One sided invertibility of matrices over commutative rings, corona problems, and Toeplitz operators with matrix symbols

Ilya M. Spitkovsky

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Based on joint work with Cristina Câmara and Leiba Rodman.
Introduction

Given $p \in (1, \infty)$, an $L^p$-factorization of a function $G \in L^\infty(\mathbb{R})^{n \times n}$ is defined as a representation

$$G = G^- + DG^-,$$

where $D$ is a diagonal rational matrix of the form

$$D = \text{diag}(r_{kj})_{j=1,2,...,n},$$

for all $j = 1, 2, ..., n$,

$$r(\xi) = \xi - i\xi + i, \quad \xi \in \mathbb{R},$$

and the factors $G_{\pm}$ are such that, for $p' = p - 1$,

$$\lambda_{\pm}(\xi) = \xi \pm i(\xi \in \mathbb{R}),$$

we have

$$\lambda_{\pm} + G_{\pm} \in (H^p)^{n \times n},$$

$$\lambda_{\pm} - G_{\pm} \in (H^{p'}^p)^{n \times n}.$$
Given $p \in (1, \infty)$, an $L_p$-factorization of a function $G \in (L_\infty(\mathbb{R}))^{n \times n}$ is defined as a representation

$$G = G_- D G_+,$$  \hspace{1cm} (1)

where $D$ is a diagonal rational matrix of the form

$$D = \text{diag} \left( r^{k_j} \right)_{j=1,2,\ldots,n}, \quad k_j \in \mathbb{Z} \text{ for all } j = 1, 2, \ldots, n,$$  \hspace{1cm} (2)

and the factors $G_{\pm}$ are such that, for $p' = \frac{p}{p-1}$,

$$\lambda_{\pm}(\xi) = \xi \pm i, \quad \text{for } \xi \in \mathbb{R},$$  \hspace{1cm} (3)

we have

$$\lambda_{\pm}^{-1} + G_{\pm} \in (H_p)^{n \times n}, \quad \lambda_{\pm}^{-1} + G_{\pm} \in (H_{p'}^{+})^{n \times n},$$  \hspace{1cm} (5)

$$\lambda_{\pm}^{-1} - G_{\pm} \in (H_{p}^{-})^{n \times n}, \quad \lambda_{\pm}^{-1} - G_{\pm} \in (H_{p'}^{-})^{n \times n}.$$  \hspace{1cm} (6)
Given $p \in (1, \infty)$, an $L_p$-factorization of a function $G \in (L_\infty(\mathbb{R}))^{n \times n}$ is defined as a representation

$$G = G_-DG_+,$$  \hspace{1cm} (1)

where $D$ is a diagonal rational matrix of the form

$$D = \text{diag} (r^{k_j})_{j=1,2,...,n}, \quad k_j \in \mathbb{Z} \quad \text{for all } j = 1, 2, \ldots n,$$  \hspace{1cm} (2)

$$r(\xi) = \frac{\xi - i}{\xi + i}, \quad \text{for } \xi \in \mathbb{R},$$  \hspace{1cm} (3)

and the factors $G_\pm$ are such that, for

$$p' = \frac{p}{p - 1}, \quad \lambda_\pm(\xi) = \xi \pm i \quad (\xi \in \mathbb{R}),$$  \hspace{1cm} (4)

we have

$$\lambda_+^{-1}G_+ - 1 \in (H_p^+)^{n \times n}, \quad \lambda_+^{-1}G_+ \in (H_p^+)^{n \times n}$$  \hspace{1cm} (5)

$$\lambda_-^{-1}G_- \in (H_p^-)^{n \times n}, \quad \lambda_-^{-1}G_-^{-1} \in (H_p^-)^{n \times n}. $$  \hspace{1cm} (6)
Under conditions (5), (6), \( G_- P^+ G_-^{-1} I \) can be considered as a closable operator on \( (L_p(\mathbb{R}))^n \) defined on a dense linear manifold \( \lambda_{-1}^{-1} G_+ \mathcal{R}^n \).

If, in addition, \( G_- P^+ G_-^{-1} \) is bounded in the metric of \( (L_p(\mathbb{R}))^n \) (and therefore extends onto \( (L_p(\mathbb{R}))^n \) by continuity), we say that (1) is a Wiener-Hopf (WH) \( p \)-factorization of \( G \).

For each \( p \), the diagonal middle factor in (1) is unique up to the order of its diagonal elements, and the integers \( k_j \) are called the partial indices of \( G \), its sum \( \text{Ind}_p(G) \) being the (total) \( p \)-index of \( G \).

In the case of a scalar symbol possessing a WH \( p \)-factorization, the partial and the total indices coincide and will be simply called the \( p \)-index of \( G \).

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One sided invertibility, corona problem, and applications
Under conditions (5), (6), $G_- P^+ G_-^{-1} I$ can be considered as a closable operator on $(L_p(\mathbb{R}))^n$ defined on a dense linear manifold $\lambda_-^{-1} G_+ \mathcal{R}^n$.

If, in addition,

$$G_- P^+ G_-^{-1} I$$

is bounded in the metric of $(L_p(\mathbb{R}))^n$ (7) (and therefore extends onto $(L_p(\mathbb{R}))^n$ by continuity), we say that (1) is a *Wiener-Hopf (WH) $p$-factorization of $G$. 
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If, in addition,

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G_-P^+G_-^{-1}I \text{ is bounded in the metric of } (L_p(\mathbb{R}))^n \quad (7)
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(and therefore extends onto \((L_p(\mathbb{R}))^n\) by continuity), we say that (1) is a \textit{Wiener-Hopf (WH) \(p\)-factorization} of \(G\).

For each \(p\), the diagonal middle factor in (1) is unique up to the order of its diagonal elements, and the integers \(k_j\) are called the \textit{partial indices} of \(G\), its sum \(\text{Ind}_p(G)\) being the (total) \textit{\(p\)-index} of \(G\).
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The factorization (1) is said to be \textit{bounded} if

\[ \begin{align*}
G_+ & \in \mathcal{G}(H_\infty^+)^{n\times n}, & G_- & \in \mathcal{G}(H_\infty^-)^{n\times n}. 
\end{align*} \] (8)

Clearly, a bounded factorization is a WH \( p \)-factorization for all \( p \in ]1, +\infty[. \)
The factorization (1) is said to be *bounded* if

\[ G_+ \in \mathcal{G}(H^+_{\infty})^{n \times n}, \quad G_- \in \mathcal{G}(H^-_{\infty})^{n \times n}. \]  

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Clearly, a bounded factorization is a WH $p$-factorization for all $p \in ]1, +\infty[.$

Any matrix function in $\mathcal{GR}^{n \times n}$ admits a factorization (1) with $G_{\pm} \in \mathcal{G}(\mathcal{R}^{\pm})^{n \times n}.$ In particular, every scalar function in $\mathcal{GR}$ is the product of functions in $\mathcal{GR}^+, \mathcal{GR}^-$, and some integer power of the function $r$ defined by (3).
The factorization (1) is said to be \textit{bounded} if

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Any matrix function in \( \mathcal{GR}^n \) admits a factorization (1) with \( G_\pm \in \mathcal{G}(\mathcal{R}^\pm)_{n \times n} \). In particular, every scalar function in \( \mathcal{GR} \) is the product of functions in \( \mathcal{GR}^+, \mathcal{GR}^- \), and some integer power of the function \( r \) defined by (3).

Factorization (1) is called \textit{canonical} if all the \( k_j \)'s are zeros, in which case \( D = I \).
Representation (1) in which $G_{\pm}$ satisfy

$$G_+ \in \mathcal{G}(AP^+)^{n \times n}, \quad G_- \in \mathcal{G}(AP^-)^{n \times n} \quad (9)$$

and the diagonal elements of $D$ have the form $e_{\mu_j}$, as opposed to (2), is called a (right) $AP$ factorization of $G$. Of course, a canonical (that is, satisfying $\mu_1 = \ldots = \mu_n = 0$) $AP$ factorization of $G$ is at the same time a bounded canonical factorization.
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and the diagonal elements of $D$ have the form $e_{\mu_j}$, as opposed to (2), is called a (right) AP factorization of $G$. An AP factorization of $G$ is by definition its $APW$ factorization if conditions (9) are strengthened to

$$G_+ \in \mathcal{G}(APW^+)^{n\times n}, \quad G_- \in \mathcal{G}(APW^-)^{n\times n}.$$ 

The real parameters $\mu_j$ are defined uniquely, provided that an AP (or APW) factorization of $G$ exists, and are called its partial AP indices. Of course, a canonical (that is, satisfying $\mu_1 = \ldots = \mu_n = 0$) AP factorization of $G$ is at the same time a bounded canonical factorization.
Representation (1) in which $G_{\pm}$ satisfy

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**Toeplitz operators** with matrix *symbol* $G \in (L_\infty(\mathbb{R}))^{n \times n}$ are defined as follows:

\[
T_G : (H_p^+)^n \rightarrow (H_p^+)^n, \quad T_G \phi^+ = P^+ G \phi^+ \quad (p \in ]1, +\infty[). \tag{10}
\]
Toeplitz operators with matrix symbol $G \in (L_\infty(\mathbb{R}))^{n \times n}$ are defined as follows:

$$T_G : (H_p^+)^n \rightarrow (H_p^+)^n, \quad T_G \phi^+ = P^+ G \phi^+ \quad (p \in ]1, +\infty[).$$

The relation between Fredholm properties of $T_G$ and factorization (1) is well known.

**Theorem**

Let $G \in (L_\infty(\mathbb{R}))^{n \times n}$, $p \in ]1, +\infty[$. Then $T_G$ is Fredholm on $(H_p^+)^n$ if and only if $G$ admits a WH $p$-factorization.
The partial indices are related to the dimension of the kernel and the cokernel of $T_G$ ($\text{coker } T_G := (H_p^+)^n/\text{Im } T_G$) by

$$\dim \ker T_G = \sum_{k_j \leq 0} |k_j|, \quad \dim \text{coker } T_G = \sum_{k_j \geq 0} k_j.$$  \hspace{1cm} (11)

Thus, the index of $T_G$, $\text{Ind } T_G$, is given by

$$\text{Ind } T_G := \dim (\ker T_G) - \dim (\text{coker } T_G) = -\text{Ind}_p G.$$
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$$\dim \ker T_G = \sum_{k_j \leq 0} |k_j|, \quad \dim \text{coker } T_G = \sum_{k_j \geq 0} k_j. \quad (11)$$

Thus, the index of $T_G$, Ind $T_G$, is given by

$$\text{Ind } T_G := \dim (\ker T_G) - \dim (\text{coker } T_G) = -\text{Ind}_p G.$$

We see thus that the existence of a canonical $p$-factorization for $G$ is particularly interesting, since it is equivalent to invertibility for $T_G$. Moreover, the inverse operator can then be defined in terms of $G_\pm$ by

$$T_G^{-1} = G_+^{-1} P^+ G_-^{-1} I. \quad (12)$$
Factorability criteria are known for several important classes of matrix functions. In particular, a matrix function $G \in C^{n \times n}$ is WH $p$-factorable if and only if it is invertible. This result does not depend on $p \in ]1, \infty [$, and also implies that $G$ is factorable only simultaneously with the scalar function $\det G$. 
Factorability criteria are known for several important classes of matrix functions. In particular, a matrix function $G \in C^{n \times n}$ is $WH$ $p$-factorable if and only if it is invertible. This result does not depend on $p \in ]1, \infty [$, and also implies that $G$ is factorable only simultaneously with the scalar function $\det G$. These nice properties both fail when passing to more general classes, starting already with piecewise continuous on $\mathbb{R}$ with at least one point of discontinuity.
Factorability criteria are known for several important classes of matrix functions. In particular, a matrix function $G \in \mathbb{C}^{n \times n}$ is $WH$ $p$-factorable if and only if it is invertible. This result does not depend on $p \in ]1, \infty[$, and also implies that $G$ is factorable only simultaneously with the scalar function $\det G$. These nice properties both fail when passing to more general classes, starting already with piecewise continuous on $\mathbb{R}$ with at least one point of discontinuity. For $AP$ (and even $APW$ matrix functions the situation is even more intriguing: while scalar $APW$ functions admit an $APW$ factorization if and only if they are invertible, starting with $n = 2$ the $AP$ factorability criteria are presently not known.
One sided invertibility of matrices with elements from an abstract commutative ring

Let \( A \) be a unital commutative ring. We say that an element \( a \in A^{n \times k}, \ k \leq n \), is left invertible over \( A \) if there exists \( b \in A^{k \times n} \) such that \( ba = I_k \), the identity matrix in \( A^{k \times k} \). The notion of right invertibility over \( A \) is introduced in a similar way.

Lemma
Let \( \Phi \in A^{n \times m} \) with \( m \leq n \), and let \( \Phi_I \) be some \( m \times m \) submatrix of \( \Phi \). Denote by \( \Delta_{\Phi_I}^{pq} \) the determinant of the matrix obtained from \( \Phi_I \) by deleting its \( p \)-th row and \( q \)-th column. Define \( \Phi^*_I \in A^{m \times n} \) by setting its \((q,p)\)-entry according to the formula

\[
\Phi^*_q \Phi = \text{det} \Phi_I \text{diag}[(-1)^p q]_{q=1}^{m}.
\]

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Let $A$ be a unital commutative ring. We say that an element $a \in A^{n \times k}$, $k \leq n$, is \textit{left invertible} over $A$ if there exists $b \in A^{k \times n}$ such that $ba = I_k$, the identity matrix in $A^{k \times k}$. The notion of right invertibility over $A$ is introduced in a similar way.
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**Lemma**

Let $\Phi \in A^{n \times m}$ with $m \leq n$, and let $\Phi_I$ be some $m \times m$ submatrix of $\Phi$. Denote by $\Delta_P^{\Phi_I}$ the determinant of the matrix obtained from $\Phi_I$ by deleting its $p$-th row and $q$-th column. Define $\Phi^*_I \in A^{m \times n}$ by setting its $(q, p)$-entry according to the formula

$$
\Phi^*_{qp} = \begin{cases} 
(-1)^{p+q} \Delta_P^{\Phi_I} & \text{if } p \in I, \\
0 & \text{otherwise.}
\end{cases}
$$

(13)

Then

$$
\Phi^*_I \Phi = \det \Phi_I \text{diag}[(-1)^q]_{q=1,\ldots,m}.
$$
For $m \leq n$, label by $I_1, I_2, \ldots, I_N$ all $N = \binom{n}{m}$ subsets of $\{1, \ldots, n\}$ with $m$ elements. If $\Phi \in \mathcal{A}^{n \times m}$, let us denote $d_k^\Phi := \det \Phi_{I_k}$.
Theorem

(i) An element $\Phi$ of $\mathcal{A}^{n \times m}$ is left invertible over $\mathcal{A}$ if and only if

$$\Delta = \begin{bmatrix} d_1^\Phi, d_2^\Phi, \ldots, d_N^\Phi \end{bmatrix}^T$$

is left invertible in $\mathcal{A}$.

(ii) If $\Phi \in \mathcal{A}^{n \times m}$ is left invertible over $\mathcal{A}$ with left inverse $\Psi \in \mathcal{A}^{m \times n}$, then $\Delta^* = \begin{bmatrix} d_{1}^\Psi, d_{2}^\Psi, \ldots, d_{N}^\Psi \end{bmatrix}$ is a left inverse of $\Delta$.

(iii) If $\Delta$ is left invertible in $\mathcal{A}$ with a left inverse $\Delta^* = \begin{bmatrix} \Delta^*_1, \Delta^*_2, \ldots, \Delta^*_N \end{bmatrix}$, then

$$\Psi = \text{diag}\left[ (-1)^q \right]_{q=1}^{nN} \sum_{k=1}^{N} \Delta^*_k \Phi^* I_k,$$

where $\Phi^* I_k$ are defined in accordance with (13) for $I = I_k$, is a left inverse of $\Phi$. 

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Theorem

(i) An element $\Phi$ of $\mathcal{A}^{n \times m}$ is left invertible over $\mathcal{A}$ if and only if

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\Delta = \begin{bmatrix} d_1^\Phi, d_2^\Phi, \ldots, d_N^\Phi \end{bmatrix}^T
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is left invertible in $\mathcal{A}$. (ii) If $\Phi \in \mathcal{A}^{n \times m}$ is left invertible over $\mathcal{A}$ with left inverse $\Psi \in \mathcal{A}^{m \times n}$, then $\Delta^* = \begin{bmatrix} d_1^{\Psi^T}, d_2^{\Psi^T}, \ldots, d_N^{\Psi^T} \end{bmatrix}$ is a left inverse of $\Delta$.
Theorem

(i) An element $\Phi$ of $\mathbb{A}^{n \times m}$ is left invertible over $\mathbb{A}$ if and only if

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$$\Psi = \text{diag}[(\frac{-1}{2})^q]_{q=1}^{\frac{n}{2}} \sum_{k=1}^{N} \Delta_k^* \Phi_{l_k}^*,$$

where $\Phi_{l_k}^*$ are defined in accordance with (13) for $l = l_k$, is a left inverse of $\Phi$. 

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One sided invertibility, corona problem, and applications
The “if” part of this theorem is an abstract form of its $H_\infty$ version contained in the proof of [Fuhrman’68], Theorem 3.1.
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The “if” part of this theorem is an abstract form of its \( H_\infty \) version contained in the proof of [Fuhrman’68], Theorem 3.1. In what follows, we adapt the notation to the special case \( m = n - 1 \) which is of particular relevance to the main results of the paper. Given \( \Phi \in \mathcal{A}^{n \times (n-1)} \) we denote by \( \Delta_p;.(\Phi) \) the determinant of the \((n-1) \times (n-1)\) matrix obtained by omitting the row \( p \) in \( \Phi \); we denote by \( \Delta_{p,s;j}(\Phi) \) the determinant of the \((n-2) \times (n-2)\) submatrix of \( \Phi \) obtained by omitting the rows \( p \) and \( s \) \((p \neq s)\) and column \( j \) (we take \( p, s \in \{1, 2, \ldots, n\}, j \in \{1, 2, \ldots, n-1\} \)). Analogously, for \( \Psi \in \mathcal{A}^{(n-1) \times n} \), we use the notation \( \Delta_.;p(\Psi) \) for the determinant of the \((n-1) \times (n-1)\) matrix obtained by omitting the column \( p \) in \( \Psi \); and \( \Delta_{j;p,s}(\Phi) \) stands for the determinant of the \((n-2) \times (n-2)\) submatrix of \( \Psi \) obtained by omitting the columns \( p \) and \( s \) \((p \neq s)\) and row \( j \).
Corollary

An element $\Phi \in \mathcal{A}^{n \times (n-1)}$ is left invertible over $\mathcal{A}$ if and only if

$$
\begin{bmatrix}
\Delta_1;(\Phi) \\
\vdots \\
\Delta_n;(\Phi)
\end{bmatrix}
$$

is left invertible over $\mathcal{A}$.

Moreover, in this case a left inverse of $\Phi$ is given by

$$
\begin{bmatrix}
\Psi_1 \\
\Psi_2 \\
\vdots \\
\Psi_{n-1}
\end{bmatrix}
$$

with

$$
\Psi_j \in \mathcal{A}^{1 \times n}, \quad j = 1, 2, \ldots, n-1.
$$
Corollary

An element $\Phi \in \mathcal{A}^{n \times (n-1)}$ is left invertible over $\mathcal{A}$ if and only if

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\begin{bmatrix}
\Delta_1;\cdot(\Phi) \\
\vdots \\
\Delta_n;\cdot(\Phi)
\end{bmatrix}
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$$
\psi = \begin{bmatrix}
\psi_1 \\
\psi_2 \\
\vdots \\
\psi_{n-1}
\end{bmatrix}, \quad \psi_j \in \mathcal{A}^{1 \times n}, \quad j = 1, 2, \ldots, n-1,
$$

with
\( \psi_j = (-1)^j \Delta^* \begin{bmatrix} 0 & \Delta_{1,2;j} & \Delta_{1,3;j} & \ldots & \Delta_{1,n;j} \\ -\Delta_{1,2;j} & 0 & \Delta_{2,3;j} & \ldots & \Delta_{2,n;j} \\ -\Delta_{1,3;j} & -\Delta_{2,3;j} & 0 & \ldots & \Delta_{3,n;j} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\Delta_{1,n;j} & -\Delta_{2,n;j} & -\Delta_{3,n;j} & \ldots & 0 \end{bmatrix} \cdot \tilde{I}_n, \)  

for \( j = 1, 2, \ldots, n - 1 \), where \( \Delta_{p,s;j} := \Delta_{p,s;j}(\Phi) \), and

\[
\Delta^* = \begin{bmatrix} \Delta_{1;(\psi^T)}, \ldots \Delta_{n;(\psi^T)} \end{bmatrix} \]

is a left inverse of (14) over \( \mathcal{A} \), and

\[
\tilde{I}_n = \text{diag} [1, -1, 1, \ldots, (-1)^{n+1}].
\]
Having these results in mind, we now establish necessary and sufficient conditions for the left invertibility of $n$-tuples in the concrete unital algebras of interest: $H^+_\infty$, $AP^+_\infty$, $M^+_\infty$. 
Having these results in mind, we now establish necessary and sufficient conditions for the left invertibility of \( n \)-tuples in the concrete unital algebras of interest: \( H_\infty^+, AP^+, M_\infty^+ \). We define \textit{corona tuples}, with respect to one of the unital algebras \( H_\infty^+, AP^+, M_\infty^+ \), as follows:

\[
HCT_n^\pm := \left\{ (h_1^\pm, h_2^\pm, \ldots, h_n^\pm): h_j^\pm \in H_\infty^\pm \text{ and } \inf_{z \in \mathbb{C}^\pm} \left( \sum_{j=1}^n |h_j^\pm(z)| \right) > 0 \right\}
\]
Having these results in mind, we now establish necessary and sufficient conditions for the left invertibility of $n$-tuples in the concrete unital algebras of interest: $H_\infty^\pm$, $AP^\pm$, $M_\infty^\pm$. We define *corona tuples*, with respect to one of the unital algebras $H_\infty^\pm$, $AP^\pm$, $M_\infty^\pm$, as follows:

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$$APCT_n^\pm := \left\{ (h_1^\pm, h_2^\pm, \ldots, h_n^\pm)_{j}^\pm \in AP^\pm \text{ and } \inf_{z \in \mathbb{C}^\pm} \left( \sum_{j=1}^{n} |h_j^\pm(z)| \right) > 0 \right\}$$
Having these results in mind, we now establish necessary and sufficient conditions for the left invertibility of $n$-tuples in the concrete unital algebras of interest: $H^+_\infty, AP^+, M^+_\infty$. We define \textit{corona tuples}, with respect to one of the unital algebras $H^\pm_\infty, AP^\pm, M^\pm_\infty$, as follows:

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\text{HCT}_n^\pm := \left\{ (h_1^\pm, h_2^\pm, \ldots, h_n^\pm) : h_j^\pm \in H^\pm_\infty \text{ and } \inf_{z \in \mathbb{C}^\pm} \left( \sum_{j=1}^{n} |h_j^\pm(z)| \right) > 0 \right\}
\]

\[
\text{APCT}_n^\pm := \left\{ (h_1^\pm, h_2^\pm, \ldots, h_n^\pm)_j^\mp \in AP^\pm \text{ and } \inf_{z \in \mathbb{C}^\pm} \left( \sum_{j=1}^{n} |h_j^\pm(z)| \right) > 0 \right\}
\]

\[
\text{MCT}_n^\pm := \left\{ (r_1 h_1^\pm, r_2 h_2^\pm, \ldots, r_n h_n^\pm) : (h_1^\pm, \ldots, h_n^\pm) \in \text{HCT}_n^\pm \text{ and } r_1, \ldots, r_n \in \mathcal{GR} \right\}
\]
Theorem

(a) Let $h_1^\pm, h_2^\pm, \ldots, h_n^\pm \in H_\infty^\pm$. Then $(h_1^\pm, h_2^\pm, \ldots, h_n^\pm) \in HCT_n^\pm$ if and only if \[
\begin{bmatrix}
h_1^\pm \\
h_2^\pm \\
\vdots \\
h_n^\pm
\end{bmatrix}
\] is left invertible over $H_\infty^\pm$, i.e. there exist $g_j \in H_\infty^\pm, j = 1, 2, \ldots, n$, such that $\sum_{j=1}^n g_j h_j = 1$. 

(b) Let $h_1^\pm, h_2^\pm, \ldots, h_n^\pm \in AP_\infty^\pm$. Then $(h_1^\pm, h_2^\pm, \ldots, h_n^\pm) \in APCT_n^\pm$ if and only if 
\[
\begin{bmatrix}
h_1^\pm \\
h_2^\pm \\
\vdots \\
h_n^\pm
\end{bmatrix}
\] is left invertible over $AP_\infty^\pm$. 

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One sided invertibility, corona problem, and applications
Theorem

(a) Let $h_1^\pm, h_2^\pm, \ldots, h_n^\pm \in H_\infty^\pm$. Then $(h_1^\pm, h_2^\pm, \ldots, h_n^\pm) \in HCT_n^\pm$ if and only if

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(b) Let $h_1^\pm, h_2^\pm, \ldots, h_n^\pm \in AP^\pm$. Then $(h_1^\pm, h_2^\pm, \ldots, h_n^\pm) \in APCT_n^\pm$ if and only if

$$\begin{bmatrix} h_1^\pm \\ \vdots \\ h_n^\pm \end{bmatrix}$$

is left invertible over $AP^\pm$. 
The following statements are equivalent for 
\( h_1^\pm, h_2^\pm, \ldots, h_n^\pm \in M_\infty^\pm: \)

(1) \((h_1^\pm, h_2^\pm, \ldots, h_n^\pm) \in \text{MCT}_n^\pm;\)

(2) There exist \( r \in G\mathcal{R} \) and \((g_1, \ldots, g_n) \in \text{HCT}_n^\pm\) such that 
\[ h_j^\pm = rg_j, \quad j = 1, 2, \ldots, n; \]

(3) \[
\begin{bmatrix}
    h_1^\pm \\
    \vdots \\
    h_n^\pm
\end{bmatrix}
\]
is left invertible over \( M_\infty^\pm. \)
(c) The following statements are equivalent for $h_1^\pm, h_2^\pm, \ldots, h_n^\pm \in M_\infty^\pm$:

1. $(h_1^\pm, h_2^\pm, \ldots, h_n^\pm) \in MCT_n^\pm$;

2. There exist $r \in G\mathcal{R}$ and $(g_1, \ldots, g_n) \in HCT_n^\pm$ such that $h_j^\pm = rg_j$, $j = 1, 2, \ldots, n$;

3. \[
\begin{bmatrix}
  h_1^\pm \\
  \vdots \\
  h_n^\pm
\end{bmatrix}
\]
is left invertible over $M_\infty^\pm$.

Part (a) is the classical corona theorem, going back to Carleson [3]. Part (b) is its almost periodic version, in principle contained already in [ArensSinger'56] and stated explicitly in [Xia'85].
The following statements are equivalent for $h_{1}^{\pm}, h_{2}^{\pm}, \ldots, h_{n}^{\pm} \in M_{\infty}^{\pm}$:

1. $(h_{1}^{\pm}, h_{2}^{\pm}, \ldots, h_{n}^{\pm}) \in MCT_{n}^{\pm}$;
2. There exist $r \in GR$ and $(g_{1}, \ldots, g_{n}) \in HCT_{n}^{\pm}$ such that $h_{j}^{\pm} = rg_{j}$, $j = 1, 2, \ldots, n$;
3. $\begin{bmatrix} h_{1}^{\pm} \\ \vdots \\ h_{n}^{\pm} \end{bmatrix}$ is left invertible over $M_{\infty}^{\pm}$.

Part (a) is the classical corona theorem, going back to Carleson [3]. Part (b) is its almost periodic version, in principle contained already in [ArensSinger'56] and stated explicitly in [Xia'85]. Corollary therefore admits the following interpretation.
Theorem

(a) Let $\Phi \in (H^\pm_\infty)^{n \times (n-1)}$. Then $\Phi$ is left invertible over $H^\pm_\infty$ if and only if $(\Delta_1,(\Phi),\ldots,\Delta_n,(\Phi)) \in HCT^\pm_n$. 
Theorem

(a) Let $\Phi \in (H_\infty^\pm)^{n \times (n-1)}$. Then $\Phi$ is left invertible over $H_\infty^\pm$ if and only if $(\Delta_1,.(\Phi), \ldots, \Delta_n,.(\Phi)) \in HCT_n^\pm$.
(b) Let $\Phi \in (AP^\pm)^{n \times (n-1)}$. Then $\Phi$ is left invertible over $AP^\pm$ if and only if $(\Delta_1,.(\Phi), \ldots, \Delta_n,.(\Phi)) \in APCT_n^\pm$. 
Theorem

(a) Let $\Phi \in (H^\pm_{\infty})^{n \times (n-1)}$. Then $\Phi$ is left invertible over $H^\pm_{\infty}$ if and only if $(\Delta_1,. (\Phi), \ldots, \Delta_n,. (\Phi)) \in HCT^\pm_n$.

(b) Let $\Phi \in (AP^\pm)^{n \times (n-1)}$. Then $\Phi$ is left invertible over $AP^\pm$ if and only if $(\Delta_1,. (\Phi), \ldots, \Delta_n,. (\Phi)) \in APCT^\pm_n$.

(c) Let $\Phi \in (M^\pm_{\infty})^{n \times (n-1)}$. Then $\Phi$ is left invertible over $M^\pm_{\infty}$ if and only if $(\Delta_1,. (\Phi), \ldots, \Delta_n,. (\Phi)) \in MCT^\pm_n$.

In all cases (a), (b), (c), formula (15) applies provided $\Phi$ is left invertible over the respective algebra.
Theorem

(a) Let $\Phi \in (H_\infty^\pm)^{n \times (n-1)}$. Then $\Phi$ is left invertible over $H_\infty^\pm$ if and only if $(\Delta_1, (\Phi), \ldots, \Delta_n, (\Phi)) \in HCT_n^\pm$.

(b) Let $\Phi \in (AP^\pm)^{n \times (n-1)}$. Then $\Phi$ is left invertible over $AP^\pm$ if and only if $(\Delta_1, (\Phi), \ldots, \Delta_n, (\Phi)) \in APCT_n^\pm$.

(c) Let $\Phi \in (M_\infty^\pm)^{n \times (n-1)}$. Then $\Phi$ is left invertible over $M_\infty^\pm$ if and only if $(\Delta_1, (\Phi), \ldots, \Delta_n, (\Phi)) \in MCT_n^\pm$.

In all cases (a), (b), (c), formula (15) applies provided $\Phi$ is left invertible over the respective algebra.
According to [2, Theorem 2.7], $\Phi \in (M_{\infty}^+)^{2 \times 1}$ is left invertible in $M_{\infty}^+$ if and only if there exists a matrix function $R \in \mathcal{GR}^{2 \times 2}$ and $f_+ \in HCT_1^+$ such that $\Phi = RF_+$. We here extend this result to include $\Phi \in (M_{\infty}^{\pm})^{n \times (n-1)}$ with arbitrary $n \in \mathbb{N}$. 
According to [2, Theorem 2.7], \( \Phi \in (M_\infty^\pm)^{2 \times 1} \) is left invertible in \( M_\infty^+ \) if and only if there exists a matrix function \( R \in \mathcal{G}\mathcal{R}^{2 \times 2} \) and \( f_+ \in HCT_1^+ \) such that \( \Phi = Rf_+ \). We here extend this result to include \( \Phi \in (M_\infty^\pm)^{n \times (n-1)} \) with arbitrary \( n \in \mathbb{N} \).

**Theorem**

Let \( \Phi \in (M_\infty^\pm)^{n \times (n-1)} \). Then \( \Phi \) is left invertible over \( M_\infty^\pm \) if and only if there exist \( R \in \mathcal{G}\mathcal{R}^{n \times n} \), \( Q \in \mathcal{G}\mathcal{R}^{(n-1) \times (n-1)} \) and \( F \in (H_\infty^\pm)^{n \times (n-1)} \), the latter being left invertible over \( H_\infty^\pm \), such that

\[
\Phi = RFQ. \tag{19}
\]
The following result will be crucial in establishing relations between left invertibility of some matrix functions and Fredholmness of Toeplitz operators.

**Theorem**

Let $\Phi \in A^{n \times (n-1)}$ be left invertible over $A$, and let $\Psi \in A^{(n-1) \times n}$ be its left inverse:

$$
\Psi \Phi = I_{n-1}.
$$

(20)

Let moreover

$$
\Phi_e = [\Phi \ N], \quad \Psi_e = \begin{bmatrix} \Psi \\ \tilde{N} \end{bmatrix}, \quad \text{with} \quad N \in A^{n \times 1}, \quad \tilde{N} \in A^{1 \times n}.
$$

(21)

Then:
(i) if

\[ N = \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_n \end{bmatrix}, \quad N_j = (-1)^{j-1} \Delta_{..j}(\psi), \quad (22) \]

then

\[ \psi_e \Phi_e = \begin{bmatrix} I_{n-1} & 0_{(n-1) \times 1} \\ \hat{N} \Phi & \hat{N} \hat{N} \end{bmatrix}. \quad (23) \]
(i) if 

$$
\mathbf{N} = \begin{bmatrix}
N_1 \\
N_2 \\
\vdots \\
N_n
\end{bmatrix}, \quad N_j = (-1)^{j-1} \Delta_{j;\cdot}(\Psi),
$$

then 

$$
\psi_e \Phi_e = \begin{bmatrix}
I_{n-1} & 0_{(n-1) \times 1} \\
\mathbf{N} \Phi & \mathbf{NN}
\end{bmatrix}.
$$

(ii) if 

$$
\tilde{\mathbf{N}} = \begin{bmatrix}
\tilde{N}_1 \\
\tilde{N}_2 \\
\vdots \\
\tilde{N}_n
\end{bmatrix} \quad \text{with} \quad \tilde{N}_j = (-1)^{j-1} \Delta_{j;\cdot}(\Phi),
$$

then 

$$
\psi_e \Phi_e = \begin{bmatrix}
I_{n-1} & \psi \mathbf{N} \\
0_{1 \times (n-1)} & \tilde{\mathbf{NN}}
\end{bmatrix}.
$$
(iii) if $N$ and $\tilde{N}$ satisfy (22) and (24), respectively, then

$$\Psi_e \Phi_e = \Phi_e \Psi_e = I_n \quad \text{and} \quad \det \Phi_e = \det \Psi_e = (-1)^{n-1}.$$  

(26)
Let $T_j : X_j \rightarrow Y_j$, $j = 1, 2$, be bounded linear operators between Banach spaces $X_j, Y_j$. We say that $T_1$ and $T_2$ are nearly Fredholm equivalent if and only if they are either both Fredholm or both not Fredholm. In the former case we say that they are Fredholm equivalent if in addition $\text{Ind} T_1 = \text{Ind} T_2$, and strictly Fredholm equivalent if their defect numbers coincide, that is, $\dim \ker T_1 = \dim \ker T_2$, $\dim \text{coker} T_1 = \dim \text{coker} T_2$.

Thus, for instance, a Toeplitz operator with matrix symbol $G \in (C(\dot{\mathbb{R}}))^{n \times n}$ is nearly Fredholm equivalent to the Toeplitz operator (with scalar symbol) $T_{\det G}$; the operators $T_G$ and $T_G^*$ are Fredholm equivalent, and if $G$ admits a Wiener-Hopf factorization (1), then $T_G$ is strictly Fredholm equivalent to $T_D$. 

Ilya M. Spitkovsky

One-sided invertibility, corona problem, and applications
One-sided invertibility and Fredholmness of Toeplitz operators

Let $T_j : X_j \rightarrow Y_j$ be bounded linear operators between Banach spaces $X_j, Y_j, j = 1, 2$. 

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One-sided invertibility and Fredholmness of Toeplitz operators

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One-sided invertibility and Fredholmness of Toeplitz operators

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Thus, for instance, a Toeplitz operator with matrix symbol $G \in C(\dot{\mathbb{R}})$ is nearly Fredholm equivalent to the Toeplitz operator (with scalar symbol) $T_{\det G}$; the operators $T_G$ and $T_{G^*}$ are Fredholm equivalent, and if $G$ admits a Wiener-Hopf factorization (1), then $T_G$ is strictly Fredholm equivalent to $T_D$. Ilya M. Spitkovsky
Let $T_j : X_j \to Y_j$ be bounded linear operators between Banach spaces $X_j, Y_j, j = 1, 2$. We say that $T_1$ and $T_2$ are \textit{nearly Fredholm equivalent} if and only if they are either both Fredholm or both not Fredholm. In the former case we say that they are \textit{Fredholm equivalent} if in addition $\text{Ind } T_1 = \text{Ind } T_2$, and \textit{strictly Fredholm equivalent} if their defect numbers coincide, that is, $\dim \ker T_1 = \dim \ker T_2$, $\dim \text{coker } T_1 = \dim \text{coker } T_2$. 

Ilya M. Spitkovsky
Let $T_j : X_j \to Y_j$ be bounded linear operators between Banach spaces $X_j, Y_j, j = 1, 2$. We say that $T_1$ and $T_2$ are nearly Fredholm equivalent if and only if they are either both Fredholm or both not Fredholm. In the former case we say that they are Fredholm equivalent if in addition $\text{Ind } T_1 = \text{Ind } T_2$, and strictly Fredholm equivalent if their defect numbers coincide, that is, $\dim \ker T_1 = \dim \ker T_2$, $\dim \coker T_1 = \dim \coker T_2$.

Thus, for instance, a Toeplitz operator with matrix symbol $G \in (C(\hat{\mathbb{R}}))^{n \times n}$ is nearly Fredholm equivalent to the Toeplitz operator (with scalar symbol) $T_{\det G}$; the operators $T_G$ and $T_{G^*}$ are Fredholm equivalent, and if $G$ admits a Wiener-Hopf factorization (1), then $T_G$ is strictly Fredholm equivalent to $T_D$. 
Here we show that one-sided invertibility over the algebras $M^+_{\infty}$ or $H^+_{\infty}$ of certain matrix functions associated with the $n \times n$ matrix function $G$ implies that the Toeplitz operators $T_G$ with the symbol $G$ and $T_{\det G}$ are at least nearly, and for some classes of $G$, even strictly, Fredholm equivalent. In particular, Coburn’s theorem holds in the latter case.
Here we show that one-sided invertibility over the algebras $M^+_{\infty}$ or $H^+_{\infty}$ of certain matrix functions associated with the $n \times n$ matrix function $G$ implies that the Toeplitz operators $T_G$ with the symbol $G$ and $T_{\det G}$ are at least nearly, and for some classes of $G$, even strictly, Fredholm equivalent. In particular, Coburn’s theorem holds in the latter case.

In what follows, we use $\tilde{P}^\pm$ to denote the projections, from $L_\infty(\mathbb{R})$ into $L^\pm_p := (\xi \pm i)H^\pm_p$, $p \in ]1, \infty[$, defined by

$$\tilde{P}^\pm \phi = (\xi + i)P^\pm \left( \frac{\phi}{\xi + i} \right). \quad (27)$$
Given $\Phi^\pm \in (M^\pm_\infty)^{n\times(n-1)}$, $\Psi^\pm \in (M^\pm_\infty)^{(n-1)\times n}$ such that $\Psi^\pm \Phi^\pm = I_{n-1}$, let moreover $\Phi^\pm_e, \Psi^\pm_e$ be defined by

$$\Phi^\pm_e = \begin{bmatrix} \Phi^\pm & N^\pm \end{bmatrix}, \quad \Psi^\pm_e = \begin{bmatrix} \Psi^\pm \\ \tilde{N}^\pm \end{bmatrix},$$

(28)

where

$$N^\pm = \begin{bmatrix} \Delta_{\cdot,1}(\Psi^\pm) \\ -\Delta_{\cdot,2}(\Psi^\pm) \\ \vdots \\ (-1)^{n-1}\Delta_{\cdot,n}(\Psi^\pm) \end{bmatrix} \in (M^\pm_\infty)^{n\times 1},$$

(29)

$$\tilde{N}^\pm = \begin{bmatrix} \Delta_{1,\cdot}(\Phi^\pm), \Delta_{2,\cdot}(\Phi^\pm), \ldots, (-1)^{n-1}\Delta_{n,\cdot}(\Phi^\pm) \end{bmatrix} \in (M^\pm_\infty)^{1\times 30}.$$
The next theorem is stated in the framework of the algebras $M^+_{\infty}$ and $H^+_{\infty}$.

Theorem
Let $G \in (L^\infty(R))^{n \times n}$, and let $\Psi$ be an $(n-1) \times n$ submatrix of $G$ obtained by omitting one row in $G$.

(a) If $\Psi \in (M^+_{\infty})^{(n-1) \times n}$, and if $\Psi$ is right invertible over $M^+_{\infty}$, then $T_G$ is nearly Fredholm equivalent to $T_{\det G}$, for every fixed $p \in [1, \infty[$.

(b) If moreover $\Psi \in (H^+_{\infty})^{(n-1) \times n}$, and if $\Psi$ is right invertible over $H^+_{\infty}$, then, for any fixed $p \in [1, \infty[$, $T_G$ is strictly Fredholm equivalent to $T_{\det G}$. In particular, it is (at least) one sided invertible, and is two sided invertible simultaneously with $T_{\det G}$.

(c) If, in the setting of (b), in addition $\text{Ind} T_{\det G} \geq 0$ and the omitted row $\hat{G}_n$ of $G$ is its last one, then a WH $p$-factorization of $G$ is given by 

\[
(1)
\]
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**Theorem**

Let $G \in (L_\infty(\mathbb{R}))^{n \times n}$, and let $\Psi$ be an $(n-1) \times n$ submatrix of $G$ obtained by omitting one row in $G$.

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(b) If moreover $\Psi \in (H_{\infty}^+)^{(n-1) \times n}$, and if $\Psi$ is right invertible over $H_{\infty}^+$, then, for any fixed $p \in \]1, \infty[,$ $T_G$ is strictly Fredholm equivalent to $T_{\det G}$. In particular, it is (at least) one sided invertible, and is two sided invertible simultaneously with $T_{\det G}$.

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One sided invertibility, corona problem, and applications $$
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(b) If moreover $\Psi \in (H_\infty^+)^{(n-1) \times n}$, and if $\Psi$ is right invertible over $H_\infty^+$, then, for any fixed $p \in ]1, \infty[$, $T_G$ is strictly Fredholm equivalent to $T_{\text{det } G}$. In particular, it is (at least) one sided invertible, and is two sided invertible simultaneously with $T_{\text{det } G}$.

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(c) If, in the setting of (b), in addition $\text{Ind } T_{\det G} \geq 0$ and the omitted row $\hat{G}_n$ of $G$ is its last one, then a WH $p$-factorization of $G$ is given by (1) with
\[ \begin{align*}
G_- &= \begin{bmatrix}
I_{n-1} & 0 \\
0 & \gamma_-
\end{bmatrix}
\begin{bmatrix}
I_{n-1} & 0 \\
\tilde{P}^-(\hat{G}_n\Phi^+\gamma_-^{-1}) & 1
\end{bmatrix}, \\
D &= \begin{bmatrix}
I_{n-1} & 0 \\
0 & r^k
\end{bmatrix}; \\
G_+ &= \begin{bmatrix}
\tilde{P}^+(\hat{G}_n\Phi^+\gamma_-^{-1}) \cdot r^{-k} & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
I_{n-1} & 0 \\
0 & (-1)^{n-1}\gamma_+
\end{bmatrix} \cdot \Psi_e^+.
\end{align*} \]
\[ G_- = \begin{bmatrix} l_{n-1} & 0 \\ 0 & \gamma_- \end{bmatrix} \begin{bmatrix} l_{n-1} & 0 \\ \tilde{P}^-(\hat{G}_n\Phi^+\gamma_-^{-1}) & 1 \end{bmatrix}, \quad (31) \]

\[ D = \begin{bmatrix} l_{n-1} & 0 \\ 0 & r^k \end{bmatrix}; \quad (32) \]

\[ G_+ = \begin{bmatrix} l_{n-1} & 0 \\ \tilde{P}^+(\hat{G}_n\Phi^+\gamma_-^{-1}) \cdot r^{-k} & 1 \end{bmatrix} \begin{bmatrix} l_{n-1} & 0 \\ 0 & (-1)^{n-1}\gamma_+ \end{bmatrix} \cdot \psi^+_e. \quad (33) \]

Here

\[ \det G = \gamma_- r^k \gamma_+ \quad (34) \]

is a WH \( p \)-factorization of \( \det G \) and \( \Phi^+ \) is a right inverse of \( \Psi \).
\[ G_- = \begin{bmatrix} l_{n-1} & 0 \\ 0 & \gamma_- \end{bmatrix} \begin{bmatrix} l_{n-1} & 0 \\ \tilde{P}^-(\hat{G}_n\Phi^+\gamma_-^{-1}) & 1 \end{bmatrix}, \quad (31) \]

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Here

\[ \det G = \gamma_- r^k \gamma_+ \quad (34) \]

is a WH p-factorization of \( \det G \) and \( \Phi^+ \) is a right inverse of \( \Psi \). Note that in (34) \( k \leq 0 \) since it is opposite to \( \text{Ind } T_{\det G} \). This condition is essential for the statement (c) to be valid, while of course omitting the \( n \)-th row (as opposed to some other row) is just to simplify the notation.
The next result is a dual version of the previous theorem.
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**Theorem**

Let $G \in (L_\infty(\mathbb{R}))^{n \times n}$, and let $\Phi$ be an $n \times (n - 1)$ submatrix of $G$ obtained by omitting one column in $G$ (it will be assumed that the $n$th column is omitted, essentially without loss of generality).

(a) If $\Phi \in (M^{-\infty})^{n \times (n - 1)}$, and if $\Phi$ is left invertible over $M^{-\infty}$, then $T_G$ is Fredholm if and only if $T_{\det G}$ is Fredholm, for every fixed $p \in [1, \infty]$.

(b) If moreover $\Phi \in (H^{-\infty})^{n \times (n - 1)}$ and $\Phi$ is left invertible over $H^{-\infty}$, then, for any fixed $p \in (1, \infty)$, the operator $T_G$ is strictly Fredholm equivalent to $T_{\det G}$, and thus either $\ker T_G = \{0\}$ or $\ker T^*_G = \{0\}$. In particular, $T_G$ is invertible if and only if so is $T_{\det G}$.

(c) If, in the setting of (b), in addition $\text{Ind} \det G \leq 0$ and the omitted column of $G$ is $\hat{G}_n$, its last one, then a WH $p$-factorization of $G$ is given by (1) with $I_{n-1}$.
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Let $G \in (L_\infty(\mathbb{R}))^{n \times n}$, and let $\Phi$ be an $n \times (n - 1)$ submatrix of $G$ obtained by omitting one column in $G$ (it will be assumed that the $n$th column is omitted, essentially without loss of generality).

(a) If $\Phi \in (M_\infty^-)^{n \times (n-1)}$, and if $\Phi$ is left invertible over $M_\infty^-$, then $T_G$ is Fredholm if and only if $T_{\det G}$ is Fredholm, for every fixed $p \in ]1, \infty[$.

(b) If moreover $\Phi \in (H_\infty^-)^{n \times (n-1)}$ and $\Phi$ is left invertible over $H_\infty^-$, then, for any fixed $p \in (1, \infty)$, the operator $T_G$ is strictly Fredholm equivalent to $T_{\det G}$, and thus either $\ker T_G = \{0\}$ or $\ker T_G^* = \{0\}$. In particular, $T_G$ is invertible if and only if so is $T_{\det G}$.

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**Theorem**

Let $G \in (L_\infty(\mathbb{R}))^{n \times n}$, and let $\Phi$ be an $n \times (n - 1)$ submatrix of $G$ obtained by omitting one column in $G$ (it will be assumed that the $n$th column is omitted, essentially without loss of generality).

(a) If $\Phi \in (M^-_\infty)^{n \times (n-1)}$ and if $\Phi$ is left invertible over $M^-_\infty$, then $T_G$ is Fredholm if and only if $T_{\det G}$ is Fredholm, for every fixed $p \in ]1, \infty[$.

(b) If moreover $\Phi \in (H^-_\infty)^{n \times (n-1)}$ and $\Phi$ is left invertible over $H^-_\infty$, then, for any fixed $p \in (1, \infty)$, the operator $T_G$ is strictly Fredholm equivalent to $T_{\det G}$, and thus either $\ker T_G = \{0\}$ or $\ker T_G^* = \{0\}$. In particular, $T_G$ is invertible if and only if so is $T_{\det G}$. 

(3) If, in the setting of (2), in addition $\text{Ind det } G \leq 0$ and the omitted column of $G$ is $\hat{G}_n$, its last one, then a WH $p$-factorization of $G$ is given by

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(c) If, in the setting of (b), in addition $\Ind \det G \leq 0$ and the omitted column of $G$ is $G_n$, its last one, then a WH $p$-factorization of $G$ is given by (1) with
Here $\det G = \gamma - r k \gamma + \gamma_0$ is a WH p-factorization of $\det G$, $\Psi$ is a left inverse of $\Phi$, and $\Phi^{-e} = (\Psi^{-e})^{-1}$ is given by (28)–(30).

Of course, formulas similar to those given in part (c) hold when the removed column is not the last one.

In the previous results we have used the one sided invertibility of a submatrix of $G$ to study the Fredholmness, and other associated properties, of the Toeplitz operator $T_G$. Now we turn to the study of the same properties of $T_G$ based on one sided invertibility of a solution to a Riemann-Hilbert problem with coefficient $G$.
\[ G_- = \Phi_e^{-} \begin{bmatrix} I_{n-1} & r^{-k} \tilde{P}_-(\gamma_+^{-1}\psi_- \hat{G}_n) \\ 0 & (-1)^{n-1}\gamma_- \end{bmatrix}, \quad D = \begin{bmatrix} I_{n-1} & 0 \\ 0 & r^k \end{bmatrix}, \]

\[ G_+ = \begin{bmatrix} I_{n-1} & \gamma_+ \tilde{P}_+(\gamma_+^{-1}\psi_- \hat{G}_n) \\ 0 & \gamma_+ \end{bmatrix}. \]

Here \( \det G = \gamma_- r^k \gamma_+ \) is a \( WH \) \( p \)-factorization of \( \det G \), \( \Psi_- \) is a left inverse of \( \Phi \), and \( \Phi_e^{-} = (\Psi_e^{-})^{-1} \) is given by (28)–(30).
\[ G_{-} = \Phi_{e}^{-} \begin{bmatrix} l_{n-1} & r^{-k} \tilde{P}_{-}(\gamma_{+}^{-1}\psi_{-}\hat{G}_{n}) \\ 0 & (-1)^{n-1}\gamma_{-} \end{bmatrix}, \quad D = \begin{bmatrix} l_{n-1} & 0 \\ 0 & r^{k} \end{bmatrix}, \]

\[ G_{+} = \begin{bmatrix} l_{n-1} & \gamma_{+}\tilde{P}_{+}(\gamma_{+}^{-1}\psi_{-}\hat{G}_{n}) \\ 0 & \gamma_{+} \end{bmatrix}. \]

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\[ G_- = \Phi_e^- \begin{bmatrix} I_{n-1} & r^{-k} \hat{P}_- (\gamma_{+}^{-1} \Psi_- \hat{G}_n) \\ 0 & (-1)^{n-1} \gamma_- \end{bmatrix}, \quad D = \begin{bmatrix} I_{n-1} & 0 \\ 0 & r^k \end{bmatrix}, \]

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Theorem

Let \( G \in (L_\infty(\mathbb{R}))^{n\times n} \), and let

\[ G\Phi^+ = \Phi^-, \quad \Phi^\pm \in (M^\pm_\infty)^{n\times(n-1)}, \]

(35)

where \( \Phi^\pm \) are left invertible over \( M^\pm_\infty \). Then:

(i) \( T_G \) is Fredholm equivalent to \( T_{\det G} \);

(ii) If moreover \( \Phi^\pm \) are left invertible over \( H^\pm_\infty \), with left inverses \( \Psi^\pm \in (H^\pm_\infty)^{(n-1)\times n} \) for \( \Phi^\pm \), respectively, then \( T_G \) is strictly Fredholm equivalent to \( T_{\det G} \) and either \( \ker T_G = \{0\} \) or \( \ker T^*_G = \{0\} \). In particular, \( T_G \) is invertible if and only \( T_{\det G} \) is invertible.
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(iii) Assuming that

$$\det G = \gamma_- r^k \gamma_+ \quad \text{with } k \geq 0 \quad (36)$$

is a WH $p$-factorization for $\det G$, a WH $p$-factorization for $G$ is given by (1) with

$$G_- = \Phi_e^- \cdot \begin{bmatrix} I_{n-1} & 0 \\ 0 & \gamma_- \end{bmatrix} \begin{bmatrix} I_{n-1} & \alpha_- \\ 0 & 1 \end{bmatrix}, \quad (37)$$

$$D = \begin{bmatrix} I_{n-1} & 0_{(n-1) \times 1} \\ 0_{1 \times (n-1)} & r^k \end{bmatrix} \quad (38)$$

$$G_+ = \begin{bmatrix} I_{n-1} & \alpha_+ \\ 0_{1 \times (n-1)} & 1 \end{bmatrix} \begin{bmatrix} I_{n-1} & 0 \\ 0 & \gamma_+ \end{bmatrix} \cdot \Psi_e^+,$$  \quad (39)

where $\Phi_e^-$, $\Psi_e^+$ are given by (28)–(30),
\[
\alpha_+ = \tilde{P}^+ (Q) \in (\mathcal{L}_p^+)^{(n-1) \times 1},
\]

\[
\alpha_- = r^{-k} \tilde{P}^- (Q) \in (\mathcal{L}_p^-)^{(n-1) \times 1},
\]

and where

\[
Q := \Psi^- G N^+ \in (L_\infty(\mathbb{R}))^{(n-1) \times 1},
\]

with \( N^+ \) as in (29).
Proof. (i) Let $\Phi_e^\pm$, $\Psi_e^\pm$ be defined as in (28), (30), where $\Psi^\pm \in (M_\infty^\pm)^{(n-1) \times n}$ is a left inverse of $\Phi^\pm$ over $M_\infty^\pm$. From Theorem it follows that $\Psi_e^\pm \in G((M_\infty^\pm)^{n \times n})$ and $(\Psi_e^\pm)^{-1} = \Phi_e^\pm$.

Defining

$$G_0 = \Psi_e^- G \Phi_e^+,$$

we can rewrite (35) as

$$G_0 \Psi_e^+ \Phi^+ = \Psi_e^- \Phi^-.$$  \hfill (44)

On the other hand, it follows from Theorem (see (25) or (26), taking (21) into account) that

$$\Psi_e^\pm \Phi^\pm = \begin{bmatrix} I_{n-1} \\ 0_{1 \times (n-1)} \end{bmatrix},$$  \hfill (45)
therefore (44) implies that $G_0$ has the form

$$G_0 = \begin{bmatrix} I_{n-1} & Q \\ 0_{1 \times (n-1)} & \det G \end{bmatrix}.$$  \hspace{1cm} (46)

In particular, $\det G = \det G_0$. From (43) it follows that $T_G$ is Fredholm if and only if $T_{G_0}$ is Fredholm, and this in turn is equivalent to $T_{\det G}$ being Fredholm (by (46)).
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$$G_0 = \begin{bmatrix}
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(ii) If $\Phi^\pm \in (H^\pm_\infty)^{n \times (n-1)}$ and $\Phi^\pm$ is left invertible over $H^\pm_\infty$, with a left inverse $\Psi^\pm$, then

$$\Psi^-_e \in \mathcal{G} \left( (H^-_\infty)^{n \times n} \right), \quad \Phi^+_e \in \mathcal{G} \left( (H^+_\infty)^{n \times n} \right),$$

and it follows that $T_G$ is strictly Fredholm equivalent to $T_{\det G}$ and that either $\ker T_G = \{0\}$ or $\ker T_G^* = \{0\}$. 

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(iii) The formulas for $G_\pm$ and $D$ follow from

$$G = \Phi_e^- G_0 \psi_e^+, \quad$$

together with (46) and (36).
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**Theorem**

Let $G \in AP^{n \times n}$ be invertible, and suppose that it contains a submatrix $\Psi \in (AP^+)^{(n-1) \times n}$ which is right invertible over $AP^+$. Then the operator $T_G$ is invertible (resp. right invertible, or left invertible) on $(H_p^+)^n$ for any (equivalently, all) $p \in (1, \infty)$ if and only if $\det G$ has zero (resp. non-positive, or non-negative) mean motion $\kappa$. If in addition $G \in APW^{n \times n}$ and $\kappa \geq 0$, then $G$ is APW factorable, and its partial AP indices are $0, \ldots, 0$ ($n - 1$ times) and $\kappa$. 

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The proof runs along the same lines as for the $H^\infty$ case, taking into consideration that $\det G$ is an invertible $AP$ function and thus the operator $T_{\det G}$ is automatically one sided invertible. To construct the $APW$ factorization, one can still use formulas (31)–(33) substituting $r^k$ by $e_{\kappa(\det G)}$ and $\tilde{P}^\pm$ by the projections of $APW$ onto $APW^\pm$. 

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Theorem

Let $G \in APW^{n \times n}$ be invertible, with $\kappa(\det G) \geq 0$. Moreover, let there exist $\Phi^\pm \in (APW^\pm)^{n-1 \times n}$ left invertible over $APW^\pm$ and such that $G\Phi^+ = \Phi^-$. Then $G$ is $APW$ factorable, with the partial $AP$ indices equal $0, \ldots, 0$ ($n - 1$ times) and $\kappa(\det G)$. 
Particular cases

Some classes of matrix functions can be clearly identified, to which the previous results apply. Let \( G \in \mathbb{L}_{n \times n}^\infty \) with all rows but one having elements in \( \mathbb{M}_\infty^+ \) (the case of all columns but one having elements in \( \mathbb{M}_\infty^- \) can be treated analogously). Assume for simplicity that \( G = [\Psi \; g_n] \) with \( \Psi \in (\mathbb{M}_\infty^+)^{(n-1) \times n} \), \( g_n \in \mathbb{L}_1^{\infty \times n} \).

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$$G = \begin{bmatrix} \Psi \\ g_n \end{bmatrix} \quad \text{with} \quad \Psi \in (M_\infty^+)^{(n-1) \times n}, \ g_n \in L_\infty^{1 \times n}. \quad (47)$$

Then the following results hold.
Theorem
(i) If $G$ is unitary with constant determinant and $g_n^T \in MCT_n^-$, then $T_G$ is Fredholm for all $p \in (1, \infty)$. If moreover $\Psi \in (H_\infty^+)^{(n-1)\times n}$ and $g_n^T \in HCT_n^-$, then $T_G$ is invertible.

(ii) If $G$ is (complex) orthogonal with constant determinant and $g_n^T \in MCT_n^+$, then $T_G$ is Fredholm for all $p \in (1, \infty)$. If moreover $\Psi \in (H_\infty^+)^{(n-1)\times n}$ and $g_n^T \in HCT_n^+$, then $T_G$ is invertible.

(iii) If one of the $(n-1)\times (n-1)$ minors of $\Psi$ is invertible in $M_{\infty}^+$, then $T_G$ is Fredholm equivalent to $T_{\det G}$. If the above mentioned minor is in fact invertible in $H_\infty^+$, then $T_G$ is strictly Fredholm equivalent to $T_{\det G}$ and either $\ker T_G = \{0\}$ or $\ker T_G^* = \{0\}$.

Of course $|\det G| = 1$ in case (i) and $\det G = \pm 1$ in case (ii).
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Proof.

(i) Observe that the $k$-th entry $g_{nk}$ of $g_n$ coincides with $(-1)^{n+k} \det G \Delta_{.,k}(\Psi)$, $k = 1, \ldots, n$. Thus, condition $g_n^T \in MCT_n^-$ can be rewritten equivalently as

$$\left( \Delta_{.,k}(\Psi) \right)_{k=1,\ldots,n} \in MCT_n^+. \tag{48}$$

By the right invertibility analogue of the results above, it follows that $\Psi$ is right invertible over $M_\infty^+$, and $T_G$ is Fredholm. The second part of (i) follows analogously.

(ii) If $G$ is orthogonal, then $g_{nk} = (-1)^{n+k} \det G \Delta_{.,k}(\Psi)$, so that $g_n^T \in MCT_n^+$ can be rewritten as (48). The rest of the proof goes as in (i).

(iii) The invertibility of any $(n-1) \times (n-1)$ minor of $\Psi \in (M_\infty^+)^{n \times n}$ implies (48).
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Remark
In the case of orthogonal (47) we automatically have $g_n \in (M_\infty^+)^{1 \times n}$, so that the relation $GG^T = I$ immediately provides the right inverse of $\Psi$ over $M_\infty^+$. It can then be used in factorization formulas (31)–(33).

A factorization of unitary matrices $G$ with $\det G = 1$, $\Psi \in (H_\infty^+)^{(n-1) \times n}$ and $g^n \in HCT_n$, as in Theorem 0.14 was by different methods considered earlier in [EphremidzeJanashiaLagvilava'98].
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A well known property of Toeplitz operators $T_G$ with matrix symbols $G$ continuous on $\hat{\mathbb{R}}$ is that they are Fredholm if and only if $\det G$ does not vanish on $\hat{\mathbb{R}}$ which in its turn is equivalent to $T_{\det G}$ being Fredholm.

We generalize this result as follows.

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Let $G$ be of the form (47) with $\Psi \in \mathbb{R}^{(n-1) \times n}$. Then $T_G$ is Fredholm equivalent to $T_{\det G}$.
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Lemma

Let $G$ be of the form (47) with $\Psi \in \mathcal{R}^{(n-1) \times n}$. Then $T_G$ is Fredholm equivalent to $T_{\det G}$.
Proof.
If the determinants $\Delta_{.,k}(\psi), \ k = 1, \ldots, n$ (which are rational functions in $\mathcal{R}$) have at least one common zero in $\dot{\mathbb{R}}$, then $\det G$ has the same zero and thus neither $T_{\det G}$ nor $T_G$ is Fredholm. Suppose now there are no common zeros of $\Delta_{.,k}(\psi)$ in $\dot{\mathbb{R}}$. Since there are at most finitely many such zeros in $\mathbb{C}^\pm$, then (48) holds again. Thus, $\psi$ is right invertible over $M_\infty^\pm$. The statement now follows.
Theorem

Let $G \in L^{n \times n}$ be such that all its elements except maybe for those located in one row or one column are continuous on $\mathbb{R}$. Then $T_G$ is Fredholm equivalent to $T_{\text{det } G}$.
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Let $G \in L^{n \times n}_\infty$ be such that all its elements except maybe for those located in one row or one column are continuous on $\mathbb{R}$. Then $T_G$ is Fredholm equivalent to $T_{\det G}$.

Proof. Without loss of generality, $G$ is of the form (47) with $\Psi \in C^{(n-1) \times n}$. 
Necessity. Suppose $T_G$ is Fredholm. Then $\det G$ is invertible in $L_\infty$. Expanding $\det G$ across the last row, represent it as

$$\det G = \sum_{j=1}^{n} f_j g_{n,j},$$

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Necessity. Suppose $T_G$ is Fredholm. Then det $G$ is invertible in $L_\infty$. Expanding det $G$ across the last row, represent it as

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Now let

$$\tilde{g}_{n,j} = g_{n,j} \det G / \det G_1, \quad \tilde{g}_n = [\tilde{g}_{n,1} \ldots \tilde{g}_{n,n}]$$

and $\tilde{G} = \begin{bmatrix} \tilde{\Psi} \\ \tilde{g}_n \end{bmatrix}$. 

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One sided invertibility, corona problem, and applications
The matrix function $\tilde{G}$ can be made arbitrarily close to $G$, so that we may suppose $T_{\tilde{G}}$ to be Fredholm. As shown earlier, the operator $T_{\det \tilde{G}}$ is Fredholm. It remains to observe that

$$\det \tilde{G} = \sum_{j=1}^{n} \tilde{f}_j \tilde{g}_{n,j} = \det G.$$
Sufficiency. Along with $T_G$, let us consider $T_{\text{adj}G}$, where $\text{adj} G$ stands for the transposed matrix of the cofactors of $G$. Recall that

$$G \text{adj} G = \text{adj} G G = (\det G)I_n,$$ \hspace{1cm} (49)

and let $I_+, I^n_+$ denote the identity operators on $H^+_p$, $(H^+_p)^n$, respectively.
Sufficiency. Along with \( T_G \), let us consider \( T_{\text{adj}} G \), where \( \text{adj} G \) stands for the transposed matrix of the cofactors of \( G \). Recall that

\[
G \text{adj} G = \text{adj} G G = (\det G) I_n, \tag{49}
\]

and let \( I_+ \), \( I_+^n \) denote the identity operators on \( H^+_p \), \( (H^+_p)^n \), respectively.

Since the first \( n - 1 \) rows of \( G \) and the last column of \( \text{adj} G \) are continuous on \( \mathbb{R} \cup \{\infty\} \), the operator

\[
k_\ell := T_{\text{adj}} G T_G - T_G \text{adj} G
\]

is compact (Corollary 3.5 in [MikhlinPresdorf'86]). Taking (49) into account, we conclude that

\[
T_{\text{adj}} G T_G = (\det G) I_+^n + k_\ell
\]

is Fredholm and therefore \( T_G \) has a left regularizer (that is, the left inverse modulo the ideal of compact operators).
To show that $T_G$ has also a right regularizer — and therefore $T_G$ is Fredholm — we consider $T_G T_{\text{adj} G}$.
To show that $T_G$ has also a right regularizer — and therefore $T_G$ is Fredholm — we consider $T_G T_{adj} G$. In this case, the difference $T_G T_{adj} G - T_{G \cdot adj} G$ may not be compact, so we have to use different (and somewhat more involved) arguments.
To show that $T_G$ has also a right regularizer — and therefore $T_G$ is Fredholm — we consider $T_G T_{\text{adj} \, G}$. In this case, the difference $T_G T_{\text{adj} \, G} - T_G \cdot \text{adj} \, G$ may not be compact, so we have to use different (and somewhat more involved) arguments. Let $[T_{ij}]$, $(i,j \in \{1,2\})$, be the block representation of the operator

$$T_G T_{\text{adj} \, G} - T_G \cdot \text{adj} \, G = T_G T_{\text{adj} \, G} - (\det G) I^n_+,$$

corresponding to the decomposition $(H^+_p)^n = (H^+_p)^{n-1} \oplus H^+_p$. The operators $T_{11}$, $T_{12}$, and $T_{22}$ are compact (by [7, Corollary 7.5]), and we can write

$$T_G T_{\text{adj} \, G} = (\det G) I^n_+ + \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

$$= \begin{bmatrix} (\det G) I_+ & 0 \\ T_{21} & (\det G) I_+ \end{bmatrix} + \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}.$$
Thus, $T_G T_{\text{adj} G}$ is a compact perturbation of a block triangular operator which is Fredholm since its diagonal elements are Fredholm (see, e.g., Corollary 1.3 in [LitvinchukSpitkovsky]). Consequently, $T_G T_{\text{adj} G}$ is Fredholm which implies that $T_G$ has a right regularizer as well.


Translated from the Russian by B. Luderer, With a foreword by B. Silbermann.
