SASAKIAN GEOMETRY AND SYMPLECTIC TOPOLOGY

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ABSTRACT. These are notes for a talk given in Academica Sinica: Sasakian manifolds are an odd-dimensional analog of Kähler manifolds; they have played an important role in the construction of new Einstein metrics. I will give a short overview on some of the key results found by researchers in Sasakian geometry. I will concentrate on results in the context of singularity theory, where remarkable constructions of so-called Sasaki- Einstein metrics have been found. Finally, I will combine these ingredients with an invariant derived from symplectic homology to study the moduli space of Sasaki-Einstein metrics. This is based on joint work with Charles Boyer and Leonardo Macarini

1. Basic definitions

By a **contact metric manifold** we mean a tuple $(Y^{2n-1}, \theta, J_\theta, g)$ satisfying the following:

- $(Y, \mathcal{D} = \ker \theta)$ is a contact manifold, so $\mathcal{D}$ is a maximally non-integrable hyperplane field. The non-integrability condition is equivalent to

$$\theta \wedge d\theta^{n-1} \neq 0.$$

  With this first piece of data we define the Reeb vector field T by

$$\iota_T d\theta = 0, \quad \theta(T) = 1.$$

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- $J_\theta : TY \to TY$ is an endomorphism of the tangent bundle that induces a compatible (with $d\theta$) complex structure on $\mathcal{D}$ and $J_\theta T = 0$. Complex structure on the bundle $\mathcal{D}$ means that

$$(J_\theta|_{\mathcal{D}})^2 = -\operatorname{Id}|_{\mathcal{D}}.$$

  Compatible with $d\theta$ means that

$$d\theta(\cdot, J_\theta \cdot)$$

  is a metric on $\mathcal{D}$.
- The tensor $\theta \otimes \theta + d\theta(\cdot, J_\theta \cdot)$ is a Riemannian metric on Y.

A typical example of a contact metric manifold is $(S^{2n-1} \subset \mathbb{C}^N, \theta, i|_{TS^{2n-1}}, g_0)$ with

- $\theta = \frac{i}{2} \sum_j z_j d\bar{z}_j - \bar{z}_j dz_j$
- $J_\theta = i|_{TS^{2n-1}}$.
- g_0 is the standard round metric.

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[1] T stands for transverse to the contact structure $\mathcal{D}$. In contact topology, the notation R is standard, but this is not suitable for use in geometry.

Definition 1.1. Call (Y, θ, J_θ) **Sasakian** if the cone

$$(\mathbb{R}_{>0} \times Y, dr^2 + r^2 g, d(r^2\theta), J)$$

is Kähler. Here J_θ is extended to a complex structure on the tangent bundle of the cone by requiring that $Jr\frac{\partial}{\partial r} = T$.

Proposition 1.2. *If (Y, θ, J_θ) is Sasakian, then T is a Killing vectorfield. In other words, Sasakian manifolds are always K-contact.*

Proof. To see why this is true, write out the Nijenhuis tensor applied to the Reeb field T and a tangent vector X to the contact manifold Y. A short computation shows that $\mathcal{L}_T J_\theta = 0$. From there we see that g is preserved. □

It is maybe interesting to mention the relation with CR-geometry: consider the Tanaka-Webster connection. A strictly pseudoconvex CR-manifold (Y, θ, J_θ) is K-contact if and only if the pseudohermitian torsion of the associated Tanaka-Webster vanishes.

The proposition tells us that the isometry group of a Sasakian manifold always contains a 1-parameter subgroup. This is obviously very special. The following gives us a topological consequence.

Proposition 1.3 (Weinstein)**.** *If (Y, θ, J_θ) is a compact K-contact manifold, then the underlying cooriented contact manifold (Y, θ) is contactomorphic to $(Y, \tilde{\theta})$, where $\tilde{\theta}$ has periodic Reeb flow.*

Proof. The argument here can be found in the appendix of the paper of Chern and Hamilton. Consider the 1-parameter subgroup $G = \{Fl_t^T\}_t$ of the isometry group $Isom(Y, g)$. As Y is assumed to be contact, $Isom(Y, g)$ is a compact Lie group. Hence the closure $\bar{G}$ is a torus $(\mathbb{R}/\mathbb{Z})^k$ for some $k > 0$. Fix a direction that is close to G near the identity; we obtain a compact subgroup of $(\mathbb{R}/\mathbb{Z})^k$. The new (periodic) direction corresponds to the Reeb vector field of a perturbed contact form $\tilde{\theta}$. This is contactomorphic to the original contact structure by Gray stability. □

Corollary 1.4. *If (Y, θ, J_θ) is K-contact, then Y is diffeomorphic to an S^1-orbibundle over a symplectic orbifold (Q, ω). If (Y, θ, J_θ) is Sasakian, then (Q, ω) is Kähler.*

The first part follows from symplectic reduction applied to a slice in the symplectization $\mathbb{R} \times Y$. The second part can be checked by using the Nijenhuis tensor.

1.1. **Links of singularities.** The above suggests that Sasakian manifolds are extremely special, so one may wonder whether any interesting examples exist. The following classical construction shows that many examples exist.

Assume that p is a weighted homogeneous polynomial with weights $(w_0, \ldots, w_n; d)$. This means that

$$p(\lambda^{w_0} z_0, \ldots \lambda z_n^{w_n}) = \lambda^d p(z_0, \ldots, z_n).$$

Assume in addition that 0 is an isolated singularity. Then $p^{-1}(0)$ is a variety with an isolated singularity at 0. The link of p is defined as

$$L(p) := p^{-1}(0) \cap S^{2n+1}$$

Remark 1.5. Usually links are defined by taking the intersection with a sufficiently small sphere, but due to the $\mathbb{C}^*$-action

$$\lambda \cdot (z_0, \ldots, z_n) = (\lambda^{w_0} z_0, \ldots \lambda z_n^{w_n}),$$

the size of the sphere does not matter.

Lemma 1.6. *The link of an isolated singularity carries a contact structure.*

Proof. To see this, note that the function $f(z) = |z|^2$ restricts to a strictly plurisubharmonic function on $p^{-1}(0)\setminus\{0\}$. The property that f is strictly plurisubharmonic means that $d(-df \circ i)(\cdot, i\cdot)$ is a Kähler metric. Regular level sets are then strictly pseudo-convex in the sense of complex analysis and this means that the contact condition holds. □

The above argument works for the link of any isolated (holomorphic) singularity, but the resulting contact form is not well-adapted to the geometric situation of weighted homogeneous polynomials. In the weighted homogeneous space there are contact forms with a periodic flow corresponding to the $\mathbb{C}^*$-action. One such form[2] is

$$\theta = \frac{i}{2}\sum_j \frac{1}{w_j}\left(z_j d\bar{z}_j - \bar{z}_j dz_j\right)$$

The Reeb flow of this contact form defines a locally free circle action. It follows that $L(p)/S^1$ is a orbifold; it is, in fact, a hypersurface in weighted projective space with the obvious defining equation $p = 0$.

Links of singularities have received attention for many reasons. Here are a couple that are relevant to this talk.

(1) many exotic spheres arise as links of singularities: this includes all boundary parallelizable spheres in dimension at least 7. To give an explicit example, the Kervaire sphere is the links of the singularity
$$z_0^3 + \sum_{j=1}^n z_j^2 = 0.$$
(2) exotic involutions on spheres
(3) new Einstein metrics on spheres. Einstein metrics on exotic spheres.

How is the latter done? The first step is the following.

Proposition 1.7. *Suppose that $(Y^{2n+1}, \theta, J_\theta, g)$ is a Sasakian manifold with a periodic Reeb flow. Denote the quotient Kähler orbifold by $Q := Y/S^1$ and its Kähler metric by h. Then g is Sasaki-Einstein if and only if h is Kähler-Einstein with scalar curvature $4n(n+1)$.*

The idea behind proposition is to think of $\pi : Y \to Q$ as a Riemannian submersion, and use a variation of O'Neill's formula, which relates sectional curvature on the base to the sectional curvature on the total space. Details can be found for example in Blair's book on "Riemannian geometry of contact and symplectic manifolds".

Remark 1.8. We see that this proposition can only yield Einstein metrics with positive Einstein constant; this positivity is related to the Reeb vector field: for contact metric manifolds, one has strong restrictions on $Ric(T, T)$ and for K-contact manifolds of dimension $2n + 1$ one has $Ric(T, T) = 2n > 0$.

[2] there are other expressions for a contact form with these properties that are more common in Sasakian geometry; with the form we have chosen the link with pseudoconvexity is not clear unless one is willing to deform the complex structure.

Hence we come to the problem of constructing Kähler-Einstein metrics on manifolds and orbifolds. There are too many people who have done important work to list here, so we just give a couple of names: Yau, Aubin and Tian. Note that Einstein constant must be positive due to the above proposition, so we are in the Fano case, i.e. $c_1(Q) > 0$; this case has additional obstructions.

1.2. **Boyer-Galicki-Kollar.** The earliest results treating the orbifold case and yielding new Einstein metrics corresponding to the above proposition is due to Boyer-Galicki-Kollar: They have proved remarkable results for weighted homogeneous polynomials of the form

$$p = \sum_j z_j^{a_j}.$$

Theorem 1.9 (Boyer-Galicki-Kollar)**.** *If*

$$1 < \sum_{j=0}^{n} \frac{1}{a_j} < 1 + \frac{n}{n-1} \min_{i,j} \left(\frac{1}{a_i}, \frac{1}{b_i b_j} \right),$$

where $b_i = \gcd(a_i, \operatorname{lcm}_{i \neq j} a_j)$*, then the links* $L(p)$ *admits a Sasaki-Einstein metric.*

We cannot prove this theorem here, but we briefly mention that the left hand inequality is needed for the Fano condition, and the right hand inequality is used to make the continuity method for solving the Monge-Ampère (the main technical step in obtaining a Kähler-Einstein metric) work.

This theorem gaves huge collections of new examples, including Einstein metrics on exotic spheres. We remark that there have been many improvements to this theorem, as well as new work on Sasaki-Einstein metrics.

In addition, the methods of the theorem gave ways to construct positive-dimensional families of Sasaki-Einstein metrics.

1.3. **Finding different components in the moduli space of Sasaki-Einstein metrics.** The idea here is to use invariants of the underlying contact structure. We sketch the construction, leaving out the technical details. Denote the Sasaki-Einstein manifold by Y and assume that it is the boundary of a Stein domain, say $Y = \partial W$ where the boundary is the regular level set of a plurisubharmonic function; we also assume that the induced contact structure is the constructure of the Sasaki manifold.

We compute a version of Floer homology of the manifold W.

- roughly speaking, the equivairant Floer homology $SH^{S^1,+}(W)$ can be thought of as the homology of a certain action functional defined on the loop space of W.
- the chain complex of Floer homology consists of periodic Reeb orbits on Y; these are encoded in the different singular strate of $Q = Y/S^1$.
- the boundary map is defined by counting solutions to a Cauchy-Riemann type PDE.

The boundary map is very difficult to compute in general, so we use the mean Euler characteristic, which in this case can be defined as

$$\chi_m(Y, \mathcal{D}) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=-N}^{N} (-1)^j \operatorname{rk} SH^{S^1,+}(W).$$

Like the usual Euler characteristic, this Euler characteristic can here be computed using only the dimensions of chain complex groups, and not the boundary map.

Theorem 1.10. *Assume that p is a weighted homogeneous polynomial. Put $Y := L(p)$. This contact manifold is filled by a Stein subdomain of $p^{-1}(1)$. In the case that the quotient orbifold $Q = Y/S^1$ is Fano or of general type, the mean Euler characteristic can be computed as*

$$\chi_m(W) = \frac{\chi(IQ)}{|\mu_P|},$$

where IQ is the inertia orbifold of Q.

References

[BG] Charles Boyer and Krzysztof Galicki, *Sasakian geometry*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2008.

[BGK] . Boyer, K. Galicki, and J. Kollár, *Einstein metrics on spheres*, Ann. of Math. (2) 162 (2005), no. 1, 557–580.

[Bl] D. Blair, *Riemannian geometry of contact and symplectic manifolds,* Progress in Mathematics, 203. Birkhäuser Boston, Inc., Boston, MA, 2002. xii+260 pp. ISBN: 0-8176-4261-7.

[BMvK] C. Boyer, L. Macarini, and O. van Koert , *Brieskorn manifolds, positive Sasakian geometry, and contact topology*, Forum Math. 28 (2016), no. 5, 943–965

[CH] S.-S. Chern, R. S. Hamilton, *On Riemannian metrics adapted to three-dimensional contact manifolds. With an appendix by Alan Weinstein,* Lecture Notes in Math., 1111, Workshop Bonn 1984 (Bonn, 1984), 279, Springer, Berlin, 1985.

[Ta] N. Tanaka, *A Differential Geometric Study on Strongly Pseudoconvex Manifolds,* 1975, Kinokuniya Co. Ltd., Tokyo

[We] S. M. Webster, *Pseudohermitian structures on a real hypersurface,* J. Diff. Geom. 13 (1978), 25-41.

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