SMOOTH AND SYMPLECTIC TOPOLOGY OF HYPERSURFACE SINGULARITIES

Abstract. Singular points on varieties can often be recognized by looking at their links. These are boundaries of small balls enclosing a singular point. In the case of surface singularities, such links are links in the sense of knot theory. In higher dimensions links provide a wide class of interesting manifolds, including exotic spheres. In this talk, we will describe both the smooth topology of links and some of their symplectic topology which captures finer detail.

1. Links of singularities

Throughout we will consider a polynomial $p \in \mathbb{C}[x_0, \dots, x_n]$. We will assume the following

A: If 0 is a singular point of p, then it is isolated.

Here we call 0 a singular point if the derivative of p at 0 vanishes, i.e. $d_0 p = 0$, or more concretely

$$\left(\frac{\partial p}{\partial x_0}, \dots, \frac{\partial p}{\partial x_n}\right) = 0.$$

The variety $p^{-1}(\varepsilon)$ will be denoted by $V_\varepsilon(p)$.

1.1. **Some examples.** We will have two running examples in these notes. The first is a quadratic singularity. In the plane, we can take $p = x^2 - y^2$ and $q = x^3 - y^2$, the cubic cusp curve.

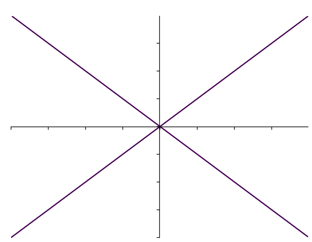

Figure 1. A plot of the level set $x^2 - y^2 = 0$

Looking the (real) pictures, we may wonder whether there are some geometric means to see that 0 is really singular. In particular, this means that we don't want to resort to directly looking at the defining polynomial and its Jacobian.

We make the following observations (to make sense of the numbers, keep in mind that the real dimension of the variety $V_0(p)$ is $2n$ and that it is embedded in $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$)

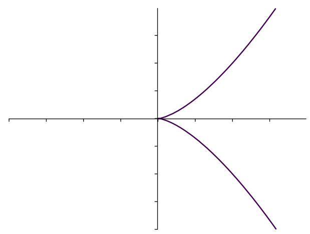

FIGURE 2. A plot of the level set $x^3 - y^2 = 0$

(1) A smooth point has a (small) neighborhood that looks like a ball, i.e. is diffeomorphic to one.
(2) In particular, the boundary of such a small neighborhood is smooth sphere.
(3) We can find a small neighborhood of the singular (or smooth) point by taking a small ball $B_\varepsilon^{2n+2}(0) \subset \mathbb{C}^{n+1}$ and intersecting it with $V_0(p)$. We will call the intersection B.
(4) In case of a smooth point, the sphere from point (3) is isotopic to the standard embedding $S^{2n-1} = \partial B \to S^{2n+1} = \partial B_\varepsilon^{2n+2}(0)$.

Figure 3 illustrates these observations schematically.

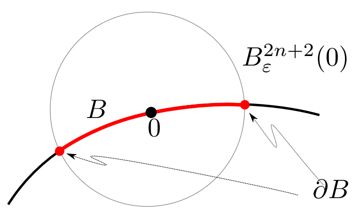

FIGURE 3. A neighborhood of a smooth point

1.1.1. *Checking the examples.* Let us check these criteria for the two examples. We first consider $V_0(x^2 - y^2)$. If we define B as in the above, then we find that it is the union of two circles

$$\begin{aligned}\partial B &= \{(x,y) \in S_\varepsilon^3 \mid x = y \text{ or } x = -y\} \\ &= \{\sqrt{\frac{\varepsilon}{2}}(e^{it}, e^{it}) \mid t \in [0, 2\pi[\} \cup \{\sqrt{\frac{\varepsilon}{2}}(e^{it}, -e^{it}) \mid t \in [0, 2\pi[\}.\end{aligned}$$

If 0 were a smooth point, then we should have gotten only a single circle, so we conclude that 0 is a singular point.

By the way, this pair of circles links as a so-called Hopf link. In fact, the two circles are two fibers of the famous Hopf fibration. Here is a brief description of the Hopf fibration. We think of $S^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$. Then we note that there is a map $pr : S^3 \to \mathbb{C}P^1$ sending $(z,w) \mapsto [z : w]$. As $\mathbb{C}P^1$ is diffeomorphic to the 2-sphere, we can now look at preimages of points in $\mathbb{C}P^1 = S^2$ under pr. These are all circles, and any pair of them links like a Hopf link. In particular, we see that S^3 is foliated (think decomposed) into circles. To visualize

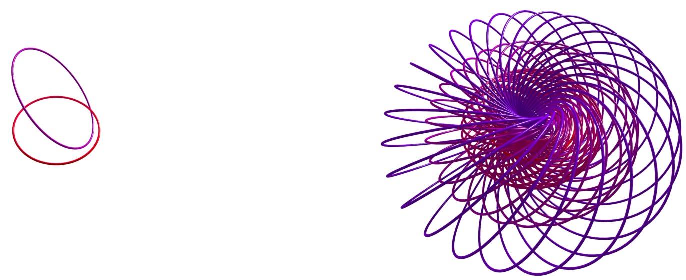

FIGURE 4. A Hopf link and some Hopf fibers

this, note that $S^3 = \mathbb{R}^3 \cup \{\infty\}$. This allows to describe most of S^3 as just Euclidean space. Explicitly, this can be done with stereographic projection. The result of this procedure is illustrated in Figure 4.

The other example, the point 0 in the variety $V_0(x^3 - y^2)$, is somewhat more subtle. As before, we can parametrize the set ∂B. In this case, we have

$$\partial B = \{(x, y) \in S^3_\varepsilon \mid (x, y) \operatorname{Im}\Big(t \mapsto (r_0^2 e^{it/3}, r_0^3 e^{it/2})\Big)\},$$

where r_0 is the positive, real solution to $|r_0^2|^2 + |r_0^3|^2 = \varepsilon^2$. We see that ∂B lies on the torus given by $|x| = r_0^2, |y| = r_0^3$. We can visualize this set using stereographic projection. Putting things in a computer, and massaging gives the following picture. Alternatively, one can look a bit longer at the parametrization, and note that ∂B

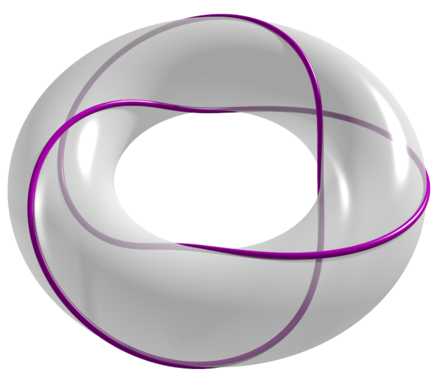

FIGURE 5. The trefoil knot

is a knot lying on torus, running around 3 times in one direction and 2 times in the other. The upshot is that ∂B is the so-called trefoil knot [1] or $2, 3$ torus knot. This is of course homeomorphic to the circle, but it is not isotopic to the standard embedding, so we conclude that the cubic cusp curve is also singular.

[1]A trefoil or clover is a small plant with (usually) three leaves.

1.1.2. *Links.* In the above two examples, the observations we made were useful in identifying the singular points. Let us now formalize this.

Definition 1.1. Call $L_\varepsilon(p) := S^{2n-1}_\varepsilon \cap V_0(p)$ the **link** of p at 0.

Theorem 1.2. *If* 0 *is an isolated singular point of* $V_0(p)$*, then* $L_\varepsilon(p)$ *is a smooth manifold for sufficiently small, positive* ε*.*

Idea. Since 0 is, at worst, an isolated singular point, we find a small ball $B_{\tilde\varepsilon}(0)$ such that

$$\mathrm{rk}_{\mathbb{R}}\, Jac(z \mapsto p(z)) = 2 \text{ on } B_{\tilde\varepsilon}(0) \setminus \{0\}$$

Now choose a positive regular value ε^2 of the map $z \mapsto |z|^2$.

By the regular value theorem, $L_\varepsilon(p)$ is a smooth manifold. □

1.2. **What does the link know about the singularity?** A common question in geometry is "what does the boundary know about the interior"? Here we also have a version of this question. We start by citing a result from the literature. We won't use this, but it serves as motivation for studying the topology of a link of a singularity.

Theorem 1.3 (Mumford+Perelman)**.** *Suppose that* $(X, 0)$ *is a normal, complex surface singularity. Then* 0 *is a smooth point of* X *if and only if* $L_\varepsilon(0) \cong S^3$*.*

In other words, the link knows quite a bit about the singularity; at least we can use it to detect whether the singular point was in fact smooth in this dimension.

Remark 1.4. Mumford proved a statement involving the fundamental group. By the Poincaré conjecture, the statement about S^3 follows. The "normality" condition is a technical condition that is needed in general. One can show that our hypersurface singularities are always normal in complex dimension at least 2.

To give an example, let us point out that the link $L_\varepsilon(x^2+y^3+z^5)$ is diffeomorphic to the Poincaré homology sphere.

One may wonder whether Mumford's theorem also holds true in higher dimensions. It turns out that this is the case if one includes additional structure, see Theorem 2.4. Without this additional structure, however, it does not: The link $L_\varepsilon(z_0^3 + \sum_{j=1}^3 z_j^2)$ is diffeomorphic to S^5, yet $z_0^3 + \sum_{j=1}^3 z_j^2$ has a genuine, isolated, normal singularity at 0.

1.3. **Topology of links.** Since we are considering hypersurface singularities $p^{-1}(0)$, we can make the singular variety smooth by smoothening it: simply modify the equation to obtain the deformed variety $p^{-1}(\varepsilon)$. As we have assumed p to have an isolated singularity at 0, the deformed variety is smooth for sufficiently small ε.

Remark 1.5. We note that smoothing of singularities is possible in our setup of hypersurface singularities, but in general this process will not work. There is a general procedure for resolving singularities which needs the so-called blowup construction. We will not consider this, though.

1.3.1. *Examples.* The most basic genuine hypersurface singularity is $V_0(\sum_j z_j^2)$. We claim that the smoothed variety is diffeomorphic to the cotangent bundle of a sphere, T^*S^n [2]. Indeed, we have the following explicit map,

$$T^*S^n = \{(q,p) \in T^*\mathbb{R}^{n+1} \; \|q\| = 1, \; \|q\| = 1, \langle q, p\rangle\} \longrightarrow V_1(\sum_{j=0}^{n} z_j^2)$$
$$(q,p) \longmapsto \sqrt{1 + |\|p\|^2} q + ip$$

We also mention the smoothing of a cusp curve. We have

$$V_1(x^3 - y^2) \cong T^2 \setminus \{pt\}.$$

Indeed, this is a so-called elliptic curve. An explicit diffeomorphism can be written down using the so-called Weierstrass $\wp$-function.

The simple, explicit description in the first case will turn out to be useful.

Definition 1.6. A **Morsification** of a function p with an isolated singularity consists of a family of of holomorphic functions $P : \mathbb{C}^{n+1} \times B_\varepsilon \to C$ such that

(1) $P(z, 0) = p(z)$
(2) for fixed $\lambda > 0$ the function $P(z, \lambda)$ is a holomorphic Morse function. This means that the Hessian of $P(\cdot, \lambda)$ is non-degenerate at every critical point.

Individual members of the family satisfying the second condition are also referred to as Morsifications. By the so-called holomorphic Morse lemma, all critical points of a Morsified function are quadratic.

Remark 1.7. In practice, Morsifications can be found by perturbing the defining function. For example, if we start with $p = x^3 + y^2$, then $\tilde{p} = x^3 + \lambda x + y^2$ has only quadratic singularities for $\lambda \neq 0$.

Definition 1.8. The **Milnor number** of p at 0 is the number of critical points of a Morsification of p. In a formula,

$$\mu(p) := \#Crit(\tilde{p})$$

for a Morsification $\tilde{p}$ of p.

It is not a priori clear that the Milnor number is well-defined, but it turns out that it is. The following proposition shows this and gives a way to compute the Milnor number.

Proposition 1.9. *If p is a polynomial with an isolated singularity at the origin, then*

$$\mu(p) = \dim_{\mathbb{C}} \mathbb{C}[[x_0, \dots, x_n]]/Jac(p),$$

where the Jacobian ideal $Jac(p)$ is generated by $\frac{\partial p}{\partial x_0}, \dots, \frac{\partial p}{\partial x_n}$

The geometric meaning of the Milnor number is captured by the following theorem.

Theorem 1.10. *Let $\tilde{p}$ a smoothing of a polynomial $p \in \mathbb{C}[x_0, \dots, x_n]$ with an isolated singularity at 0. Then $V_0(\tilde{p})$ is homotopy equivalent to a wedge of $\mu(p)$ n-spheres, meaning the union of spheres glued along a single point, see Figure 6. The standard notation for this wedge is $\bigvee_\mu S^n$.*

[2] One can also say tangent bundle to the sphere, but it turns out that the cotangent bundle carries a canonical symplectic structure. This structure is needed at the end of these notes.

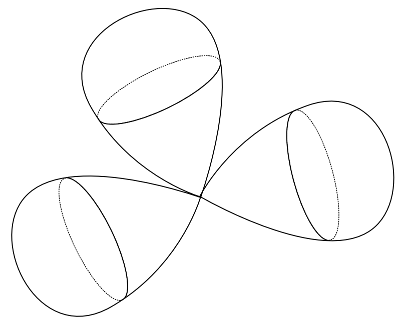

FIGURE 6. A wedge of spheres

Rough idea of the proof. Each quadratic singularity contributes a single copy of S^n as we have seen in the above example. To get the statement about the full variety, the idea is to use the Mayer-Vietoris sequence to compute the homology. The proof is completed with the Hurewicz and Whitehead theorem. □

Unfortunately, homotopy equivalence does not provide enough information for many purposes. For instance, let us consider the following two surfaces: a pair of pants and a torus with an open disk removed. Both spaces are homotopy equivalent

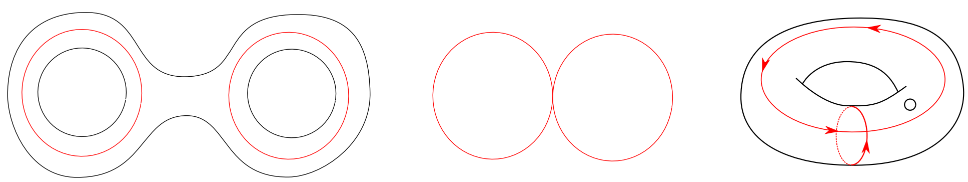

FIGURE 7. Homotopy equivalent but non-homeomorphic spaces: left a pair of pants, right a punctured torus. Center: a figure-eight. The generators of the homology are indicated in red.

to a figure-eight, but the number of boundary components differs. We will see that we can recover this information with algebraic topology, too.

1.4. **Intersection product.** To obtain this more refined information we need a tool from topology. First some definitions.

Definition 1.11. Suppose that A and B are submanifolds in a smooth manifold W. We say that A and B intersect **transversely** if for all $p \in A \cap B$ we have $T_pA + T_pB = T_pW$.

In other words, we want the two vector spaces T_pA and T_pB to generate the full vector space. We usually write $A \pitchfork B$ if A and B intersect transversely in W.

Suppose now that A and B are closed submanifolds in a smooth, compact manifold W, and assume that A and B are intersecting transversely. The regular value theorem can be used to show that $A \cap B$ is a submanifold, and if A and B have complementary dimensions, so $\dim A + \dim B = \dim W$, then A and B intersect in a finite set of points.

FIGURE 8. A transverse intersection in a point (left), and a non-transverse intersection in an interval (right)

We will consider this last case now, and make one additional assumption. If A, B, W are orientable, then we can associate a sign with each (transverse) intersection point. To get this sign, we can use the following procedure. Take an ordered basis of T_pA, say $e_1, \ldots, e_k$, an ordered basis of T_pB, say $f_1 \ldots, f_\ell$ matching the orientations of A and B. By the transversality assumption, $e_1, \ldots, e_k, f_1, \ldots, f_\ell$ forms an ordered basis of T_pW. If this ordered basis matches the orientation on W, assign to the intersection point $+1$. If not, assign it -1.

These signs can be used to compute the intersection product via the formula

$$[A] \cdot [B] := \sum_{p \in A \cap B} sign(p).$$

Here $[A]$ denotes the homology class represented by A. In fact, the intersection product is a bilinear form on homology,

$$H_k(W; \mathbb{Z}) \times H_\ell(W; \mathbb{Z}) \longrightarrow \mathbb{Z}, ([A], [B]) \longmapsto [A] \cdot [B].$$

and the above procedure provides a recipe to compute it for homology classes that are represented by submanifolds. We will denote the intersection product by S, so $S([A], [B]) = [A] \cdot [B]$.

1.4.1. *Some examples.* The variety $V_1(x^3+y^2)$ is a punctured torus. Taking a basis as indicated in the Figure 9 on the left, we find the intersection product

$$S(V_1(x^3 + y^2)) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In this dimension (the real dimension of the variety is 2), the intersection product is a skew-symmetric bilinear form, but in real dimension 4, 8, etc, it is a symmetric bilinear form; just look at the orientations at the intersection point to see the skew-symmetry in the surface case. On the right in Figure 9 we have a more complicated example with 8-dimensional homology in degree 1. The associated matrix is hence a little big, so we will describe the intersection form using a graph, which is here more economical. We get this intersection graph from the intersection form S by associating a vertex with each basis element in homology, and an edge if two homology classes have intersection number $+1$ (so we need to take care with the order). Self-intersection can be encoded by an additional number for each vertex, but in the case of smoothed hypersurface singularities, this number is either 0 or ± 2 depending on the dimension; the reason for this is that we can choose so-called Lagrangian spheres as a basis of homology, and these have neighborhoods that are isomorphic to T^*S^n. This gives in turn the claimed self-intersection numbers.

The following theorem tells us how the intersection form helps us.

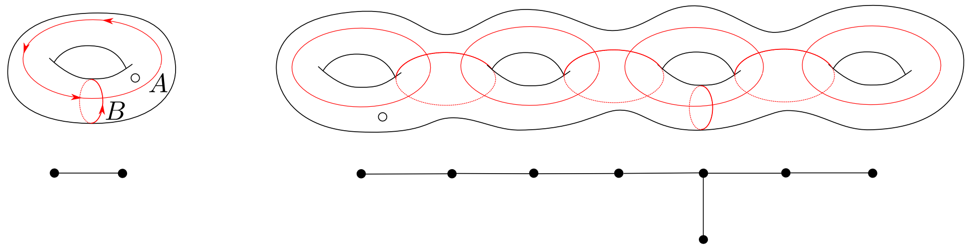

Figure 9. A basis of the homology of $V_1(x^3 + y^2)$ (left) and of $V_1(x^3 + y^5)$ (right) with their corresponding intersection graphs. The graph on the right is the E_8-graph.

Theorem 1.12. *Suppose that $p \in \mathbb{C}[x_0, \dots, x_n]$ is a polynomial with an isolated singularity at 0. Denote its Morsification by $\tilde{p}$. Let S denote the intersection product of $V_0(\tilde{p})$. We have the following:*

- *the link $L_\delta(p)$ is $n-2$-connected. This means that homotopy groups $\pi_0, \dots \pi_{n-2}$ all vanish.*
- *$\tilde{H}_{n-1}(L_\delta(p); \mathbb{Z}) \cong \operatorname{coker} S$.*

One can prove this theorem by combining the results from Milnor's book [M2] (section on the "Milnor" number μ) with the story in the book of Bredon, [B, Chapter VI, 18 (plumbing)].

From this theorem we see in particular that we can obtain the number of boundary components. It is simply $\operatorname{rk} \operatorname{coker} S + 1$. Another corollary, which we will need later is the following.

Corollary 1.13. *The link $\Sigma := L_\delta(z_0^3 + z_1^5 + \sum_{j=2}^4 z_j^2)$ is homeomorphic to S^7.*

Proof. Here is a quick way to see this using some very powerful theorems. [3] We apply the above theorem and see that we need the intersection form S. This turns out to be the E_8-graph, too, but now in dimension 8. Its cokernel is trivial, so the middle-dimensional homology vanishes. By Poincaré duality, we see that Σ has the same homology as that of S^7, and in addition it is simply-connected, again by the theorem. By the generalized Poincaré conjecture the claim follows. □

2. Exotic spheres as links

We will now see that links of singularities can be exotic spheres. Exotic spheres are manifolds that are homeomorphic, but not diffeomorphic to spheres. The existence of such spaces was found by Milnor, and we adapt his first argument here.

To describe Milnor's setup, let us consider a smooth, oriented 7-manifold M that is boundary of an 8-manifold W. One may believe that this is a restriction, but cobordism theory tells us that this is not the case. In our case, we have even explicit fillings, so we don't need to rely on cobordism theory directly.

Given this data, define

$$\lambda(M) := \big(2\langle p_1(TW)^2, [W]\rangle - \sigma(W)\big) \mod 7.$$

[3] You may view this as a mathematical "crime" since we are using theorems to prove a lemma. A direct proof is longer though.

Here $p_1(TW)$ is the first Pontryagin class of TW, a characteristic class that you have seen in the course on characteristic classes. In our case TW is actually trivial, so for us $p_1(TW) = 0$. The number $\sigma(W)$ is the signature of W, the number of positive eigenvalues minus the number of negative eigenvalues of the intersection form of W.

Claim: the number $\lambda(M)$ is independent of the choice of W.

Proof. The main ingredient is the Hirzebruch signature theorem. In dimension 8 it says the following. Suppose that we are given a closed, oriented manifold C^8. Then its signature can be computed as

$$\sigma(C^8) = \langle \frac{1}{45}(7p_2(TC) - p_1(TC)^2, [C]\rangle.$$

From this theorem we see that $45\sigma \equiv -\langle p_1^2, [C]\rangle \mod 7$. Rewriting gives as

$$6\sigma + 2\langle p_1^2, [C]\rangle \mod 7 \equiv 0 \text{ or } 2\langle p_1^2, [C]\rangle - \sigma \mod 7 \equiv 0.$$

How do we get the claimed independence? Suppose that W and W' are fillings for M inducing the same orientation on M. Form $C := W \cup_\partial \overline{W'}$ (we flip the orientation on W' to obtain an oriented manifold). Since σ and p_1^2 are additive

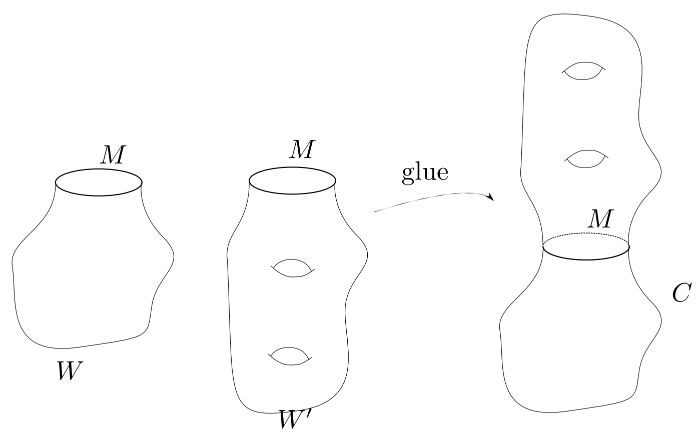

FIGURE 10. Gluing two fillings in order to apply Hirzebruch's theorem

under this operation, the claim follows. □

Corollary 2.1. *For the standard sphere we have $\lambda(S^7) = 0$. For the link Σ of a singularity with intersection graph E_8, we have $\lambda(\Sigma) \neq 0$ In particular, S^7 is not diffeomorphic to Σ.*

The first claim is clear, since $S^7 = \partial B^8$, and $H^4(B^8; \mathbb{Z}) = 0$. For the second claim, we use that $\Sigma = \partial V_\delta$ and the fact that TV_δ is trivial. The signature of the E_8-graph is $+8$, so the claim follows.

Remark 2.2. To see that Σ is really an exotic sphere, we need to check that S^7 and Σ are homeomorphic. We did this in Corollary 1.13. Milnor checked this in a different setup using Reeb's theorem which asserts that a smooth function on a closed manifold with two critical points is homeomorphic to a sphere: this is a much more direct proof than the one we sketched.

Remark 2.3. One may wonder whether every exotic sphere is the link of a singularity. This is not the case, but it is true that every exotic sphere that is boundary-parallelizable (meaning the boundary of a parallelizable manifold) is the link of a singularity.

2.1. **Symplectic and contact topology.** As a final remark, let us mention that Mumford's theorem can be made to work in higher dimensions by including extra structure. Namely the contact structure. Links of singularities always carry a contact structure, and McLean has recently shown in [Mc] how this can be used in a general setup.

Theorem 2.4 (McLean). *Suppose that* $(X, 0)$ *is a normal* 3*-fold singularity. Then* 0 *is a smooth point if and only if* $(L_\delta(0), \xi) \cong (S^5, \xi_0)$.

In other words, the link of a point needs to be the standard smooth sphere equipped with the standard contact structure for a point to be smooth.

Remark 2.5. In the special setup of hypersurface singularities proving McLean's theorem can be done with simpler means than the ones McLean used.

3. A guide to the literature

The book of Milnor, [M2], is very good, concise and to the point. Milnor is an excellent writer, so this book is highly recommended. The book of Dimca, [D], contains several proofs of more advanced theorems such as Mumford's theorem. The rather unusual definition of the Milnor number in these notes and proofs of relevant theorems can be found in Ebeling's book, [E].

The discussion of the intersection product can be found in more detail in the book of Bredon, [B]. For the exposition on exotic spheres, I have two recommendations. One is Milnor's original paper, [M1]. Although Milnor had a different setup consisting of sphere bundles over spheres, I have just applied his argument here. For a detailed exposition of links and exotic spheres, see the book of Hirzebruch and Mayer, [HM]. It is really an excellent book, but it is in German. I am not aware of any translation.

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