Generalized inverses of products of operators on Hilbert C*-modules

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Cheongpung, Korea, August 7-12, 2014
Some results of this talk has been prepared according to the following:


Dijana Mosić and Dragan S. Djordjević, Moore-Penrose invertible normal and Hermitian elements in rings, Linear Algebra Appl. 431 (5-7) (2009), 732-745.


Dijana Mosić and Dragan S. Djordjević, Some results on the reverse order law in rings with involutions, Aequationes Mathematicae 83 (3) (2012), 271-282.

Nebojša C. Dinčić and Dragan S. Djordjević, Basic reverse order law and its equivalencies, Aequationes Mathematicae 85 (3) (2013), 505-517.


Dijana Mosić and Dragan S. Djordjević, Mixed-type reverse order laws for generalized inverses in rings with involution, Publicationes Mathematicae Debrecen 82 (3-4) (2013), 641-650.

Nebojša C. Dinčić, Dragan S. Djordjević, Hartwig’s triple reverse order law revisited, Linear Multilinear Algebra (to appear).


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Chapter 1

Generalized inverses

1.1 Basic properties of generalized inverses

Let $X, Y$ be Banach spaces. We use $\mathcal{L}(X, Y)$ to denote the set of all linear bounded operators from $X$ to $Y$. Also, $\mathcal{L}(X) = \mathcal{L}(X, X)$. For $A \in \mathcal{L}(X, Y)$, we use $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively, to denote the null-space and the range of $A$.

Consider an operator $A \in \mathcal{L}(X, Y)$. It is often convenient to know that $A$ is invertible. However, in most applications only some properties of invertible operators are used. These properties can be described by some equations involving the operator $A$ and its ”pseudo“ inverse. In the case when $A$ is invertible, it turns out that these equations are trivially satisfied.

Definition 1.1.1. Let $A \in \mathcal{L}(X, Y)$. If there exists some $B \in \mathcal{L}(Y, X)$, such that $ABA = A$ holds, then $B$ is an inner generalized inverse of $A$, and the operator $A$ is inner regular, or relatively regular.

If $CAC = C$ holds for some $C \in \mathcal{L}(Y, X)$, $C \neq 0$, then $C$ is an outer generalized inverse of $A$. In this case $A$ is outer regular.

An operator $D \in \mathcal{L}(Y, X)$ is a reflexive generalized inverse of $A$, if $D$ is both inner and outer generalized inverse of $A$.

The following result is elementary.

Lemma 1.1.1. Let $A \in \mathcal{L}(X, Y)$. Then the following hold:

(1) If $A$ is invertible, then $A^{-1}$ is the only one inner generalized inverse of $A$. 

(2) If $B, C \in \mathcal{L}(Y, X)$ are inner generalized inverses of $A$, then $BAC$ is a reflexive generalized inverse of $A$.

(3) If $B \in \mathcal{L}(Y, X)$ is an inner, or an outer generalized inverse of $A$, then $AB$ is a projection in $\mathcal{L}(Y)$, and $BA$ is a projection in $\mathcal{L}(X)$.

Inner regularity implies outer regularity.

Inner inverses carry more invertibility information that outer inverses.

Inner inverses were first used by Fredholm in solving integral equations.

**Theorem 1.1.1.** If $B \in \mathcal{L}(Y, X)$ is an inner generalized inverse of $A \in \mathcal{L}(X,Y)$, then $AB$ is a projection from $Y$ on $\mathcal{R}(A)$, and $I - BA$ is a projection from $X$ on $\mathcal{N}(A)$.

Consequently, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are closed and complemented subspaces of $Y$ and $X$ respectively.

The opposite result also holds.

**Theorem 1.1.2.** Let $A \in \mathcal{L}(X,Y)$. If $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are closed and complemented subspaces of $Y$ and $X$ respectively, then $A$ is inner regular.

Under the conditions of this theorem, there exist closed subsets $T$ of $X$ and $S$ of $Y$, such that $X = T \oplus \mathcal{N}(A)$ and $Y = \mathcal{R}(A) \oplus S$.

The general form of the operator $A$ is given by

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} T \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ S \end{bmatrix}, \quad (1.1)$$

and $A_1$ is an invertible operator from $T$ to $\mathcal{R}(A)$.

Hence, any inner generalized inverse $B$ of $A$ has the matrix form:

$$B = \begin{bmatrix} A_1^{-1} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ S \end{bmatrix} \rightarrow \begin{bmatrix} T \\ \mathcal{N}(A) \end{bmatrix}, \quad (1.2)$$

where $B_{12}, B_{21}, B_{22}$ are arbitrary linear bounded operators on corresponding subspaces.

This is the **geometric approach to generalized inverses**.
Corollary 1.1.1. An operator $A \in \mathcal{L}(X, Y)$ is inner regular, if and only if $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively, are closed and complemented subspaces of $X$ and $Y$.

Corollary 1.1.2. If $X, Y$ are Hilbert spaces, then $A \in \mathcal{L}(X, Y)$ is inner regular if and only if $\mathcal{R}(A)$ is closed in $Y$.

It is interesting to consider inner generalized inverses related to given complementary subspaces of $\mathcal{N}(A)$ and $\mathcal{R}(A)$. Equivalently, if $B$ is an inner generalized inverse of $A$, we want to find the matrix forms of $A$ and $B$ with respect to decompositions $X = \mathcal{R}(BA) \oplus \mathcal{N}(A)$ and $Y = \mathcal{R}(A) \oplus \mathcal{N}(AB)$.

**Theorem 1.1.3.** Let $A \in \mathcal{L}(X, Y)$ be inner regular, and let $T$ and $S$ be closed subspaces of $X$ and $Y$ respectively, such that $X = T \oplus \mathcal{N}(A)$ and $Y = \mathcal{R}(A) \oplus S$. Then $A$ has the following matrix form:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} T \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ S \end{bmatrix},$$

(1.3)

where $A_1$ is invertible.

Moreover, if $B$ is any inner generalized inverse of $A$ such that $\mathcal{R}(BA) = T$ and $\mathcal{N}(AB) = S$, then $B$ has the form

$$B = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & W \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ S \end{bmatrix} \rightarrow \begin{bmatrix} T \\ \mathcal{N}(A) \end{bmatrix},$$

(1.4)

where $W \in \mathcal{L}(S, \mathcal{N}(A))$ is arbitrary.

From the proof of Theorem 1.1.3 we see that an inner generalized inverse is not unique if we fix its range and null space. We have to assume that $B$ is a reflexive generalized inverse of $A$ to obtain the following result.

**Corollary 1.1.3.** $B$ is a reflexive generalized inverse of $A$ if and only if $W = 0$.

We traditionally use $A\{1\}$, $A\{2\}$, and $A\{1,2\}$ respectively, to denote the set of all inner, outer and reflexive generalized inverses of an inner regular operator $A$.

Now, we list some properties of outer generalized inverses.

**Theorem 1.1.4.** Let $A \in \mathcal{L}(X, Y)$. Then there exists an outer generalized inverse $B \in \mathcal{L}(Y, X)$ of $A$ ($B \neq 0$) if and only if $A \neq 0$. 
We are able to find the matrix form of outer generalized inverses.

**Theorem 1.1.5.** Let $A \in \mathcal{L}(X,Y)$ be a non zero operator, and let $T$ and $S$, respectively, be subspaces of $X$ and $Y$. Then the following statements are equivalent:

1. There exists some non zero $B \in \mathcal{L}(Y,X)$ such that $BAB = B$ holds, $\mathcal{R}(B) = T$ and $\mathcal{N}(B) = S$.

2. $T$ and $S$ are closed and complemented subspaces of $X$ and $Y$ respectively, $A(T)$ is closed, $A(T) \oplus S = Y$, and the reduction $A|_T : T \rightarrow A(T)$ is invertible.

If (1) or (2) is satisfied, then the operator $B$ in the part (a) is unique.

If conditions of Theorem 1.1.5 are satisfied, then it follows that there exists the unique outer generalized inverse $B$ of $A$ with the prescribed range $T$ and the null space $S$. We denote such $B$ as $A^{(2)}_{T,S}$. We prove the following matrix representation of $A$ and its outer generalized inverse $A^{(2)}_{T,S}$.

**Corollary 1.1.4.** Let $B = A^{(2)}_{T,S}$ be the corresponding outer generalized inverse of $A$. Then $A$ has the following matrix form:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} T \\ \mathcal{N}(BA) \end{bmatrix} \rightarrow \begin{bmatrix} A(T) \\ S \end{bmatrix},$$

where $A_1$ is invertible. Moreover, $A^{(2)}_{T,S}$ has the following form

$$A^{(2)}_{T,S} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} A(T) \\ S \end{bmatrix} \rightarrow \begin{bmatrix} T \\ \mathcal{N}(BA) \end{bmatrix}.$$

### 1.2 Linear equations

Let $A \in \mathcal{L}(X,Y)$ and $b \in Y$. Consider the linear equation

$$Ax = b. \quad (1.5)$$

If $A$ is invertible, then $A^{-1}b$ is the unique solution of 1.5. If $A$ is not invertible and $b \in \mathcal{R}(A)$, then there exists a solution (possible several solutions) of the equation 1.5. If $b \notin \mathcal{R}(A)$, then there are no solutions of 1.5. In the last two cases it is possible to use generalized inverses of $A$ in order to obtain generalized solutions, or pseudo solutions of 1.5. There are several classes of pseudo solutions.
Definition 1.2.1. A vector $x_0 \in X$ is the best approximate solution of the equation 1.5, if the following holds:

$$\|Ax_0 - b\| = \min_{x \in X} \|Ax - b\|.$$ 

The following result concerns the the best approximation problem.

Theorem 1.2.1. Let $A \in \mathcal{L}(X,Y)$ be inner regular, and let $B \in \mathcal{L}(Y,X)$ be an inner generalized inverse of $A$.

If $b \in Y$ is given and $\|I - AB\| = 1$, then $x_0 = Bb$ is the best approximate solution of the equation $Ax = b$.

On the other hand, if $x_0 = Bb$ is the best approximate solution of the equation $Ax = b$ for all $b \in Y$, then $\|I - AB\| = 1$.

1.3 Moore–Penrose inverse

Let $H, K$ be Hilbert spaces and $A \in \mathcal{L}(H, K)$. Find $A^\dagger \in \mathcal{L}(K, H)$ satisfying the following:

$$\begin{align*}
(1) \quad & AA^\dagger A = A, \\
(2) \quad & A^\dagger AA^\dagger = A^\dagger, \\
(3) \quad & (AA^\dagger)^* = AA^\dagger, \\
(4) \quad & (A^\dagger A)^* = A^\dagger A.
\end{align*}$$

Since closed subspaces of a Hilbert space are always complemented, it follows that $A$ is inner regular if and only if $\mathcal{R}(A)$ is closed.

If (1) or (2) holds, then:

(3) implies that the projection $AA^\dagger$ is orthogonal;

(4) implies that the projection $A^\dagger A$ is orthogonal.

If $A^\dagger$ exists, then $AA^\dagger$ is the orthogonal projection onto $\mathcal{R}(A)$, and $A^\dagger A$ is the orthogonal projection onto $\mathcal{R}(A^*)$.

Theorem 1.3.1. Let $H, K$ be Hilbert spaces and let $A \in \mathcal{L}(H, K)$ have a closed range. Then there exists the unique operator $A^\dagger \in \mathcal{L}(K, H)$ (known as the Moore-Penrose inverse of $A$) satisfying previous (1.6) equations.
The very nice assumption that $\mathcal{R}(A)$ is closed can be avoided in general. However, in that case the Moore-Penrose inverse is not bounded.

Since the orthogonal projections have the norm equal to 1, it follows that the Moore-Penrose inverse is important in solving linear equations.

**Theorem 1.3.2.** Let $A \in \mathcal{L}(H,K)$ have a closed range and let $b \in K$. Then $x_0 = A^\dagger b$ is the best approximate solution of the linear equation $Ax = b$. Moreover, if $M$ is the set of all best approximate solutions of the equation $Ax = b$, then $x_0 = \min \{ \| x \| : x \in M \}$.

Previous result allows the applications of the Moore-Penrose inverse in many approximation problems: solving operator equations, statistics, numerical linear algebra, image recovering, CAGD,...

### 1.4 Moore–Penrose inverse in $C^*$-algebras and rings with involution

Analogously to the case of bounded linear operators on Hilbert space, it is possible to define the Moore-Penrose inverse of an element in a ring, or in a $C^*$-algebra.

**Definition 1.4.1.** Let $\mathcal{R}$ be a ring with involution, and $a \in \mathcal{R}$. The Moore-Penrose inverse of $a$ is the element $a^\dagger \in \mathcal{R}$ which satisfies:

\[(1) \quad aa^\dagger a = a, \quad (2) \quad a^\dagger aa^\dagger = a^\dagger, \quad (3) \quad (aa^\dagger)^* = aa^\dagger, \quad (4) \quad (a^\dagger a)^* = a^\dagger a. \quad (1.7)\]

If $a \in \mathcal{R}$ and $a^\dagger$ exist, then $a$ is MP-invertible. In this case $a^\dagger$ is unique.

The following result is a consequence of a direct computation.

**Theorem 1.4.1.** For any $a \in \mathcal{R}^\dagger$ and $\lambda \in \mathbb{C}$, the following is satisfied:

\[(1) \quad (a^\dagger)^\dagger = a; \quad (2) \quad (a^*)^\dagger = (a^\dagger)^*; \quad (3) \quad (\lambda a)^\dagger = \lambda^\dagger a^\dagger \text{ where } \lambda^\dagger = \begin{cases} \lambda^{-1}, & \lambda \neq 0, \\ 0, & \lambda = 0; \end{cases} \quad (4) \quad (a^\dagger a)^\dagger = a^\dagger (a^\dagger)^*; \quad (5) \quad (aa^*)^\dagger = (a^*)^\dagger a^\dagger; \]
(6) \( a^* = a^\dagger a^* = a^* a a^\dagger; \)

(7) \( a^\dagger = (a^* a)^\dagger a^* = a^* (a a^*)^\dagger; \)

(8) \( (a^*)^\dagger = a(a^* a)^\dagger = (a a^*)^\dagger a. \)

From the last result we actually see that the following chain of equivalences hold:

\[ a \in \mathcal{R}^\dagger \iff a^* \in \mathcal{R}^\dagger \iff a a^* \in \mathcal{R}^\dagger \iff a^* a \in \mathcal{R}^\dagger. \]

We do not need all properties of \( C^* \)-algebras to prove nice properties of generalized inverses. We list some useful properties.

A ring \( \mathcal{R} \) is called \( * \)-reducing if for all \( a \in \mathcal{R} \) the following implication holds:

\[ a^* a = 0 \implies a = 0. \]

All \( C^* \)-algebras are \( 8 \)-reducing.

A ring \( \mathcal{R} \) is a Rickart \( * \)-ring, if for every \( a \in \mathcal{R} \) there exists an selfadjoint projection \( p = p^* = p^2 \in \mathcal{R} \) such that \( o a = \mathcal{R} p. \)

All von-Neumann algebras are Rickart \( * \)-rings.

A ring \( \mathcal{R} \) with an involution has the Gelfand–Naimark property if

\[ 1 + a^* a \in \mathcal{R}^{-1} \text{ for all } a \in \mathcal{R}. \]

Every \( C^* \)-algebra possesses the Gelfand-Naimark property.

**Definition 1.4.2.** Let \( \mathcal{R} \) be a ring with involution. An element \( a \in \mathcal{R} \) is well-supported if there exists a selfadjoint idempotent \( p \) such that

\[ a p = a, \quad a^* a + 1 - p \in \mathcal{R}^{-1} \]

The idempotent \( p \) is called the support of \( a \).

If \( \mathcal{R} = \mathcal{L}(H) \) and \( H \) is Hilbert space, then \( A \in \mathcal{R} \) is well-supported if and only if \( \mathcal{R}(A^*) \) is closed in \( H \). In this case \( P \) is the orthogonal projection onto \( \mathcal{R}(A^*) \).

The question of the Moore-Penrose invertibility in a unital \( C^* \)-algebra \( \mathcal{A} \) is solved by the following theorem.
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Theorem 1.4.2. In a unital $C^*$-algebra $\mathcal{A}$ the following conditions on $a \in \mathcal{A}$ are equivalent:

1. $a$ is well-supported;
2. $a$ is Moore–Penrose invertible;
3. $a$ is inner invertible.

The equivalence (2) $\iff$ (3) is the well-known result of R. Harte and M. Mbekhta (1992).

The present form is proved by J. Koliha (1999).

This result can also be obtained using the GNS Theorem and analytic functional calculus.

It is possible to extend the notion of the Moore-Penrose inverse to Banach algebra elements.

Definition 1.4.3. Let $\mathcal{A}$ be a Banach algebra. An element $a \in \mathcal{A}$ is Hermitian, if $\|e^{ita}\| = 1$ for all real $t$.

Once we have Hermitian elements, we can present the definition of the Moore-Penrose inverse as follows:

Definition 1.4.4. (V. Rakočević) The element $a \in \mathcal{A}$ is MP-invertible, if there exists an element $a^\dagger \in \mathcal{A}$ satisfying the following:

$$aa^\dagger a = a, \quad a^\dagger aa^\dagger = a^\dagger, \quad \text{anbd } aa^\dagger, a^\dagger a \text{ are Hermitian.}$$

If $a^\dagger$ exists, then it is unique.

There is a different approach, requiring directly $\|aa^\dagger\| = 1$ and $\|a^\dagger a\| = 1$.

Somehow, the first definition in Banach algebras is more convenient and applicable.

1.5 Drazin inverse in Banach algebras

An element $a \in \mathcal{A}$ is generalized Drazin invertible, or Koliha-Drazin invertible, if 0 is not an accumulation point of the spectrum $\sigma(a)$.
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Equivalently, \( a \in \mathcal{A} \) is generalized Drazin invertible, if and only if there exists an idempotent \( p \) satisfying:

\[
a + p \in \mathcal{A}^{-1}, \; ap \text{ is quasinilpotent, } p \text{ commutes with } a.
\]

Obviously, \( p \) is the spectral idempotent of \( a \) corresponding to the spectral set \( \{0\} \).

Define \( f(z) \) to be equal \( \frac{1}{z} \) in a neighborhood of \( \sigma(a) \setminus \{0\} \), and \( f(z) = 0 \) in a neighborhood of \( \{0\} \).

Then \( f(a) = a^D \) is the generalized Drazin inverse of \( a \).

**Theorem 1.5.1.** Let \( a \in \mathcal{A} \). There exists the unique element \( a^D \in \mathcal{A} \) satisfying

\[
aa^D = a^Da, \quad a^Da^D = A^D, \quad a(1 - aa^D) \text{ is quasinilpotent.}
\]

In this case \( a^D \) is the generalized Drazin inverse of \( a \), and \( p = 1 - aa^D \) is the spectral idempotent.

If we require that \( a(1 - aa^D) \) is nilpotent, or equivalently that \( ap \) is nilpotent, then \( a^D \) reduces to the (ordinary) Drazin inverse. In this case \( a \) is Drazin invertible.

The smallest \( n \) such that \( [a(1 - aa^D)]^n = 0 \), i.e., called the Drazin index of \( a \), and it is denoted by \( \text{ind}(a) \).

\[
\text{ind}(a) = 0 \text{ if and only if } a \in \mathcal{A}^{-1}.
\]

\[
\text{ind}(a) \leq 1 \text{ if and only if } a \text{ is group invertible.} \text{ In this case the multiplicative semigroup of } \mathcal{A} \text{ generated by } \{a, a^D\} \text{ is actually a group with the unit } aa^D.
\]

**Example 1.5.1.** Let \( \mathcal{A} = \mathbb{C}^{n \times n} \) is the algebra of square matrices of a fixed dimension. Then (Jordan theorem)

\[
A = B^{-1} \begin{bmatrix} A_1 & 0 \\ 0 & N \end{bmatrix} B,
\]

where \( A_1 \) is invertible and \( N \) is nilpotent block matrix.

The Drazin index of \( A \) is equal to the nilpotency index of \( N \), and

\[
A^D = B^{-1} \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}.
\]
Theorem 1.5.2. Let $X$ be a Banach space and $A \in \mathcal{L}(X)$ be Drazin invertible. If $P$ is the spectral idempotent of $A$ corresponding to $\{0\}$, then

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \end{bmatrix},$$

where $A_1$ is invertible and $A_2$ is nilpotent.

Also,

$$A^D = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \end{bmatrix} \to \begin{bmatrix} \mathcal{N}(P) \\ \mathcal{R}(P) \end{bmatrix}.$$

If $A$ is generalized Drazin invertible, then $A_2$ is quasinilpotent.

Drazin inverse find applications in solving singular systems of differential equations.
Chapter 2

Reverse order rule

2.1 Basic theorems

Let $a, b$ be invertible elements of a semigroup $S$ with the unit $1$. Then the formula

$$(ab)^{-1} = b^{-1}a^{-1}$$

is called the reverse order rule (for the ordinary inverse).

It is important to notice that, in general:

$$(ab)^\dagger \neq b^\dagger a^\dagger,$$

$$(ab)^D \neq b^Da^D,$$

$$(ab)^- \neq b^-a^-,$$

where $a^-$ stands for an arbitrary generalized inverse of $a$.

There are many necessary and sufficient conditions such that

$$(ab)^\dagger = b^\dagger a^\dagger$$

holds.

There are sufficient conditions that

$$(ab)^D = b^Da^D$$

holds.
In several papers we were interested in the reverse order rule for the Moore-Penrose inverse.

Notice that even in the case of Hilbert spaces, the product of two bounded operators with closed ranges is not necessarily an operator with closed range.

Let $X$ be a Banach space and let $A, B \in \mathcal{L}(X)$ have Drazin inverses. Bouldin (1982) proved the following result:

**Theorem 2.1.1.** If $B^D B$ commutes with $A$, $A^D A$ commutes with $B$ and
\[
\mathcal{N}((AB)^j) \supset \mathcal{N}(A^D) \cup \mathcal{N}(B^D)
\] (2.1)
holds for some nonnegative integer $j$, then
\[
(AB)^D = B^D A^D,
\]
\[
\mathcal{N}(AB)^D = \text{span}\{\mathcal{N}(A^D), \mathcal{N}(B^D)\}, \quad \mathcal{R}((AB)^D) = \mathcal{R}(A^D) \cap \mathcal{R}(B^D)
\]
and the least $j$ for which (1) holds is the Drazin index of $AB$.

We prove the following result concerning the generalized Drazin inverse.

**Corollary 2.1.1.** If $A, B, AB \in \mathcal{L}(X)$ have generalized Drazin inverses, $AA^d$ commutes with $BB^d$, $\mathcal{R}(B^d A^d) = \mathcal{R}((AB)^d)$ and $\mathcal{N}(B^d A^d) = \mathcal{N}((AB)^d)$, then $(AB)^d = B^d A^d$.

**Theorem 2.1.2.** Let $A, B, AB \in \mathcal{L}(X)$ have group inverses. Then the following statements are equivalent:

1. $AA^D$ commutes with $BB^D$, $\mathcal{R}((AB)^D) = \mathcal{R}(B^D A^D)$ and $\mathcal{N}((AB)^D) = \mathcal{N}(B^D A^D)$.
2. $(AB)^D = B^D A^D$. 

2.2 Products of generalized inverses

If $A, B$ are bounded linear operators, we seek for the conditions of the form:

\[ B\{1,3\} \cdot A\{1,3\} \subset (BA)\{1,3\}, \]
\[ B\{1,4\} \cdot A\{1,3\} \subset (BA)\{1,4\}, \]
\[ B^\dagger A^\dagger \in (AB)\{1,2,3\}, \]
\[ B^\dagger A^\dagger \in (AB)\{1,2,4\}, \]
\[ B^\dagger A^\dagger \in (AB)\{1,3\}, \]
\[ B^\dagger A^\dagger \in (AB)\{1,4\}. \]

**Theorem 2.2.1.** Let $A \in \mathcal{L}(K, L)$ and $B \in \mathcal{L}(H, K)$ be such that $A, B, AB$ have closed ranges. Then the following statements are equivalent:

1. $\mathcal{R}(A^*AB) \subset \mathcal{R}(B)$;
2. $B\{1,3\} \cdot A\{1,3\} \subset (AB)\{1,3\}$;
3. $B^\dagger A^\dagger \in (AB)\{1,3\}$;
4. $B^\dagger A^\dagger \in (AB)\{1,2,3\}$.

**Theorem 2.2.2.** Let $A \in \mathcal{L}(K, L)$ and $B \in \mathcal{L}(H, K)$ be such that $A, B, AB$ have closed ranges. Then the following statements are equivalent:

1. $\mathcal{R}(BB^*A^*) \subset \mathcal{R}(A^*)$;
2. $B\{1,4\} \cdot A\{1,4\} \subset (AB)\{1,4\}$;
3. $B^\dagger A^\dagger \in (AB)\{1,4\}$;
4. $B^\dagger A^\dagger \in (AB)\{1,2,4\}$.

2.3 Reverse order rule in rings with involution

**Theorem 2.3.1.** Let $\mathcal{R}$ be a ring with involution, let $a, b \in \mathcal{R}$ be MP-invertible and let $(1 - a^\dagger a)b$ be left $*$-cancellable. Then the following condition are equivalent:

1. $ab$ is MP-invertible and $(ab)^\dagger = b^\dagger a^\dagger$;
2. $[a^\dagger a, bb^*] = 0$ and $[bb^*, a^*a] = 0$.

If $\mathcal{R} = \mathcal{L}(X)$ is the ring of all bounded linear operators on a Hilbert space $H$, and if $A, B \in \mathcal{L}(X)$ are two closed range operators such that

\[ [A^\dagger A, BB^*] = 0 \quad \text{and} \quad [B^\dagger B, A^*A] = 0, \]
then 6.3.2 Theorem implies that $AB$ is also a closed range operator.

Some typical theorems have the following forms:

**Theorem 2.3.2.** Let $a, b, a^\dagger abb^\dagger \in \mathcal{R}^\dagger$, let $ab \in \mathcal{R}^\#$ and let $(1 - a^\dagger a)b$ be left $^*$-cancellable. Then $(ab)^\# = b^\dagger a^\dagger$ if and only if $(ab)^\# = b^\dagger (a^\dagger abb^\dagger)^\dagger a^\dagger$ and any one of the following equivalent conditions holds:

(a) $abb^\dagger a^\dagger ab = ab$;
(b) $b^\dagger a^\dagger abb^\dagger a^\dagger = b^\dagger a^\dagger$;
(c) $a^\dagger abb^\dagger = bb^\dagger a^\dagger a$;
(d) $a^\dagger abb^\dagger$ is an idempotent;
(e) $bb^\dagger a^\dagger$ is an idempotent;
(f) $b^\dagger (a^\dagger abb^\dagger)^\dagger a^\dagger = b^\dagger a^\dagger$;
(g) $(a^\dagger abb^\dagger)^\dagger = bb^\dagger a^\dagger a$.

**Theorem 2.3.3.** If $a, b, a^\# ab, a^\# abb^\# \in \mathcal{R}^\#$, then the following statements are equivalent:

(i) $(a^\# ab)^\# = b^\# (a^\# abb^\#)^\#$,
(ii) $b^\# (a^\# abb^\#)^\# \in (a^\# ab)\{5\}$,
(iii) $bb^\# a^\# ab = a^\# ab$ and $ba^\# abb^\# (a^\# abb^\#)^\# = (a^\# abb^\#)^\# a^\# ab$,
(iv) $b\{1, 5\} \cdot (a^\# abb^\#)\{1, 5\} \subseteq (a^\# ab)\{5\}$,
(v) $(a^\# abb^\#)^\# = b(a^\# ab)^\#$,
(vi) $b(a^\# ab)^\# \in (a^\# abb^\#)\{5\}$,
(vii) $b \cdot (a^\# ab)\{1, 5\} \subseteq (a^\# abb^\#)\{5\}$.

**Theorem 2.3.4.** Let $\mathcal{A}$ be a unital $C^*$–algebra and let $a, b \in \mathcal{A}^-$. Then the following conditions are equivalent:
(a) $ab^\dagger a^\dagger ab = ab$;

(b) $b^\dagger a^\dagger abb^\dagger a^\dagger = b^\dagger a^\dagger$;

(c) $a^\dagger abb^\dagger = bb^\dagger a^\dagger a$;

(d) $a^\dagger abb^\dagger$ is an idempotent;

(e) $bb^\dagger a^\dagger a$ is an idempotent;

(f) $b^\dagger (a^\dagger abb^\dagger)^\dagger a^\dagger = b^\dagger a^\dagger$;

(g) $(a^\dagger abb^\dagger)^\dagger = bb^\dagger a^\dagger a$. 
Chapter 3

Adjointable operators on Hilbert $C^*$-modules

Parts of these books are taken for the presentation.


3.1 Hilbert modules

Let $\mathcal{A}$ be a complex $C^*$-algebra, not necessarily unital. If $\mathcal{A}$ has the unit, then its unit is denoted by 1. If $\mathcal{A}$ does not have the unit, then $\mathcal{A}_1 = \mathcal{A} \oplus \mathbb{C}$ will denote the unitization of $\mathcal{A}$.

**Definition 3.1.1.** Let $\mathcal{M}$ be a complex linear space, and let $\mathcal{A}$ be a $C^*$-algebra. $\mathcal{M}$ is a right $\mathcal{A}$-module, provided that there is an exterior multiplication of elements in $\mathcal{M}$ by elements in $\mathcal{A}$ (the action of $\mathcal{A}$ to $\mathcal{M}$), denoted by $\cdot$, satisfying the following conditions, for all $x, y \in \mathcal{M}$, $a, b \in \mathcal{A}$, and $\lambda \in \mathbb{C}$:

1. $(x + y) \cdot a = x \cdot a + y \cdot a$;
2. $x \cdot (a + b) = x \cdot a + y \cdot b$;
3. $x \cdot (ab) = (x \cdot a) \cdot b$;
4. $\lambda(xa) = (\lambda x)a = x(\lambda a)$.
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It is easy to see that $x \cdot 0 = 0 \cdot a = 0 \in \mathcal{M}$ for all $x \in \mathcal{M}$ and $a \in A$. Moreover, if $A$ is unital, then $x \cdot 1 = x$ for all $x \in \mathcal{M}$.

We see that the axioms for modules are the same as those for linear spaces, with one major difference: we take "scalars" form a $C^*$-algebra $A$, instead of a field.

In an $A$-module $\mathcal{M}$, the $A$-valued semi inner product can be defined.

Definition 3.1.2. Let $A$ be a $C^*$-algebra and let $\mathcal{M}$ be a right $A$-module. An $A$-valued semi inner product in $\mathcal{M}$ is the following function $\langle \cdot, \cdot \rangle: \mathcal{M} \times \mathcal{M} \to A$,
which satisfies the following conditions for all $x, y, z \in \mathcal{M}$, $a \in A$ and $\lambda, \mu \in \mathbb{C}$:

1. $\langle x, x \rangle \geq 0$ in $A$;
2. $\langle x, x \rangle = 0$ if and only if $x = 0$;
3. $\langle x, y \rangle = \langle y, x \rangle^*$;
4. $\langle x, \lambda y + \mu z \rangle = \lambda \langle x, y \rangle + \mu \langle x, z \rangle$;
5. $\langle x, y \cdot a \rangle = \langle x, y \rangle a$.

In this case, $\mathcal{M}$ is a pre-Hilbert right $A$-module.

Also, if $x, y \in \mathcal{M}$ and $a \in A$, then $\langle x \cdot a, y \rangle = a^* \langle x, y \rangle$.

If $\mathcal{M}$ satisfies all previous axioms, only weaker form then the axiom (2):

(2') if $x = 0$, then $\langle x, x \rangle = 0$, 
then $\mathcal{M}$ is a semi pre-Hilbert $A$-module.

From now on, $\mathcal{M}$ is a pre-Hilbert $A$-module.

For $x \in \mathcal{M}$ define $\|x\|_{\mathcal{M}} := \|\langle x, x \rangle\|^{1/2}$.
The function $\| \cdot \|_{\mathcal{M}}$ is a norm on $\mathcal{M}$, and it has several interesting properties.

Lemma 3.1.1. If $\mathcal{M}$ is a pre-Hilbert $A$-module, then for all $x \in \mathcal{M}$, $x \mapsto \|x\|_{\mathcal{M}} = \|\langle x, x \rangle\|^{1/2}$ is the norm on $\mathcal{M}$, which also obeys the following properties:

1. $\langle x, y \rangle \langle y, x \rangle \leq \|y\|^2_{\mathcal{M}} \langle x, x \rangle$, for all $x, y \in \mathcal{M}$;
2. $\|x \cdot a\|_{\mathcal{M}} \leq \|x\|_{\mathcal{M}} \|a\|$, for all $x \in \mathcal{M}$ and all $a \in A$;
3. $\|\langle x, y \rangle\| \leq \|x\|_{\mathcal{M}} \|y\|_{\mathcal{M}}$ for all $x, y \in \mathcal{M}$.

The inequality (3) of Lemma 3.1.1 is known as the Cauchy-Bunyakovsky-Schwarz inequality in Hilbert $C^*$-modules.
Definition 3.1.3. Let \( \mathcal{M} \) be a pre-Hilbert right \( \mathcal{A} \)-module. If it is complete with respect to the norm \( \| \cdot \| \), then \( \mathcal{M} \) is as a Hilbert right \( \mathcal{A} \)-module, or simply a Hilbert \( \mathcal{A} \)-module.

Let \( \mathcal{N} \) be a closed submodule of a Hilbert \( C^* \)-module \( \mathcal{M} \). The orthogonal complement of \( \mathcal{N} \) is defined as \( \mathcal{N}^\perp = \{ x \in \mathcal{M} : (\forall y \in \mathcal{N}) \langle x, y \rangle = 0 \} \).

Example 3.1.1. Let \( A = C[a, b] \) be the \( C^* \)-algebra of all complex continuous functions on a segment \([a, b]\), and let \( \mathcal{M} = A \). Notice that \( \langle f, g \rangle = fg \) for all \( f, g \in \mathcal{M} \).

Let \( \mathcal{N} = C_0(a, b) \) be the subset of \( A \) containing all functions that vanishes at the end of interval. In other words, if \( f \in C[a, b] \), then \( f \in \mathcal{N} \) if and only if \( f(a) = f(b) = 0 \).

It is easy to see that \( \mathcal{N} \) is a submodule of \( \mathcal{M} \). Obviously, \( \mathcal{N}^\perp = \{ 0 \} \).

3.2 Bounded and adjointable operators

Let \( \mathcal{M}, \mathcal{N} \) be Hilbert \( \mathcal{A} \)-modules, where \( \mathcal{A} \) is a \( C^* \)-algebra. A mapping \( T : \mathcal{M} \to \mathcal{N} \) is called operator if \( T \) is a \( \mathbb{C} \)-linear \( \mathcal{A} \)-homomorphism from \( \mathcal{M} \) to \( \mathcal{N} \); i.e. \( T \) satisfies:

\[
T(x + y) = T(x) + T(y), \quad T(\lambda x) = \lambda T(x), \quad T(x \cdot a) = T(x) \cdot a,
\]

for all \( x, y \in \mathcal{M} \), \( a \in \mathcal{A} \), and \( \lambda \in \mathbb{C} \).

Operator \( T \) is bounded, if there exists some \( M \geq 0 \) such that

\[
\| T(x) \|_\mathcal{N} \leq M \| x \|_\mathcal{M}, \; x \in \mathcal{M}.
\]

The norm of \( T \) is given by

\[
\| T \| = \inf \{ M \geq 0 : \| T(x) \|_\mathcal{N} \leq M \| x \|_\mathcal{M}, \; \text{for all} \; x \in \mathcal{M} \}.
\]

The set of all bounded operators from \( \mathcal{M} \) to \( \mathcal{N} \) is denoted by \( \text{Hom}_\mathcal{A}(\mathcal{M}, \mathcal{N}) \equiv \text{Hom}(\mathcal{M}, \mathcal{N}) \). Particularly, \( \text{End}(\mathcal{M}) \equiv \text{End}_\mathcal{A}(\mathcal{M}) = \text{Hom}_\mathcal{A}(\mathcal{M}, \mathcal{M}) \).

Lemma 3.2.1. \( \text{Hom}(\mathcal{M}, \mathcal{N}), \| \cdot \| ) \) is a Banach space. \( \text{End}_\mathcal{A}(\mathcal{M}) \) is a Banach algebra.
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Notice that from $M^\perp = \{0\}$, for all $y, z \in M$ we have the implication:

$$\left( \forall x \in M \right) \langle x, y \rangle = \langle x, z \rangle \implies y = z.$$  

Also, if $x_n \to x$ in $M$, then for all $y \in N$ we have $\langle x_n, y \rangle \to \langle x, y \rangle$.

Lemma 3.2.2. Let $M, N$ be Hilbert $A$-modules, and let $T : N \to M$ and $T^* : M \to N$ be mappings such that

$$\langle x, Ty \rangle = \langle T^* x, y \rangle$$

for all $x \in M$, $y \in N$.

Then $T \in \text{Hom}(M, N)$ and $T^* \in \text{Hom}(N, M)$.

Definition 3.2.1. An operator $T \in \text{Hom}(M, N)$ is adjointable, if there exists and operator $T^* \in \text{Hom}(N, M)$ such that for all $x \in M$ and all $y \in N$ the following holds:

$$\langle Tx, y \rangle = \langle x, T^* y \rangle.$$

There exists operators that are not adjointable.

The set of all adjointable operators from $M$ to $N$ is denoted by $\text{Hom}^*(M, N) \equiv \text{Hom}^*_A(M, N)$.

Lemma 3.2.3. $\text{End}^*(M)$ is a $C^*$-algebra.

Theorem 3.2.1. For $T \in \text{End}^*_A(M)$ the following conditions are equivalent:

1. $T$ is a positive element in the $C^*$-algebra $\text{End}^*_A(M)$;
2. For all $x \in M$ the element $\langle Tx, x \rangle$ is positive in the $C^*$-algebra $A$.

Important result follows.

Theorem 3.2.2. (Misčenko) Let $M, N$ be Hilbert $A$-modules, and let $T \in \text{Hom}^*_A(M, N)$ such that $R(T)$ is closed in $N$. Then the following hold:

1. $N(T)$ is a complemented submodule of $M$ and $N(T)^\perp = R(T^*)$;
2. $R(T)$ is a complemented module of $N$ and $R(T)^\perp = N(T^*)$;
3. $T^*$ also has a closed range.

This theorem is fundamental for the geometric investigation of the Moore-Penrose inverse.
Chapter 4

New results

This section contains new and unpublished results.

4.1 Reverse order law for adjointable bounded operators

The notion for the commutators follows: 

\[ [U, V] = UV - VU, \]

for appropriate choice of operators \( U \) and \( V \).

Let \( \mathcal{M}, \mathcal{N}, \mathcal{L} \) be Hilbert modules, and let \( A \in \text{Hom}^*(\mathcal{N}, \mathcal{L}) \) and \( B \in \text{Hom}^*(\mathcal{M}, \mathcal{N}) \) have closed ranges, such that \( AB \) also has a closed range. Find necessary and sufficient conditions such that the reverse order law holds:

\[ (AB)^\dagger = B\dagger A\dagger. \]

**Theorem 4.1.1.** If \( A \in \text{Hom}^*(\mathcal{N}, \mathcal{L}), B \in \text{Hom}^*(\mathcal{M}, \mathcal{N}) \) and \( AB \in \text{Hom}^*(\mathcal{M}, \mathcal{N}) \) have closed ranges, then the following statements are equivalent:

1. \( (AB)^\dagger = B\dagger A\dagger; \)
2. \( [A\dagger A, BB^*] = 0 \) and \( [A^* A, BB^\dagger] = 0; \)
3. \( R(A^*AB) \subset R(B) \) and \( R(BB^*A^*) \subset R(A^*); \)
4. \( A^*ABB^* \) has a commuting Moore-Penrose inverse.
This result is well-known for complex matrices, bounded operators on Hilbert spaces, and in partially in specific rings with involution.