A mathematical model for measurements

Aki Kishimoto, at Cheongpung, Aug. 2014

My apology: There is nothing new mathematically in this talk.

Just a proposal to the measurement problem, which I believe is new but does not seem to entail any consequences (on physics).

My hope is advance study of semigroup of endomorphisms on C*-algebras which may be valid to describe certain physical models like measurement apparatus.

1 Measuring process due to von Neumann

- The quantum system: $\mathcal{H}$, a separable Hilbert space or $\mathcal{K}(\mathcal{H})$ the C*-algebra of compact operators on $\mathcal{H}$.
- The measuring process: $(\mathcal{L}, \phi, M, U)$, where

  $\mathcal{L}$ is a separable Hilbert space,

  ϕ is a state of the compact operators $\mathcal{K}(\mathcal{L})$,

  M is a self-adjoint operator on $\mathcal{L}$,

  U is a unitary on $\mathcal{H} \otimes \mathcal{L}$.

Comment: Since I am not comfortable enough with this subject I want to add I am following Masanao Ozawa's papers.

Let E_M denote the spectral measure of M.

The measuring process applied to the quantum system in the state φ produces

$$\mathcal{E}(\Delta, \varphi) \in \mathcal{K}(\mathcal{H})^*_+$$

for each Borel subset Δ of $\mathbb{R}$ by

$$\mathcal{E}(\Delta, \varphi)(x) = \varphi \otimes \phi(U^*(x \otimes E_M(\Delta))U), \quad x \in \mathcal{K}.$$

All what we get from the quadruple $(\mathcal{L}, \phi, M, U)$ with U a unitary on $\mathcal{H} \otimes \mathcal{L}$ is

$$\mathcal{E}(\Delta, \varphi), \ \Delta \subset \mathbb{R}, \ \varphi \in \mathcal{K}(\mathcal{H})^*,$$

which is called a *Davies-Lewis instrument* or DL-instrument for short.

The self-adjoint operator M is called a *meter observable* or a *probe observable*.

When the observed system $\mathcal{K}(\mathcal{H})$ is under the state φ and is applied this measuring process, the state $\varphi \otimes \phi$ of the combined system will turn to $(\varphi \otimes \phi)\mathrm{Ad}\, U^*$;
then we are supposed to be able to perform a *precise local* measurement of M to get a real number x whose distribution is given by $\Delta \mapsto \mathcal{E}(\Delta, \varphi)(1)$
so that the ensemble of all states of $\mathcal{K}(\mathcal{H})$ after observing $x \in \Delta$ is given by

$$\mathcal{E}(\Delta, \varphi)/\mathcal{E}(\Delta, \varphi)(1)$$

for each Δ.

(There is a justification for the last statement.)

Given a self-adjoint operator Q on $\mathcal{H}$, observing Q is supposed to be choosing a suitable $(\mathcal{L}, \phi, M, U)$ and applying the above process.

There is a map $X : \Delta \mapsto X(\Delta) \in \mathcal{B}(\mathcal{H})_+$ such that $\mathcal{E}(\Delta, \varphi)(1) = \varphi(X(\Delta))$, $\varphi \in \mathcal{K}(\mathcal{H})^*$. This is a positive operator valued measure on $\mathbb{R}$ such that $X(\mathbb{R}) = 1$. If this is a projection-valued measure, what we observe by $(\mathcal{L}, \phi, M, U)$ is exactly

$$Q = \int \lambda X(d\lambda).$$

X is called a *semiobservable* in general.

This seems to be all well-established except for how to drive the quantum effect on M to the macroscopic level for observation.

Here we propose a (hopefully new) mathematical model for measurements of a quantum system in a C*-algebra setting, which incorporates a mechanism of *magnifying quantum effects to the classical level* into the measuring apparatus.

In this scheme the state of the quantum system transforms to new ones according to a certain probability law just like the phase does to a new one in phase transition we encounter in equilibrium quantum statistical mechanics.

2 A new measuring process

A measuring apparatus is something you can reset, i.e., the state of the apparatus will be identical in the beginning; also something you can extract information from after having the one interact with the quantum system which is prepared in a prescribed way (say, by recording a digital number shown on it), therefore different observed values represent mutually disjoint (classically different) states;

thus the combined system of the apparatus and the quantum system will endure an irreversible time development during a short period of measurement, which will be described by a proper endomorphism, not an automorphism in the C^*-algebra setting.

Hence the state reduction or the collapse of wave-function (of the combined system) is a primary event. Through which one can make an observation of the quantum system.

- The quantum system: $\mathcal{K}(\mathcal{H})$ with the norm topology.
- The measuring process: (A, ϕ, γ, U) where

  A is a unital non-type I nuclear simple C*-algebra.

  ϕ is a state of A,

  γ is an asymptotically inner endomorphism of A such that $\pi_\phi\gamma(A)'$ is a non-trivial abelian von Neumann algebra (or $\pi_\phi\gamma(A)''$ has non-trivial center),

  U is a unitary multiplier of $\mathcal{K}(\mathcal{H}) \otimes A$.

The condition on A is quite arbitrary, which is imposed to guarantee the existence of (ϕ, γ). A general result says that there is such γ for an arbitrary pure state ϕ.

This arbitrariness and not knowing which A should correspond to an apparatus seems to be a fault; and ϕ should also be allowed to be factorial (but then I do not know how to construct γ).

U dictates an interaction between $\mathcal{K}(\mathcal{H})$ and A; U should be a multiplier of $\mathcal{K}(\mathcal{H}) \otimes A$ to make $\mathrm{Ad}U$ leaves $\mathcal{K}(\mathcal{H}) \otimes A$ invariant.

After applying $\mathrm{Ad}U$ and γ we reach a situation similar to the above; instead of M (or the von Neumann algebra generated by M) we will obtain an abelian von Neumann algebra with zero intersection with the compact operators, as the center of the observable algebra, as an outcome of this process. Central elements can be regarded as classical observables.

Let us explain some basic of states and how the endomorphism works.

When ϕ is a state of A, we denote π_ϕ the GNS representation of A on a Hilbert space $\mathcal{H}_\phi$ with a specific unit vector Ω_ϕ such that

$$\phi(x) = \langle \Omega_\phi, \pi_\phi(x)\Omega_\phi \rangle, \ x \in A$$

and $\pi_\phi(A)\Omega_\phi$ is dense in $\mathcal{H}_\phi$.

If ϕ is a pure state, the weak closure of $\pi_\phi(A) = \mathcal{B}(\mathcal{H}_\phi)$. We call ϕ (or π_ϕ) *factorial* if $\mathcal{M} = \pi_\phi(A)''$ is a factor. In particular a pure state is factorial.

If γ is a unital endomorphism of A with $\gamma(A) \neq A$ and ϕ is a factorial state then $\phi\gamma$ is a state but may not be factorial, i.e., $\mathcal{M} \cap \mathcal{M}'$ may not be $\mathbb{C}1$ where $\mathcal{M} = \pi_{\phi\gamma}(A)''$. (If $\phi\gamma$ is factorial $\pi_\phi\gamma$ may not be factorial.) In this case $\phi\gamma$ is centrally decomposed in the sense that there is a unique probability measure ν on the Borel subset $\mathcal{F}$ of factorial states in A^* with

$$\phi\gamma = \int_{\mathcal{F}} \psi \mathrm{d}\nu(\psi),$$

where $\mathcal{M} \cap \mathcal{M}'$ on the left could be identified with $L^\infty(\mathcal{F}, \nu)$ on the right behind this equality.

A factorial state is supposed to correspond to a *phase* (or *pure phase*) in statistical mechanics; if ϕ transforms to $\phi\gamma$ causally but in a irreversible way then it would immediately jump to $\psi \in \mathcal{F}$ acausally according to the probability ν on $\mathcal{F}$.

We also assume that γ is asymptotically inner. Namely we assume that there is a continuous unitary path u_t, $t \in [0,1)$ in A such that

$$\gamma(x) = \lim_{t \to 1} u_t x u_t^*, \ \ x \in A$$

and $u_0 = 1$. Then it follows that there is a bounded sequence (h_n) of self-adjoint elements of A such that $\lim_n [h_n, x] = 0$ and

$$\gamma(x) = \lim_n \mathrm{Ad}(e^{ih_1} e^{ih_2} \cdots e^{ih_n})(x), \quad x \in A.$$

We regard γ as a time development as being a limit of Hamiltonean induced time developments which are cascading quantum effects to the visible classical level within a small time interval. Thus γ describes an irreversible process.

If the quantum system is in a state φ and the measuring apparatus A is in a pure state ϕ, then we suppose the following transition of states of $\mathcal{K}(\mathcal{H}) \otimes A$:

$$\varphi \otimes \phi \rightsquigarrow (\varphi \otimes \phi)\mathrm{Ad}\, U^* \rightsquigarrow (\varphi \otimes \phi)\mathrm{Ad}\, U^*(\mathrm{id} \otimes \gamma),$$

which may not be factorial and then will be centrally decomposed as explained above, or we will witness collapse of the wave function.

We formally give the definition of DL instrument following M. Ozawa.

Let $\mathcal{M}$ be an abelian von Neumann algebra with separable predual and $\mathcal{M}_+$ the cone of positive elements of $\mathcal{M}$. Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space and $\mathcal{K} = \mathcal{K}(\mathcal{H})$ be the C*-algebra of compact operators on $\mathcal{H}$. We call a map $\mathcal{E}$ from $\mathcal{M} \times \mathcal{K}^*$ into $\mathcal{K}^*$ a *DL instrument* based on $\mathcal{M}$ if it satisfies

1. For each $\varphi \in \mathcal{K}^*_+$ the map $\mathcal{M} \ni Q \mapsto \mathcal{E}(Q, \varphi) \in \mathcal{K}^*$ is a positive continuous linear map on $\mathcal{M}$,
2. For each $Q \in \mathcal{M}_+$ the map $\mathcal{K}^* \ni \varphi \mapsto \mathcal{E}(Q, \varphi) \in \mathcal{K}^*$ is a completely positive (CP for short) linear map,
3. $\mathcal{E}(1, \varphi)(1) = \varphi(1)$ for all $\varphi \in \mathcal{K}^*$,

where $\mathcal{M}$ is equipped with the weak*-topology.

If we denote by $\mathcal{E}(Q)$ the linear map $\mathcal{K}^* \ni \varphi \mapsto \mathcal{E}(Q, \varphi) \in \mathcal{K}^*$ with $Q \in \mathcal{M}_+$, the dual map $\mathcal{E}(Q)^* : \mathcal{K}^{**} = \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is completely positive or CP, i.e., the natural extension of $\mathcal{E}(Q)^*$ to a map from $M_k \otimes \mathcal{B}(\mathcal{H})$ into $M_k \otimes \mathcal{B}(\mathcal{H})$ is positive for any $k \in \mathbb{N}$, which follows from the complete positivity of $\mathcal{E}(Q)$.

We denote $\mathcal{E}(Q)^* b$ by $\mathcal{E}^*(Q, b)$ for $b \in \mathcal{B}(\mathcal{H})$; then for each $b \in \mathcal{B}(\mathcal{H})_+$ the map

$$Q \mapsto \mathcal{E}^*(Q, b)$$

is a positive continuous linear map when $\mathcal{M}$ and $\mathcal{B}(\mathcal{H})$ are endowed with the weak*-topology. For $Q \in \mathcal{M}_+$ the map

$$b \mapsto \mathcal{E}^*(Q, b)$$

is a CP continuous linear map when $\mathcal{B}(\mathcal{H})$ is endowed with the weak*-topology. The third condition of the above definition is equivalent to

$$\mathcal{E}^*(1, 1) = 1.$$

Let (A, ϕ, γ, U) be a measuring process and let $\mathcal{M} = \pi_\phi \gamma(A)'$. For each $Q \in \mathcal{M}$ and $\varphi \in \mathcal{K}^*$ define an $\mathcal{E}(Q, \varphi) \in \mathcal{K}^*$ by

$$\mathcal{E}(Q, \varphi)(x) = \overline{\varphi \otimes \phi}(\mathrm{Ad}\, \bar{U}^*)(x \otimes Q), \quad x \in \mathcal{K},$$

where $\overline{\varphi \otimes \phi}$ is a unique extension of the positive functional $\varphi \otimes \phi \pi_\phi^{-1}$ of $\mathcal{K} \otimes \pi_\phi(A)$ to a weak*-continuous one of $(\mathcal{K} \otimes \pi_\phi(A))'' = \mathcal{B}(\mathcal{H} \otimes \mathcal{H}_\phi)$ and $\bar{U} = (\mathrm{id} \otimes \pi_\phi)(U)$. Then $\mathcal{E}$ is a DL instrument based on $\mathcal{M}$. We call $\mathcal{E}$ the DL instrument *obtained from (A, ϕ, γ, U).*

If the observed system is a general separable C*-algebra B, we specify an irreducible representation π of B and denote by V_π as the linear space consisting of $\varphi \in B^*$ such that $\varphi = f\pi$ for some $f \in \pi(B)''_*$. Then as $V_\pi^* = \pi(B)'' = \mathcal{B}(\mathcal{H}_\pi)$ each π and a measuring process (A, ϕ, γ, U) for B define a DL instrument $\mathcal{E}(\varphi, Q) \in V_\pi$ for $\varphi \in V_\pi, Q \in \mathcal{M}$ just as above. (But in this case the choice of U may be hampered.) If $B = \mathcal{K}$ then there is essentially only one π.

Note that we only use $\mathcal{M} = \pi_\phi\gamma(A)'$ for construction of the DL instrument $\mathcal{E}$, not γ directly. In this sense the present scheme is not much different from the original one due to von Neumann on the technical level. But we hope that the present model makes a contribution for a clarification on the conceptual level.

When $\mathcal{E}_1$ and $\mathcal{E}_2$ are DL instruments based on $\mathcal{M}$ and (ξ_n) is a dense sequence in the unit sphere of $\mathcal{H}$ we define $d(\mathcal{E}_1, \mathcal{E}_2) \geq 0$ by

$$d(\mathcal{E}_1, \mathcal{E}_2) = \sum_n 2^{-n} \|\mathcal{E}_1(\,\cdot\,, \psi_n) - \mathcal{E}_2(\,\cdot\,, \psi_n)\|,$$

where ψ_n is the vector state of $\mathcal{K}$ defined by ξ_n. It follows that d is a distance on the set of DL instruments based on $\mathcal{M}$.

We can show the following:

Let $\mathcal{M}$ be an abelian von Neumann algebra with separable predual. Then in the set of all DL instruments based on $\mathcal{M}$ is dense the set of DL instruments obtained from the measuring processes (A, ϕ, γ, U) with $\mathcal{M} = \pi_\phi \gamma(A)'$.

We will sketch how to prove this. First of all we have to show that there is an asymptotically inner endomorphism γ and an irreducible representation π of some unital separable non-type I nuclear C*-algebra A such that $\mathcal{M} \cong \pi\gamma(A)'$. This is indeed possible for any unital separable non-type I nuclear C*-algebra, whose proof requires Glimm's result (which shows UHF algebras are typical examples of non-type I C*-algebras), the existence result on endomorphisms (for non-type I nuclear C*-algebras), and the following well-known statement on UHF algebras: For any such $\mathcal{M}$ as above there is a representation ρ such that $\rho(A)' \cong \mathcal{M}$.

Thus we prepare (A, γ) and some irreducible representation π with $\pi\gamma(A)' \cong \mathcal{M}$.

Secondly by Ozawa's results all the DL instruments are realized by the measuring processes (stated in the beginning). For the proof we use the fact that $\mathcal{M} \times \mathcal{B}(\mathcal{H}) \ni (Q, b) \mapsto \mathcal{E}^*(Q, b) \in \mathcal{B}(\mathcal{H})$ is a completely positive, weak*-continuous bilinear map and express this map as the restriction of a *faithful* weak*-continuous representation of $\mathcal{M} \otimes \mathcal{B}(\mathcal{H})$. Namely for a DL instrument $\mathcal{E}$ based on $\mathcal{M}$ one finds a separable Hilbert space $\mathcal{L}$, a pure state ϕ of $\mathcal{K}(\mathcal{L})$, a normal unital representation ρ of $\mathcal{M}$ on $\mathcal{L}$, and a unitary U on $\mathcal{H} \otimes \mathcal{L}$ such that

$$\mathcal{E}(Q, \varphi)(x) = \overline{\varphi \otimes \phi}(\mathrm{Ad}\, U^*(x \otimes \rho(Q))), \quad Q \in \mathcal{M}, \ x \in \mathcal{K}.$$

We may assume that $\rho(\mathcal{M}) \cap \mathcal{K}(\mathcal{L}) = \{0\}$ by tensoring $\mathcal{L}$ by another infinite-dimensional separable Hilbert space if necessary and making obvious arrangements. Then we outfit an irreducible representation π of A on $\mathcal{L}$ such that $\pi\gamma(A)' = \rho(\mathcal{M})$.

Since this is done independently of U, we cannot expect that $U \in M(\mathcal{K} \otimes \pi(A))$. But, noting that $(\mathcal{K} \otimes \pi(A))'' = \mathcal{B}(\mathcal{H} \otimes \mathcal{L})$, Kadison's transitivity (or Kaplansky's density theorem) tells us that one can approximate U by a unitary $u \in \mathcal{K} \otimes \pi(A) + \mathbb{C}1$. Thus we can replace U by a unitary in $M(\mathcal{K} \otimes A)$ so that the resulting DL instrument is arbitrarily close to $\mathcal{E}$.

3 The case $\pi\gamma(A)' \cong \ell^\infty(\mathbb{N})$

Let A be a unital separable non-type I nuclear simple C*-algebra and let ϕ be a pure state of A. Let γ be an asymptotically inner endomorphism of A such that $\pi_\phi\gamma(A)'$ is commutative. The existence of such γ has been proven. Let U be a unitary in $M(\mathcal{K} \otimes A)$. We will describe how the system (A, ϕ, γ, U) works as a measuring apparatus for the observed quantum system $\mathcal{K}$.

We denote by id the identity representation of $\mathcal{K} = \mathcal{K}(\mathcal{H})$ on $\mathcal{H}$. Let φ be a state of $\mathcal{K}$, which extends to a normal state of $\mathcal{B}(\mathcal{H}) = \mathcal{K}(\mathcal{H})''$. Then through the interaction with (A, ϕ) the state $\varphi \otimes \phi$ of the combined system $\mathcal{K} \otimes A$ changes to $(\varphi \otimes \phi)\mathrm{Ad}\, U^*$, and then to $T(\varphi) = (\varphi \otimes \phi)\mathrm{Ad}\, U^*(\mathrm{id} \otimes \gamma)$. Let $\pi_0 = (\mathrm{id} \otimes \pi_\phi)\mathrm{Ad}\, U^*(\mathrm{id} \otimes \gamma)$, which is a representation of $\mathcal{K} \otimes A$ on the Hilbert space $\mathcal{H} \otimes \mathcal{H}_\phi$. Then

$$\pi_0(\mathcal{K} \otimes A)' = \mathrm{Ad}\, \bar{U}^*(\mathbb{C}1 \otimes \pi_\phi\gamma(A)'),$$

where $\bar{U} = (\mathrm{id} \otimes \pi_\phi)(U)$.

Suppose that $\pi_\phi\gamma(A)' \cong \ell^\infty(\mathbb{N})$, i.e., it is generated by minimal projections $E_1, E_2, \ldots$ on $\mathcal{H}_\phi$. Since $x \mapsto \pi_\phi\gamma(x)E_i$ is an irreducible representation of A on $E_i\mathcal{H}_\phi$, E_i is of infinite rank. Let $F_i = \mathrm{Ad}\,\bar{U}^*(1 \otimes E_i)$, which is a minimal projection of the center of $\pi_0(\mathcal{K} \otimes A)''$. If $\overline{\varphi \otimes \phi}$ denotes the natural extension to a normal state of $\mathcal{B}(\mathcal{H} \otimes \mathcal{H}_\phi)$ then

$$T(\varphi) = \sum_{i=1}^{\infty} \overline{\varphi \otimes \phi}(F_i\pi_0(\,\cdot\,)).$$

Since F_i is a minimal projection in $\pi_0(\mathcal{K} \otimes A)'$, the state

$$\omega_i = \overline{\varphi \otimes \phi}(F_i\pi_0(\,\cdot\,))/\overline{\varphi \otimes \phi}(F_i)$$

is a pure state of $\mathcal{K} \otimes A$ for $\overline{\varphi \otimes \phi}(F_i) > 0$. Since F_i's are mutually orthogonal central projections, ω_i's are mutually disjoint. Hence $T(\varphi)$ is the sum of phases with weights and Nature will pick up one according to the probability specified by $(\overline{\varphi \otimes \phi}(F_i))$.

There is a positive operator P_i in $\mathcal{B}(\mathcal{H}) = \mathcal{K}(\mathcal{H})^{**}$ such that

$$\varphi(P_i) = \overline{\varphi \otimes \phi}(F_i), \quad \varphi \in \mathcal{K}^*,$$

where $\sum_i P_i = 1$. Note that restriction of $\overline{\varphi \otimes \phi}(F_i \pi_0(\,\cdot\,))$ to $\mathcal{K}$ is

$$\mathcal{E}(E_i, \varphi) = \overline{\varphi \otimes \phi} \mathrm{Ad}\, \bar{U}^*(\,\cdot\, \otimes E_i)$$

and $\varphi(P_i) = \mathcal{E}(E_i, \varphi)(1)$.

Hence it follows that

$$T(\varphi)|\mathcal{K} = \sum_i \varphi(P_i) \frac{\mathcal{E}(E_i, \varphi)}{\varphi(P_i)},$$

where the sum is over i with $\varphi(P_i) > 0$ and $\varphi_i = \mathcal{E}(E_i, \varphi)/\varphi(P_i)$ is a state of $\mathcal{K}$, not necessarily a pure state. Here is our conclusion: After applying this measuring process to $\mathcal{K}$ Nature will transform φ to φ_i with probability $\varphi(P_i)$ for each $i = 1, 2, \ldots$.

Note that if $U = 1$ then $P_i = \overline{\varphi \otimes \phi}(1 \otimes E_i)1$ is independent of φ. Suppose that ϕ is given as a vector state by a unit vector $\psi_1 \in E_1\mathcal{H}_\phi$. If $U = 1$ then $T(\varphi) = \varphi \otimes \phi\gamma$ is pure and $P_1 = 1$ (and other $P_i = 0$); no information is gained.

We choose a unit vector $\psi_i \in E_i\mathcal{H}_\phi$ for each $i > 1$ and choose a unitary $u_i \in A$ (or $A + \mathbb{C}1$ if A is non-unital) for $i \geq 1$ such that $\pi_\phi(u_i)\psi_1 = \psi_i$. (The existence of such u_i follows from Kadison's transitivity.) We set

$$U = \sum_i e_{ii} \otimes u_i$$

as a multiplier of $\mathcal{K} \otimes A$, where (e_{ij}) are matrix units generating $\mathcal{K}$. Since

$$\bar{U}\xi_i \otimes \psi_1 = \xi_i \otimes \psi_i$$

where (ξ_i) is an orthnormal basis of $\mathcal{H}$ with $e_{ii}\xi_i = \xi_i$, it follows that

$$\overline{\varphi \otimes \phi}(F_i) = \overline{\varphi \otimes \phi}(\bar{U}^*(1 \otimes E_i)\bar{U}) = \varphi(e_{ii})$$

and

$$\varphi_i(x) = \mathrm{Tr}(e_{ii}x), \quad x \in \mathcal{K} \ \text{ if } \varphi(e_{ii}) > 0.$$

Hence for this choice of ϕ and U we obtain

$$T(\varphi)|\mathcal{K} = \sum_i \varphi(e_{ii})\mathrm{Tr}(e_{ii}\,\cdot\,),$$

which is what we expect by measuring e.g., the unbounded observable $\sum_n ne_{nn} \in M(\mathcal{K})$.