SIMPLE COMPUTATIONS IN PREQUANTIZATION BUNDLES

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Abstract. We work out some details of the well-known relation between the Maslov indices of periodic orbits in a prequantization bundle and the Chern class of the base symplectic manifold.

1. Prequantization bundles (or Boothby-Wang bundles)

Given an integral, symplectic manifold, one can define a so-called Boothby-Wang bundle or prequantization bundle. Instead referring to the literature, we try to explain this construction by reviewing Kobayashi's differential geometric construction.

Let us first define the notion of integral symplectic form. We have the natural homomorphism $\mathbb{Z} \to \mathbb{R}$, and this induces a homomorphism

$$H^2(Q;\mathbb{Z}) \longrightarrow H^2(Q;\mathbb{R}).$$

We call the image of this homomorphism the **Betti-part**, denoted $H^2_b(Q;\mathbb{Z})$. If the cohomology class $[\omega]$ lies in $H^2_b(Q;\mathbb{Z})$, then we call (Q,ω) integral.

The following construction due to Kobayashi, [K], associates with a symplectic form, a principal circle bundle with a contact form. Actually, Kobayashi's construction works for closed, integral 2-forms, but the resulting connection form is not contact, in general.

Construction 1.1. *Let (Q^{2n},ω) be an integral symplectic manifold. Then there is a principal circle bundle $p : P \to Q$ with a connection form θ such that*

- *θ is contact, and the Reeb field of θ is the generator of the circle action.*
- *the curvature form of θ is equal to $d\theta = -2\pi p^*\omega$.*

Begin construction. We first construct a principal circle bundle following Kobayashi. For this, take a good cover $\{U_i\}_i$. By definition $d\omega = 0$, so on the chart U_i, we find a 1-form $\omega|_{U_i} = d\omega_i$ by the Poincaré lemma. On a non-empty intersection $U_i \cap U_j$, we have $d(\omega_i - \omega_j) = \omega - \omega = 0$, so we find a function ω_{ij} on the set $U_i \cap U_j$ such that $d\omega_{ij} = \omega_i - \omega_j$.

We now consider non-empty triple intersections, $U_i \cap U_j \cap U_k$. On such a triple intersection, define the function $f_{ijk} := \omega_{ij} + \omega_{jk} + \omega_{ki}$. Then $df_{ijk} = \omega_i - \omega_j + \omega_j - \omega_k + \omega_k - \omega_i = 0$, so f_{ijk} is constant.

We now claim that the f_{ijk} can chosen to be integer. To see this, note first of all that the $\{f_{ijk}\}$ form a Čech cocycle, and hence an element in $\check{H}^2(Q;\mathbb{R})$ We recall a theorem due to Weil.

Theorem 1.2 (Weil)**.** *The correspondence $[\omega] \longmapsto [\{f_{ijk}\}]$ induces an isomorphism between $H^2_{dR}(Q)$ and $\check{H}^2(Q;\mathbb{R})$. Furthermore, this isomorphism sends the Betti-part to the Betti-part.*

Since Q is a manifold, we also have $\check{H}^2(Q;\mathbb{R}) \cong H^2(Q;\mathbb{R})$. By assumption $[\omega]$ represents an integer class, so $[\{f_{ijk}\}]$ also represents an integer class. Therefore, we can find a representant $\{\tilde{f}_{ijk}\}$ that is integer valued. We have $\tilde{f} = f + \partial g$ for some Čech 1-cochain g. Replace the ω_{ij} by $\omega_{ij} + g_{ij}$, and show the above procedure to find integer valued $\{f_{ijk}\}$. It follows that the following claim holds.

Claim: When regarded as transformations of $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, the ω_{ij} satisfy the cocycle condition.

Hence we define a principal circle bundle by

$$P := \coprod_i U_i \times S^1 / \sim,$$

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where $(x,[\varphi]) \in U_i \times S^1 \sim (y,[\psi]) \in U_j \times S^1$ if and only if $x=y$ and $[\psi]=[\varphi - 2\pi\omega_{ij}]$.

We now construct a connection form. By construction, the principal bundle comes with a trivializing cover $Q = \cup_{i\in I} U_i$. On the set $P|_{U_i} \cong U_i \times S^1$, define the 1-form by

$$\theta := dt - 2\pi\omega_i,$$

where t is the angular coordinate on S^1. We claim that θ is a well-defined 1-form on P, and, in fact, it is a connection form on P For this, we introduce the map

$$\begin{aligned} \Psi_{ij} : U_i \cap U_j \times S^1 \subset U_i \times S^1 &\longrightarrow U_i \cap U_j \times S^1 \subset U_j \times S^1 \\ (x,[t]) &\longrightarrow (x,[t-2\pi\omega_{ij}]) \end{aligned}$$

On $P|_{U_j}$, we have $\theta = dt - 2\pi\omega_j$, and with the above equivalence relation we find $\Psi_{ij}^*(dt - 2\pi\omega_j) = d(t-2\pi\omega_{ij}) - 2\pi\omega_j = dt - 2\pi\omega_i$.

Claim: The connection form θ is a contact form.

We compute using the trivializing charts U_i. On the set $P|_{U_i} \cong U_i \times S^1$ we find

$$\theta \wedge d\theta^n = (dt - 2\pi\omega_i) \wedge (-2\pi d\omega_i)^n = (-2\pi)^n dt \wedge p^*\omega^n \neq 0.$$

We also see that

$$d\theta = -2\pi p^*\omega.$$

This completes the construction. □

Definition 1.3. *The principal circle bundle together with the contact form obtained in the above construction is called* **prequantization bundle** *of (Q,ω). It is also called* **Boothby–Wang bundle**.

Example 1.4. *A basic example of a prequantization bundle is the Hopf fibration over $S^2 \cong \mathbb{CP}^1$. Give $\mathbb{CP}^1$ the standard Fubini-Study form. The associated prequantization bundle is then S^3 with its standard contact form.*

Remark 1.5. We made some seemingly unusual choices which led to

$$d\theta = -2\pi p^*\omega.$$

It is easy to adapt the construction to get rid of the sign. The factor 2π also seems awkward. However, the factor 2π is natural because we want to have circle fibers with length 2π: we have chosen $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. In the original construction of Kobayashi $S^1 = \mathbb{R}/\mathbb{Z}$.

That leaves the minus sign. This is also a matter of taste, but we point out that this choice has the advantage that it is compatible with Chern-Weil theory. In that theory the curvature of a connection is used to define a certain cohomology class, the Chern class. With our convention, we can use the contact form as this connection form, and we have $c_1(P) = [\omega]$. A different sign convention will change this.

Consider a prequantization bundle over a symplectic manifold (Q,ω):

$$S^1 \longrightarrow (P,\alpha) \xrightarrow{p} (Q,\omega).$$

Choose a compatible almost complex structure J_Q for (Q,ω). This complex structure lifts to compatible complex structure for $(\xi = \ker\alpha, d\alpha)$. Note that this contact structure is the horizontal lift of TQ. We have

$$d\alpha = p^*\omega \quad \text{and} \quad J_P = p^*J_Q.$$

Define the metrics

$$g_Q = \omega(\ldots, J_Q\ldots) \quad \text{and} \quad g_P = \alpha\otimes\alpha + d\alpha(\ldots, J_P\ldots).$$

2. Maslov indices of orbits in a prequantization bundle

We begin with a review of the properties of the Maslov index following the results of Robbin and Salamon. We then work out some simple computations related to the Maslov indices of periodic Reeb orbits in a prequantization bundle.

2.1. **Conley-Zehnder index and Robbin-Salamon index (or Maslov index).** Let $\psi : I \to \mathrm{Sp}(2n)$ be a path of symplectic matrices starting at the identity. There is an invariant, the **Conley-Zehnder index** $\mu_{CZ}(\psi)$, which we shall define via the crossing formula. Let ω_0 denote the standard symplectic form on $\mathbb{R}^{2n}$, namely $\omega_0 = dx \wedge dy$.

Definition 2.1. *Let $\psi : [0,T] \to Sp(2n)$ be a path of symplectic matrices. We call a point $t \in [0,T]$ a* **crossing** *if $\det(\psi(t) - \mathrm{Id}) = 0$. For a crossing t, let $V_t = \ker(\psi(t) - \mathrm{Id})$ and define for $v \in V_t$ the quadratic form*

$$Q_t(v,v) := \omega_0(v, \dot{\psi}(t)v).$$

The quadratic form Q_t is called the **crossing form** *at t.*

Let us now define the Maslov index for symplectic paths in the following steps.

(1) Take a path of symplectic matrices $\psi : [0,T] \to \mathrm{Sp}(2n)$ and suppose that all crossings are non-degenerate, i.e. the crossing form Q_t at the crossing t is non-degenerate as a quadratic form.
(2) Then we define the Maslov index for such paths ψ as

$$\mu(\psi) = \frac{1}{2}\operatorname{sgn} Q_0 + \sum_{t \in (0,T) \text{ crossing}} \operatorname{sgn} Q_t + \frac{1}{2}\operatorname{sgn} Q_T \tag{2.1}$$

Here sgn denotes the signature (i.e. the difference of the number of positive minus the number of negative eigenvalues) of a quadratic form. For $* = 0$ or T, $\operatorname{sgn} Q_* = 0$ if $*$ is not a crossing.

According to Robbin and Salamon, $\mu(\psi)$ is invariant under homotopies of the path ψ with fixed endpoints.
(3) For a general path of symplectic matrices $\psi : [0,T] \to Sp(2n)$, we choose a perturbation $\tilde{\psi}$ of ψ while fixing the endpoints such that $\tilde{\psi}$ has only nondegenerate crossings.
(4) Define

$$\mu(\psi) := \mu(\tilde{\psi}).$$

This is well defined according to Robbin and Salamon, [RS].

2.2. **A simple case: Conley-Zehnder indices of critical points.** Let H be a Hamiltonian function on $(\mathbb{R}^{2n}, \omega_0)$ that is Morse. Define the Hamilton vector field X_H by

$$i_{X_H}\omega = -dH.$$

Given a compatible complex structure J, we have

$$X_H = J \operatorname{grad} H.$$

Now take any critical point p of H. Then $X_H(p) = 0$. To linearize X_H near p, it is convenient to choose a Darboux ball around p, and the 'standard' compatible complex structure $J = J_0$. We then compute

$$\nabla X_H = \nabla J \operatorname{grad} H = J \nabla \operatorname{grad} H = J \operatorname{Hess}(H).$$

Hence we obtain the following ODE for the linearized flow,

$$\dot{y} = J \operatorname{Hess}(H) y,$$

with $y(t) \in T_pM$. A path of symplectic matrices $\psi(t)$ describing the linearized flow, can be obtained by solving the ODE

$$\dot{\psi} = J \operatorname{Hess}(H)\psi.$$

This is an autonomous ODE, since we linearized at a critical point. We obtain therefore

$$\psi(t) = \exp(J \operatorname{Hess}(H) t).$$

We make the following observations:

- The path of symplectic matrices ψ has a crossing at $t = 0$.
- Since we assumed H to be Morse, it follows that $t = 0$ is an isolated crossing. In particular, we find $\varepsilon > 0$ such that $\psi|_{[-\varepsilon,\varepsilon]}$ has only a crossing at $t = 0$.

- The crossing form at $t = 0$ is given by
$$Q_0(v,v) = \omega_0(v, J\,\mathrm{Hess}(H)v) = v^t\,\mathrm{Hess}(H)v,$$
so its signature is $2n - \operatorname{ind} H - \operatorname{ind} H$.

We plug this into Formula 2.1 to conclude

Lemma 2.2. *Let (M,ω) be a symplectic manifold and $H : M \to \mathbb{R}$ a Morse function. Suppose that p is a critical point of H. Then for all $\varepsilon > 0$ that is sufficiently small, we have*
$$\mu_{CZ}(TFl_t^{X_H}; t = 0, \dots, \varepsilon) = n - \operatorname{ind} H.$$

2.3. **From degenerate to non-degenerate Reeb orbits.** The prequantization bundle has a beautiful periodic flow, but unfortunately that also means that the setup is degenerate. Let us perturb the contact form to make Reeb orbits with small action non-degenerate. Take a Morse function $f : Q \to \mathbb{R}$. Lift this function to an S^1-invariant function $\bar f = f \circ p$. Define the contact form
$$\alpha_{\bar f} = \bar f \alpha_1 = \bar f \alpha.$$
We compute the Reeb field of this perturbed contact form.

Lemma 2.3. *The Reeb field of $\alpha_{\bar f}$ can be written as*
$$R_{\bar f} = \frac{1}{\bar f} R_1 + \tilde X_{1/\bar f},$$
where $\tilde X_{1/f}$ is the horizontal lift of the Hamiltonian vector field $X_{1/f}$ on Q defined by
$$i_{X_{1/f}}\omega = -d(1/f).$$

Proof. Since $T_xP \cong \operatorname{Span} R_1 \oplus \xi$, we can assume that the Reeb field of $\alpha_{\bar f}$ is given by
$$R_{\bar f} = aR_1 + \tilde X,$$
where $X \in \xi$. Since the defining equations for the Reeb vector field are
$$i_{R_{\bar f}} d\alpha_{\bar f} = 0 \quad \text{and} \quad i_{R_{\bar f}}\alpha_{\bar f} = 1,$$
we conclude that $a = 1/\bar f$. Furthermore, we have
$$i_{R_{\bar f}} d\alpha_{\bar f} = i_{1/\bar f R_1 + \tilde X}\left(d\bar f \wedge \alpha + \bar f d\alpha\right) = (1/\bar f R_1(\bar f) + \tilde X(\bar f)\,)\alpha - \frac{d\bar f}{\bar f} + \bar f i_{\tilde X} d\alpha$$
Since $d\alpha(\tilde X, \tilde X) = 0$, it follows that $\tilde X(\bar F) = 0$. We conclude that
$$i_{\tilde X} d\alpha = -d\frac{1}{\bar f}.$$
Because of the S^1-symmetry, $\tilde X$ projects down to a well-defined vector field X satisfying the equation
$$i_X\omega = -d\frac{1}{f}.$$
□

With the above lemma, we can understand the linearized flow near a periodic orbit also as the linearized flow of the Hamiltonian function $1/f$ near a critical point. This leads to the following lemma.

Lemma 2.4 (Indices after perturbation)**.** *Fix $T > 0$. Then there is a Morse function f on Q that is ε-close to 1 in C^2-norm such that*
- *all periodic orbits γ of $\alpha_{\bar f}$ (as given in the above construction) with action $\mathcal{A}(\gamma) < T$ correspond to critical points of f.*
- *The Conley-Zehnder indices of such periodic Reeb orbits are given by*
$$\mu_{CZ}(\gamma) = \mu(\gamma) - n + \operatorname{ind}_q f.$$
if γ corresponds to the critical point q of f.

In the above $\mu(\gamma)$ denotes the Robbin-Salamon index of the degenerate Reeb orbit γ. This orbit is part of a Morse-Bott family of periodic Reeb orbits.

Proof. Choose any Morse function g such that $|g|_{C_2} < 1$. Define $f := 1 + \varepsilon g$. The first statement follows from the previous lemma. Indeed, the fiber of the projection p over a critical point of f is a periodic Reeb orbit. Furthermore, if ε is sufficiently small, then other periodic orbits must have large action: the Reeb flow projects down to the flow of the Hamilton vector field of $1/f$, and this flow needs sufficient large time to close up.

For the second statement, take a trivialization ε of the contact structure along the Reeb orbit γ such that $\mu(\gamma, \varepsilon) = 0$ for the unperturbed flow of α_1. This can be done since that Reeb flow is periodic. Now continue to work with that trivialization, and consider the flow of $\alpha_{\bar{f}}$. We have

$$\mu_{CZ}(\gamma_{\alpha_f}, \varepsilon) = \mu_{CZ}(Fl_t^{X_{1/f}}) = n - \operatorname{ind}_p \frac{1}{f} = n - (2n - \operatorname{ind}_p f) = \operatorname{ind}_p f - n,$$

where we have used Lemma 2.2. By passing through a trivialization that comes from a disk, we obtain the claimed result. □

3. Prequantization bundles over monotone symplectic manifolds

Let (Q, ω) be an integral symplectic manifold. Suppose that

- $[\omega]$ is primitive.
- $c_1(Q) = c[\omega]$ for some $c \in \mathbb{Z}$.
- Q is simply-connected.

Consider the prequantization bundle P over $(Q, k\omega)$.

Lemma 3.1. *Then $\pi_1(P) \cong \mathbb{Z}_k$, and a circle fiber of $P \to Q$ is a generator.*

It is important to observe that, $(\xi, d\theta)$ is isomorphic, as a symplectic vector bundle, to the pullback bundle $\pi^*(TM, \omega)$. Another useful property is the following.

Lemma 3.2. *Let $L = P \times_{S^1} \mathbb{C}$ be a the associated complex line bundle. Then $L - \sigma_0$ is diffeomorphic to the symplectization $(\mathbb{R} \times P)$. Furthermore, the* positive *end $\{\infty\} \times P$ corresponds to the zeroset σ_0 of L.*

Lemma 3.3. *Suppose now that (P, λ) is a prequantization bundle over (Q, ω) (so $k = 1$). Let $\gamma = \pi^{-1}(q)$ be a circle fiber of $P \to Q$. Then $\mu(\gamma) = 2c$. Furthermore, the Maslov index of the N-fold cover is $2Nc$.*

Proof. Extend $\beta_1 := [\omega]$ to a basis $\beta_1, \ldots, \beta_m$ of $H^2(Q; \mathbb{Z})$. Take a dual basis $B_1, \ldots, B_m$ of $H_2(Q; \mathbb{Z})$. We may represent $B_1, \ldots, B_k$ by spheres as $H_2(Q; \mathbb{Z}) \cong \pi_2(Q)$.

Define the complex line bundle

$$L := P \times_{S^1} \mathbb{C}.$$

Let σ_0 denote the zero section of L. Note that $L - \sigma_0$ can be identified with the symplectization $\mathbb{R} \times P$.

Note that $L|_{B_1} \cong \mathcal{O}(1) \to \mathbb{CP}^1$, since the first Chern class is $k \cdot [\omega]$ with $k = 1$. Hence we can take a section $\sigma : B_1 \to L$ that vanishes only in q, with multiplicity 1, where $q \in B_1 \subset Q$. We get an induced section $\tilde{\sigma} : \mathring{D}^2 \cong B_1 - \{q\} \to L - \{0\}$. We extend the projection of $\tilde{\sigma}$ to P to a continuous map

$$\bar{\sigma} : D^2 \longrightarrow P.$$

The boundary of D^2 maps to the fiber $\pi^{-1}(q)$, so $\bar{\sigma}$ is a capping disk for the periodic Reeb orbit γ_q, the fiber over q.

Choose a trivialization

$$\varphi_1 : \bar{\sigma}(D^2) \times \mathbb{C}^n \longrightarrow \xi|_{\bar{\sigma}(D^2)}.$$

Define the "product" trivialization

$$\begin{aligned} \varphi_2 : \gamma_q \times \mathbb{C}^n &\longrightarrow \xi|_{\gamma_q} \\ (t; v) &\longmapsto H_{\gamma_q(t)} \left(T\pi\varphi_1(\gamma_q(0), v) \right), \end{aligned}$$

where $H_{\gamma_q(t)}$ gives the horizontal lift of a tangent vector $w \in T_{\pi(q)}Q$ to $\gamma_q(t)$. Given these trivializations, we define the following paths of symplectic matrices,

$$\psi_1(t) := \varphi_1^{-1} \circ TFl_t^R \circ \varphi_1$$
$$\psi_2(t) := \varphi_2^{-1} \circ TFl_t^R \circ \varphi_2.$$

We want to compute

$$\mu(\gamma_q) := \mu(\psi_1).$$

We first relate the two trivializations

$$\psi_1(t) = \varphi_1^{-1} \circ \varphi_2 \circ \varphi_2^{-1} \circ TFl_t^R \circ \varphi_2 \circ \varphi_2^{-1} \circ \varphi_1 = \varphi_1^{-1} \circ \varphi_2 \circ \psi_2 \circ \varphi_2^{-1} \circ \varphi_1.$$

Since $\varphi_2^{-1} \circ \varphi_1(\gamma_q(0), v) = (\gamma_q(0), v)$, we also have

$$\psi_1(t) = \varphi_1^{-1} \circ \varphi_2 \circ \psi_2(t).$$

By the loop axiom for the Robbin-Salamon version of the Maslov index, we find

$$\mu(\psi_1) = \mu(\psi_2) + 2\mu_l(\varphi_1^{-1} \circ \varphi_2).$$

Observe that $\mu(\psi_2) = 0$ as the linearized flow in the product trivialization is constant.

Finally we relate the Maslov index for loops to the Chern class via the standard procedure in McDuff-Salamon. For this, note that there is a bundle map

$$\tilde{\pi} : \xi \longrightarrow TQ$$

covering π and that is a fiberwise isomorphism. This can be done since ξ is the horizontal lift of TQ. We obtain "trivializations"

$$\tilde{\varphi}_1 = \tilde{\pi} \circ \varphi_1 \quad \text{and} \quad \tilde{\varphi}_2 = \tilde{\pi} \circ \varphi_2$$

for TQ over the disk $\pi(\bar{\sigma}(D^2)\,)$, and over q (the notation is not correct here because we need the projected domains of φ_i). Note that the trivialization of TQ over $\pi(\bar{\sigma}(D^2)\,)$ is actually not well-defined: the boundary of $\bar{\sigma}(D^2)$ projects to a point making this "trivialization" into a multivalued one.

The trivialization over q is just a basis of T_qQ. To make the first "trivialization" into an honest one, we need to shrink the disk a little: the resulting loop of symplectic matrices will be homotopic, and we therefore have the same Maslov index for loops.

We now continue with the main argument, and take the modified loop, still denoted by $\tilde{\varphi}_1^{-1} \circ \tilde{\varphi}_2$. This map corresponds to the overlap map as described in section 2.6 of McDuff-Salamon ([MS], page 75): alternatively, we can think of this as the transition function for the bundle $TQ|_{B_1}$ from one disk to the other.

By Theorem 2.69 from McDuff-Salamon, [MS], we find

$$c = \langle c_1(TQ), B_1 \rangle = \mu_l(\tilde{\varphi}_1^{-1} \circ \tilde{\varphi}_2).$$

We conclude that $\mu(\psi_1) = 2c$. Since the flow is periodic, we find the following index for an N-fold cover of γ_q,

$$\mu(\gamma_{q,N}) = 2cN.$$

□

By a covering trick we can prove the following.

Corollary 3.4. *Let $(Q, k\omega)$ be an integral symplectic manifold with*

- *$[\omega]$ is primitive, and $k \in \mathbb{Z}_{>0}$.*
- *$c_1(Q) = c[\omega]$ for some $c \in \mathbb{Z}$.*
- *Q is simply-connected.*

Let (P, λ) denote a prequantization bundle over $(Q, k\omega)$.

Suppose that γ is a simple fiber. Then a k-fold cover of γ is contractible and $\mu(\gamma_k) = 2c$.

Indeed, just take the k-fold cover of P to reduce this setup to the one of Lemma 3.3.

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