On the uniform Opial property

By: Gh. Sadeghi

We consider the noncommutative modular function spaces of measurable operators affiliated with a semifinite von Neumann algebra and show that they are complete with respect to their modular.

We prove that these spaces have the uniform Opial property with respect to convergence $\widetilde{\rho}$–a.e. for both the Luxemburg norm and the Amemiya norm.

The above geometric properties will enable us to obtain some results in noncommutative Orlicz spaces.

The Opial property originates from the fixed point theorem proved by Opial (1967).
The uniform Opial property with respect to the weak topology was defined by Prus (1992).
It is well known that the Opial property of a Banach space X play an important role in metric fixed point theory for nonexpansive mappings, as well as in the theory of differential and integral equations. The Opial property also plays an important role in the study of weak convergence of iterates, random products of nonexpansive mappings.

Definition

We say that a Banach space X has the Opial property w.r.t. a topology τ if for any τ-null sequence $\{x_n\}$ in X and every $x \neq 0$ in X there holds

$$\liminf_{n\to} \|x_n\| < \liminf_{n\to\infty} \|x_n + x\|.$$

Opial proved that l^p $(1 < p < \infty)$ has this property, but $L^p[0, 2\pi]$ does not have it if $(1 < p < \infty)$, $p \neq 2$, w.r.t. weak topology.

Definition

We say that X has the uniform Opial property with respect a topology τ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x \in X$ with $\|x\| \geq \varepsilon$ and τ-null sequence $\{x_n\}$ in the unit sphere $S(X)$ of X there holds

$$1 + \delta \leq \liminf_{n \to \infty} \|x_n + x\|.$$

It is obvious that the uniform Opial property implies the Opial property.

Associated to the uniform Opial property, we can define the following modules:

$$o_\tau(\alpha) = \inf\{\liminf_{n\to\infty} \|x_n + x\| - 1\},$$

where the infimum is taken over all $x \in X$ with $\|x\| \geq \alpha$ and all sequence $\{x_n\}$ such that $\tau - \lim_n x_n = 0$ and $\liminf_n \|x_n\| \geq 1$. It is clear that X has the uniform Opial property with respect to τ iff $o_\tau(\alpha) > 0$ for every $\alpha > 0$.

A trace $\tau : \mathfrak{M}^+ \to [0, \infty]$ is called :
i) faithful if $\tau(x) > 0$ whenever $0 < x \in \mathfrak{M}$;
ii) semi–finite if for every $x \in \mathfrak{M}^+$ with $\tau(x) > 0$ there exists $0 \leq y \leq x$ such that $0 < \tau(y) < \infty$;
iii) normal if $\tau(x_\alpha) \uparrow \tau(x)$ whenever $x_\alpha \uparrow x$ in $\mathfrak{M}^+$.
Where $x_\alpha \uparrow x$ means that $\alpha \hookrightarrow x_\alpha$ is an increasing net converging to x in the strong operator topology.
A von Neumann algebra equipped with a semi–finite, faithful, normal trace is called a semi–finite von Neumann algebra.

Example

Let $\mathfrak{H}$ be a Hilbert space and $\mathfrak{M} = \mathfrak{B}(\mathfrak{H})$. Given an orthonormal basis $\{e_i\}$ in $\mathfrak{H}$ we define

$$\tau(x) = \Sigma_{i \in I} \langle x e_i, e_i \rangle, \quad x \in \mathfrak{B}(\mathfrak{H})^+.$$

It is clear that $\tau : \mathfrak{B}(\mathfrak{H})^+ \to [0, \infty]$ is a semi–finite faithful normal trace on $\mathfrak{B}(\mathfrak{H})$. This is called the standard trace on $\mathfrak{B}(\mathfrak{H})$.

A linear operator $x : \mathcal{D}(x) \to \mathfrak{H}$ with domain $\mathcal{D}(x) \subseteq \mathfrak{H}$ is called affiliated with $\mathfrak{M}$, if $ux = xu$ for all unitaries u in the commutant $\mathfrak{M}'$ of $\mathfrak{M}$. This is denoted by $x\eta\mathfrak{M}$. Note that the equality $ux = xu$ involves the equality of the domains of the operators ux and xu, that is, $\mathcal{D}(x) = u^{-1}(\mathcal{D}(x))$.
If x is in the algebra $\mathcal{B}(\mathfrak{H})$ of all bounded linear operators on the Hilbert space $\mathfrak{H}$, then x is affiliated with $\mathfrak{M}$ if and only if $x \in \mathfrak{M}$.
If x is a self–adjoint operator affiliated with $\mathfrak{M}$, then the spectral projection $\chi_B(x)$ is an element of $\mathfrak{M}$ for any Borel set $B \subseteq \mathbb{R}$.

Here χ denotes the usual indicator function. The closed and densely defined operator x, affiliated with $\mathfrak{M}$, is called τ–measurable if and only if there exist a number $s \geq 0$ such that

$$\tau\left(\chi_{(s,\infty)}(|x|)\right) < \infty.$$

The collection of all τ–measurable operators is denoted by $\widetilde{\mathfrak{M}}$.

Measure topology

Given $0 < \varepsilon, \delta \in \mathbb{R}$, we define $\mathcal{V}(\varepsilon, \delta)$ to be the set of all $x \in \widetilde{\mathfrak{M}}$ for which there exists $p \in \mathcal{P}(\mathfrak{M})$ such that $\|xp\|_{\mathcal{B}(\mathfrak{H})} \leq \varepsilon$ and $\tau(\mathbf{1} - p) \leq \delta$. An alternative description of this set is given by

$$\mathcal{V}(\varepsilon, \delta) = \left\{ x \in \widetilde{\mathfrak{M}} \quad ; \tau\left(\chi_{(\varepsilon,\infty)}(|x|)\right) < \delta \right\}.$$

The collection $\{\mathcal{V}(\varepsilon, \delta)\}_{\varepsilon,\delta>0}$ is a neighborhood base at 0 for a vector space topology τ_m on $\widetilde{\mathfrak{M}}$.

For $x \in \widetilde{\mathfrak{M}}$, the generalized singular value function $\mu(x,.) = \mu(|x|,.)$ is defined by

$$\mu(x;t) = \inf\left\{\lambda \geq 0; \quad \tau\left(e^{|x|}(\lambda,\infty)\right) \leq t\right\}, \quad t \geq 0.$$

It follows directly that the generalized singular value function $\mu(x)$ is a decreasing right–continuous function on the positive half-line $[0,\infty)$.

In the following proposition, we list some properties of the rearrangement mapping $\mu(.;t)$.

Theorem

Let x, y and z be τ–measurable operators.
(i) the map $t \in (0,\infty) \mapsto \mu(x;t)$ is non–increasing and continuous from the right. Moreover,

$$\lim_{t\downarrow 0} \mu(x;t) = \|x\| \in [0,\infty].$$

(ii) $\mu(x;t) = \mu(|x|;t) = \mu(x^*;t)$.
(iii) $\mu(x;t) \leq \mu(y;t), \quad t > 0$, *if* $0 \leq x \leq y$.
(iv) $\mu(x+y;t+s) \leq \mu(x;t) + \mu(y;s), \quad t,s > 0$.
(v) $\mu(zxy;t) \leq \|z\|\|y\|\mu(x;t), \quad t > 0$.
(vi) $\mu(xy;t+s) \leq \mu(x;t)\mu(y;s) \quad t,s > 0$.

Let Ω be a nonempty set and Σ be a nontrivial σ–algebra of subsets of Ω. Let $\mathcal{P}$ be a δ–ring of subsets of Ω, such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \bigcup K_n$. By $\mathcal{E}$ we denote the linear space of all simple functions with supports from $\mathcal{P}$.

By $\mathcal{M}$ we will denote the space of all measurable functions, i.e., all functions $f : \Omega \to \mathbb{R}$ such that there exists a sequence $\{s_n\} \subset \mathcal{E}$, $|s_n| \leq |f|$ and $s_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$.

Let us recall that a set function $\nu : \Sigma \to [0, +\infty]$ is called a σ–subadditive measure if $\nu(\emptyset) = 0$, $\nu(A) \leq \nu(B)$ for any $A \subseteq B$ and $\nu(\cup_n A_n) \leq \Sigma_n \nu(A_n)$ for any sequence of sets $\{A_n\} \subseteq \Sigma$.

Definition

A functional $\rho : \mathcal{E} \times \Sigma \to [0, \infty]$ is called a function modular if
(i) $\rho(0, A) = 0$ for any $A \in \Sigma$;
(ii) $\rho(f, A) \leq \rho(g, A)$ whenever $|f(\omega) \leq |g(\omega)|$ for anr $\omega \in \Omega$, $f, g \in \mathcal{E}$, $A \in \Sigma$;
(iii) $\rho(f, \cdot) : \Sigma \to [0, +\infty]$ is a σ–subadditive measure for every $f \in \mathcal{E}$;
(iv) $\rho(\alpha, A) \to 0$ as α decreases to 0 for every $A \in \mathcal{P}$, where $\rho(\alpha, A) = \rho(\alpha\chi_A, A)$;
(v) If there exists $\alpha > 0$ such that $\rho(\alpha, A) = 0$, then $\rho(\beta, A) = 0$ for every $\beta > 0$;
(vi) For any $\alpha > 0$, $\rho(\alpha, \cdot)$ is order continuous on $\mathcal{P}$, i.e., $\rho(\alpha, A_n) \to 0$ if $\{A_n\} \subseteq \mathcal{P}$ and decreases to $\emptyset$.

When ρ satisfies
$(iii)'$ $\rho(f,\cdot):\Sigma\to[0,+\infty]$ is a σ–subadditive measure, we say that ρ is a additive if $\rho(f,A\cup B)=\rho(f,A)+\rho(f,B)$ whenever $A,B\in\Sigma$ such that $A\cap B=\emptyset$ and $f\in\mathcal{M}$.
The definition of ρ is extended to all $f\in\mathcal{M}$ by

$$\rho(f,A)=\sup\{\rho(s,A);s\in\mathcal{E},|s(x)|\leq|f(x)|\text{ for every }\omega\in\Omega\}.$$

In the previous conditions, we define the functional $\rho : \mathcal{M} \to [0, +\infty]$ given by $\rho(f) = \rho(f, \Omega)$. Then it is easy to check that ρ is a modular. A function modular ρ defines a corresponding modular function space, i.e., the vector space $\mathcal{L}_\rho$ given by

$$\mathcal{L}_\rho = \{x \in \mathcal{M} : \quad \rho(\lambda x) \to 0 \qquad \text{as } \lambda \to 0\}.$$

When ρ is convex, the formulas

$$\|f\|_\rho = \inf\left\{\lambda > 0 : \quad \rho\left(\frac{f}{\lambda}\right) \leq 1\right\},$$

and

$$\|f\|_\rho^A = \inf\left\{\frac{1}{\alpha}\rho\left(1 + \alpha f\right) \quad : \alpha > 0\right\},$$

define two complete norms on $\mathcal{L}_\rho$ which are called the Luxemburg norm and Amemiya norm, respectively..

Example

Orlicz–Musielak spaces: Let (Ω, Σ, ν) be a measure space, where ν is a positive σ–finite measure. Let us denote by $\mathcal{P}$ the δ–ring of all sets of finite measure. Define the modular ρ by the formula

$$\rho_\varphi(f, E) = \int_E \varphi(t, |f(t)|) d\nu(t),$$

where φ is an Orlicz function.

Definition

Similarly as in the case of measure spaces, we say that a set $A \in \Sigma$ is ρ–null if $\rho(\alpha, A) = 0$ for every $\alpha > 0$. A property $p(\omega)$ is said to hold ρ–almost everywhere (ρ–a.e.) if the set $\{\omega \in \Omega : p(\omega) \text{ does not hold}\}$ is ρ–null. For example, we will frequently say $f_n \to f$ ρ-a.e., which means that $\{\omega \in \Omega : f(\omega) \neq \lim_{n\to\infty} f_n(\omega)\}$ is ρ–null.
As usual we identify any pair of measurable sets whose symmetric difference is ρ–null as well as any pair of measurable functions differing only on a ρ–null set.

Definition

A function modular ρ is said to satisfy the Δ_2–condition if there exists $k > 0$ such that for any $f \in \mathcal{L}_\rho$ we have $\rho(2f) \leq k\rho(f)$. Notice that the Δ_2–condition implies that $0 < \rho(kf) < \infty$ for every $k > 0$ provided $0 < \rho(f) < \infty$

In this section, by $\mathcal{L}_0(\nu)$ we denote the linear space of all (equivalence class of) real valued Lebesgue measurable functions on the Ω. Let ρ be a convex additive function modular on $\mathcal{L}_0(\nu)$ with the Δ_2–condition such that for every $f, g \in \mathcal{L}_0(\nu)$, if f is submajorized by g ($f \prec\prec g$) in the sense of Hardy, Littlewood and Polya implies that $\rho(f) \leq \rho(g)$ and $\mathcal{L}_\rho$ denoted by the modular function space associated to ρ on $\mathcal{L}_0(\nu)$.

The noncommutative modular function space $\mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}}, \tau)$ is defined by

$$\mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}}, \tau) = \left\{ x \in \widetilde{\mathfrak{M}} : \quad \widetilde{\rho}(\lambda x) \to 0 \quad \text{as} \quad \lambda \to 0 \right\},$$

where the functional $\widetilde{\rho}$ on $\widetilde{\mathfrak{M}}$ is defined by $\widetilde{\rho}(x) = \rho(\mu(|x|))$. The vector space $\mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}}, \tau)$ can be equipped with the Luxemburg norm defined by

$$\|x\|_{\widetilde{\rho}} \quad = \quad \inf \left\{ \lambda > 0 : \widetilde{\rho}\left(\frac{x}{\lambda}\right) \leq 1 \right\}.$$

Similar to the commutative case one can define the Amemiya norm as follows:

$$\|x\|_{\widetilde{\rho}}^{A} = \inf\left\{\frac{1}{\lambda}(1+\widetilde{\rho}(\lambda x)) : \quad \lambda > 0\right\}.$$

We have also the following relation between these norms

$$\|x\|_{\widetilde{\rho}} \leq \|x\|_{\widetilde{\rho}}^{A} \leq 2\|x\|_{\widetilde{\rho}}.$$

Theorem

The noncommutative modular function space $\mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}}, \tau)$ *is a Banach space.*

Definition

Let $\{x_n\}$ and x be in $\widetilde{\mathfrak{M}}$. Then the sequence $\{x_n\}$ is said to be $\widetilde{\rho}$–a.e. convergent to x if $\mu(x_n - x) \to 0$ ρ–a.e.

Theorem

The noncommutative modular function space $\mathcal{L}_{\widetilde{\rho}}(\widetilde{\mathfrak{M}}, \tau)$ equipped with the Luxemburg norm $\|\cdot\|_{\widetilde{\rho}}$ (Amemiya norm $\|\cdot\|_{\rho}^{A}$) has the uniform Opial property with to respect convergence $\widetilde{\rho}$–a.e.

Corollary

Let φ satisfy the Δ_2–condition. Then the noncommutative Orlicz space $L^{\varphi}(\widetilde{\mathfrak{M}}, \tau)$ equipped with the Luxemburg norm (Amemiya norm $\|\cdot\|_{\rho}^{A}$) has the uniform Opial property with respect to convergence in measure.

Corollary

The noncommutative L^p–space $L^p(\widetilde{\mathfrak{M}}, \tau)$ has the uniform Opial property with respect to convergence in measure.

[1] B. de Pagter, *Noncommutative Banach function spaces*, Positivity, 197-227, Trends Math., Birkhäuser, Basel, (2007).
[2] Th. Fack and H. Kosaki, *Generalized s–numbers of τ–measurable operators*, Pacific J. Math. **123** (1986), 269–300.
[3] M.A. Japón, *Some geometric properties in modular spaces and application to fixed point theory*, J. Math. Anal. Appl. **295** (2005), 576–594.
[4] E. Nelson, *Notes on non–commutative integration*, J. Funct. Anal. 15 (1974), 103–116.
[5] Gh. Sadeghi, *Non-commutative Orlicz spaces associated to a modular on τ–measurable operators*, J. Math. Anal. Appl. 479 (2012), 710–715.

Thanks a lot for your attention