sigma_1\ell_1, \quad \sigma_{-1},\ \sigma_0,\ \sigma_1 \in \mathbb{R};$$

  such that $\sigma_{-1} \neq 0$.
- Let $\{h_t\}_{t\in\mathbb{R}}$ be the flow of automorphisms of D generated by $\sigma(z)$.
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In this case we call $\{G_t\}_{t\geq 0}$ a **slit holomorphic stochastic flow** driven by b and σ.

SLE type	b	σ
Chordal	$2\ell_{-2}$	ℓ_{-1}
Radial	$2\ell_{-2} + \frac{1}{2}\ell_0$	$\ell_{-1} + \frac{1}{4}\ell_1$
Dipolar	$2\ell_{-2} - \frac{1}{2}\ell_0$	$\ell_{-1} - \frac{1}{4}\ell_1$

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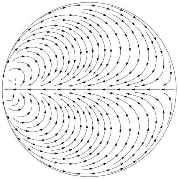
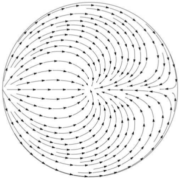
Radial	$2\ell_{-2}+\frac{1}{2}\ell_0$	$\ell_{-1}+\frac{1}{4}\ell_1$
Dipolar	$2\ell_{-2}-\frac{1}{2}\ell_0$	$\ell_{-1}-\frac{1}{4}\ell_1$



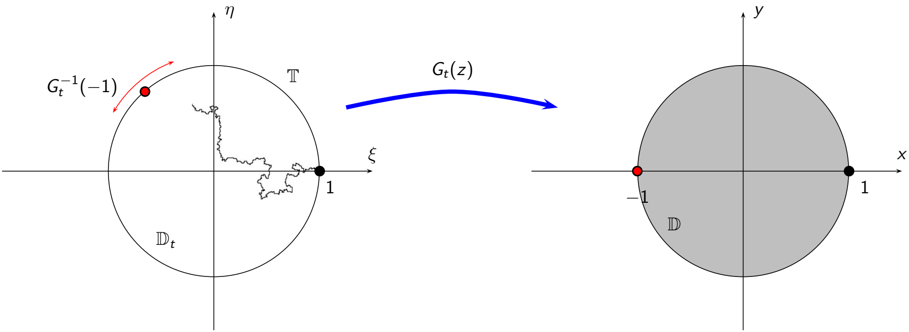
Vector field $2\ell_{-2}$



Vector field $2\ell_{-2}+\frac{1}{2}\ell_0$

ABP SLE

Some other examples: **Attractive Boundary Point SLE:**

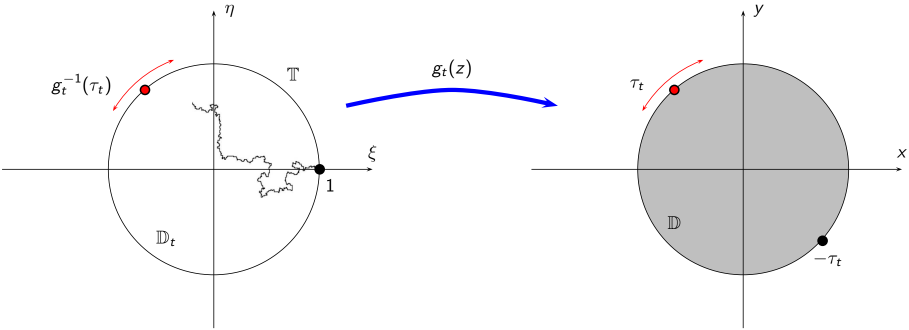


$b = 2\ell_{-2}$ as in the chordal SLE, $\sigma = \ell_{-1} + \frac{1}{4}\ell_1$ as in the radial SLE,

$$dG_t = -2\ell_{-2}G_t dt + \sqrt{\varkappa}(\ell_{-1} + \ell_1)(G_t) \circ dB_t,$$

ABP SLE

In terms of the solution of ODE with a random entry:



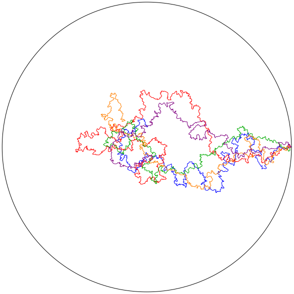
$w = g_t(z)$ is a solution to

$$\frac{dw}{dt} = \frac{1}{4}(\tau_t - w)(\bar{\tau}_t w - 1)\frac{\tau_t - w}{\tau_t + w},$$

with the initial condition $w\big|_{t=0} = z$, $\tau_t = e^{i\sqrt{\kappa}B_t}$.

ABP SLE

Some sample ABP curves:

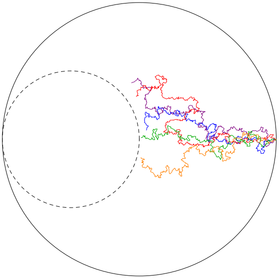


Another example of SLE

$b(z)$ as in the radial case, and $\sigma(z)$ as in the chordal case:

$$b = 2\ell_{-2} + \frac{1}{2}\ell_0, \quad \sigma = \ell_{-1}.$$

Some sample curves:



Normalization

If $b_{-2} = 2$ and $\sigma_{-1} = 1$ we call **normalized slit Löwner chains**.

'Simple' transforms

- Scaling of the driving function $(b, \sigma, u_t) \mapsto (b, c\sigma, \frac{1}{c} u_t)$;
- Time scaling $(b, \sigma, u_t) \mapsto (cb, \sigma, u_{(ct)})$.

Normalization can be always achieved.

Relationship between SLEs

Motivated by G. Lawler, O. Schramm, and W. Werner (Acta Math. 2001):

Theorem

Let $\{K_t\}_{t\geq 0}$ be the family of hulls generated by a slit Löwner chain driven by b, σ and u_t. Given another pair of vector fields, $\tilde{b}$ and $\tilde{\sigma}$, there exists and unique a driving function $\tilde{u}_{\tilde{t}}$ and a time reparametrization $\tilde{t} = \lambda(t)$, so that the corresponding slit Löwner chain generates the same family of hulls *at least locally in time* $0 \leq t \leq T_{\max}$.

Theorem

Let $\{K_t\}_{t\geq 0}$ be the family of random hulls generated by a normalized slit Löwner chain driven by $\sqrt{\kappa}\, B_t$. Let $\{\tilde{K}_{\tilde{t}}\}_{\tilde{t}\geq 0}$ be the family of radial SLE_κ-hulls. There exists a family of positive stopping times $\{T_n\}_{t\in\mathbb{N}}$, $T_n \to T_{\max}$, such that the laws of $(K_t, t \in [0, T_n])$ and $\left(\tilde{K}_{\lambda(t)}, t \in [0, T_n]\right)$ are absolutely continuous with respect to each other.

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Relationship between SLEs

As a corollary:

Let $\{G_t\}_{t\geq 0}$ be a normalized slit holomorphic stochastic flow driven by b, σ and $u(t) = \sqrt{\kappa}B_t$. Let γ denote the curve generating the hulls $\{K_t\}_{t\geq 0}$. Then, with probability 1,

- if $0 \leq \kappa \leq 4$, γ is a simple curve,
- if $4 < \kappa < 8$, γ has self-intersections,
- if $\kappa \geq 8$, γ is a space-filling curve.

The Hausdorff dimension of the curve generating the hulls of a normalized slit Löwner chain driven by $\sqrt{\kappa}\,B_t$ is equal to $\min(2, 1+\kappa/8)$ with probability 1.

- We have no fixed points $\Rightarrow$ no triples of type (Ω, a, b);
- BUT we have embeddable curves.

Theorem

Let $\xi\colon [0, T] \to \mathbb{D}$ be a simple curve in $\mathbb{D}$ starting at 1. For a given pair of vector fields b and σ there exists t_0 and at most one function g_{t_0} embeddable into a slit Löwner chain $\{g_t\}_{t\geq 0}$ driven by b and σ, such that $g_{t_0}^{-1}(\mathbb{D}) = \mathbb{D} \setminus \xi$.

We are able to formulate conformal invariance and domain Markov property for $0 \leq \kappa \leq 4$.

- Family of curves in $\mathbb{D}$:

$$\begin{aligned}\mathcal{F}_\xi \quad = \quad & \{\gamma : \gamma \text{ is a non-self-traversing curve starting from the tip of} \\ & \text{a } (b, \sigma)\text{-embeddable curve } \xi\};\end{aligned}$$

- $\mathcal{T}$ is the set of all possible (b, σ)-embeddable curves ξ;
- M is a family of measures indexed by $\mathcal{T}$, that is,

$$M = \{\mu_\xi : \xi \in \mathcal{T},\ \mu_\xi \text{ is a measure on } \mathcal{F}_\xi\}.$$

Conformal invariance

Definition

The family of measures M indexed by $\mathcal{T}$ is conformally invariant if for each pair of curves ξ and ξ' from $\mathcal{T}$

$$\mu_{\xi'} = \phi_* \mu_\xi,$$

where $\phi \colon \mathbb{D} \setminus \xi \to \mathbb{D} \setminus \xi'$ is the unique conformal isomorphism, $\phi_* \mu$ is pushforward of μ by ϕ.

(The unique ϕ is defined by $\phi = g_{\xi'} \circ g_\xi^{-1}$)

Domain Markov property

Definition

The family of measures M indexed by $\mathcal{T}$ satisfies the domain Markov property if for any Borel set B the conditional law $\mu_0(\cdot \mid \gamma[0,t] = \xi)$ is such that

$$\mu_0(B' \mid \gamma[0,t] = \xi) = \mu_\xi(B).$$

(Here 0 stands for the degenerate curve)

- This is equivalent to the fact that the slit holomorphic flow is a time-homogeneous diffusion;
- In the chordal, radial and dipolar cases CI and DMP coincide with the classical ones.

GFF

Ongoing project (G.I., A.V., Nam-Gyu Kang)

Problem

Traces of which slit holomorphic stochastic flows can be regarded as 'level' lines of Gaussian Free Field modifications?

More precise, which slit holomorphic stochastic flows G_t admit coupling with Dirichlet modifications $\tilde{\Phi}$ of the GFF such that

$$\text{Law}\left(\tilde{\Phi}_{\mathbb{D}} \,\middle|\, \mathcal{F}_t\right) = \text{Law}(\tilde{\Phi}_{\mathbb{D}} \circ G_t \,|\, \mathcal{F}_t), \qquad \text{for all } t > 0,$$

where $\mathcal{F}_t$ is the sigma algebra of G_t.

We found that for $\kappa = 4$ a 2-parameter family of HSFs admits such coupling. This includes, in particular, chordal and dipolar SLE_4 with drifts and some other new SLEs.

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Thank you!

감사합니다.