Constructing the duals of quantum groups and semigroups without the Haar weight

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Cheongpung, Korea, August 10, 2014

Duality of non-commutative groups

Pontryagin's duality : $G \simeq \widehat{\widehat{G}}$

where G is a locally compact abelian group and $\widehat{G}$ is the dual group

If G is non-commutative : $\widehat{G}$ is no more a group
so we pass to group algebras : $C_0(G)$ and $C^*(G)$,
or $L^\infty(G)$ and $\mathcal{L}(G)$, or another pair

Kac algebras (1970s : Vainerman, Kac, Enock, Schwartz)
Locally compact quantum groups (LCQG) 2000s, Kustermans, Vaes
I am not giving a historical overview !

Duality for locally compact quantum groups

Von Neumann algebraic LCQG is a von Neumann algebra M with a comultiplication $\Delta : M \to M \otimes M$, which is a normal unital *-homomorphism such that $(\mathbb{I} \otimes \Delta)\Delta = (\Delta \otimes \mathbb{I})\Delta$; and a pair of Haar (invariant) weights ϕ, ψ

Classical example : $M = L^\infty(G)$, $\widehat{M} = \mathcal{L}(G)$

Duality (Kustermans–Vaes)

If M is a von Neumann algebraic LCQG, then the following diagram commutes :

$$\begin{tikzcd} M \arrow[r, mapsto, "\text{predual}"] & M_* \arrow[d, "\lambda_\varphi"] \\ \widehat{M}_* \arrow[u, "\lambda_{\widehat{\varphi}}"] & \widehat{M} \arrow[l, mapsto, "\text{predual}"'] \end{tikzcd}$$

The weight φ is used both to define the dual algebra and to guarantee the duality : $\widehat{\widehat{M}} \simeq M$

Definition of algebras we are working with

M is a VN bialgebra with antipode (VNBA) if :
M is a von Neumann algebra ; there exists a comultiplication and an antipode $S : D(S) \subset M \to M$ such that

1. $S : D(S) \to M$ is an anti-homomorphism ;
2. $(*S)^2 = \mathbb{I}$;
3. $D(S)$ is σ-weakly dense in M ;
4. S is a "coalgebra morphism" : if $\mu, \nu \in M_*$ are such that $\mu \circ S, \nu \circ S \in M$, then for $x \in D(S)$ $(\Delta(x))(\nu \circ S \otimes \mu \circ S) = (\Delta S(x))(\mu \otimes \nu)$.

Example : LCQG, but also quantum semigroups
If S is bounded, M is a coinvolutive Hopf-VNA,
a class which includes Kac algebras

Duality without the Haar measure

Multiplicative unitaries :
Baaj, Skandalis : regular multiplicative unitaries
Woronowicz : manageable multiplicative unitaries

Recall that for a LCQG, both M and $\widehat{M}$ are generated by a unitary $W \in M\bar{\otimes}\widehat{M}$:

$$M = \{\omega \otimes \mathrm{id})(W) : \omega \in \widehat{M}_*\}$$

$$\widehat{M} = \{\omega \otimes \mathrm{id})(W) : \omega \in M_*\}$$

One can start with a W and define M, $\widehat{M}$ as above.
If W is "good" then these are Hopf algebras

but how to construct such unitaries ?

Representations of M_*

Let M be a von Neumann bialgebra with an antipode
Recall that M_* is a Banach algebra, with
$(\mu * \nu)(x) = (\mu \otimes \nu)(\Delta(x))$ for $\mu, \nu \in M_*$, $x \in M$

The natural involution is defined by $\mu^*(x) = \overline{\mu\big((Sx)^*\big)}$,
and is defined only for a subset $M_{**} \subset M_*$
Kustermans, 2000 : if $M = L^\infty(\mathbb{G})$ is a LCQG, then $M_{**} = L_1(\mathbb{G})^\sharp$
is dense in M_*.

In general, M_{**} might not be dense in M_*, but one can pass to
$M^r = M/(M_{**})^\perp$, then $M^r_* = [M_{**}]$ and M^r is still a VNBA

A *-representation of M_* is by definition repn which is involutive on M_{**}
We consider only representations $\pi : M_* \to B(H)$ which are
completely bounded, and this is equivalent to the existence of
$U \in M \bar{\otimes} B(H)$ (generator) such that
$U(\mu, \omega) = \omega(\pi(\mu))$ for all $\mu \in M_*, \omega \in B(H)_*$

Unitary representations

Definition

Call a *-representation π of M_* on a Hilbert space H unitary if (i) or (ii) holds :
(i) the generator U is unitary ; (ii), equivalently, in some basis (e_α) of H, with $\pi_{\alpha\beta}(\mu) = \langle \pi(\mu) e_\beta, e_\alpha \rangle$,

$$\sum_\gamma \pi^*_{\gamma\alpha} \cdot \pi_{\gamma\beta} = \sum_\gamma \pi_{\alpha\gamma} \cdot \pi^*_{\beta\gamma} = \begin{cases} 1, & \alpha = \beta \\ 0, & \alpha \neq \beta \end{cases}$$

for every α, β, the series converging absolutely in the M_*-weak topology of M.

Commutative case

If M_* is commutative, then its irreducible representations are characters ;

π is unitary $\Leftrightarrow$ it is a unitary element in M :

$\pi^*\pi = \pi\pi^* = 1$

Consider $M_* = B(G) = C^*(G)^*$, the Fourier–Stieltjes algebra of G.
$M = B(G)^* = W^*(G) = C^*(G)^{**}$, the "big group algebra" of J. Ernest

Theorem (M. Walter)

For a character u of $B(G)$ TFAE :

- u is unitary in $W^*(G)$
- $u(f) = f(t)$ for some $t \in G$
- $u|_{A(G)} \not\equiv 0$.

Representations of the measure algebra

Theorem

An irreducible representation π of $M(G)$ is unitary if and only if it is generated by a continuous unitary representation $\tilde{\pi}$ of G by the integral formula

$$\pi(\mu) = \int_G \tilde{\pi}(t) d\mu(t), \qquad (*)$$

The unitary dual and the universal C^*-algebra

Theorem (E. Kirchberg, 1977)

There is a functor $M \mapsto W^*U(M)$ on the category of CHvNA such that

$\{$normal reps of $W^*U(M)\} \leftrightarrow \{$ unitary reps of $M_*\}$

If M has a Haar weight, then $\widehat{M} = W^*(M_*)$ and $\widehat{M} \simeq \widehat{\widehat{\widehat{M}}}$.

C^*-version : Ch.-K. Ng (2002)

Theorem (J. Kustermans, 2000)

Let $\mathbb{G}$ be a LCQG. Then there exists a C^*-algebra $C^*_u(\widehat{\mathbb{G}})$ whose *-representations are in bijection with unitary *-representations of $L_1(\mathbb{G})$.

Absolutely continuous ideal

Definition

$M_*^0 = \bigcap\{\ker \pi : \pi \text{ is irreducible and non-unitary}\}$

By definition, if an irrep $\pi \not\equiv 0$ on M_*^0 then π is unitary.
If M is weakly separable, by disintegration one can show this
for general π which is non-degenerate on M_*^0.
We need more :
if π is unitary (on M_*) then it is non-degenerate on M_*^0 $\quad (*)$
Solution : reset $M_*^0 = 0$ if this does not hold ; then either $(*)$ or $M_*^0 = 0$.
In fact : set $M_*^0 = 0$ if the ideal generated by $(M_*^0)^\perp$ equals M

Finally : if $M_*^0 \neq 0$ then
π is unitary iff π is non-degenerate on M_*^0

If M is a LCQG : $M_*^0 = M_*$
If $M = L^\infty(S)$ for a semigroup S with only e invertible, then $M_*^0 = 0$

Group case

If $M_* = M(G)$, this ideal is the common kernel of all reps which annihilate $L_1(G)$ and has thus the same representations as $L_1(G)$; $C^*(M_*^0) = C^*(G)$.

J. Taylor denoted it by $L^{1/2}(G)$ and proved that $M_*^0 \neq L_1(G)$ unless G is discrete.

If $M_* = B(G)$, it is the common kernel of all non-evaluation characters ; has the same characters as $A(G)$, i.e. evaluations at points of G ; $C^*(M_*^0) = C_0(G)$

$$\begin{tikzcd}
C_0(G)^{**} \arrow[rr, mapsto, squiggly, "predual"] & & M(G) \\
{[A(G)]} \arrow[u, "W^*-env"] \arrow[d, hook] & & {[L_1(G)]} \arrow[u, hook] \arrow[d, "W^*-env"] \\
B(G) & & C^*(G)^{**} \arrow[ll, mapsto, squiggly, "predual"']
\end{tikzcd}$$

Duality : definition

Assume M is weakly separable

Definition

Set $\widehat{M} = W^*(M_*^0)$: the enveloping von Neumann algebra of $C^*(M_*^0)$. Then $\widehat{M}$ has a canonical structure of a VNBA and is called the dual algebra of M.

Theorem

The map $M \mapsto \widehat{M}$ is a functor on the category of weakly separable VNBA.

If M is a LCQG, then $\widehat{M} = W^*(M_*)$;

if M is a Kac algebra then $\widehat{M} \simeq \widehat{\widehat{M}}$ (probably true for all LCQG).

Other examples

$$M = \widehat{N} \text{ and } \left\{ \begin{array}{l} M \neq 0 \text{ is commutative} \Rightarrow M \simeq C_0(G)^{**} \\ \widehat{M} \neq 0 \text{ is commutative} \Rightarrow M \simeq C^*(G)^{**} \end{array} \right\} \text{ for some } G$$

(G is a locally compact group)

Various semigroups M give : $\widehat{M} = 0$ or $\widehat{M} = \mathbb{C}$

$W^*(\mathbb{R}_+)$, $L^\infty(\mathbb{R})$ with $Sf = f$,

$SU_q(N)$ with a bounded antipode ;

$C^*(\Sigma)$ where Σ is a discrete sub-semigroup of a group with $x \in \Sigma$ invertible iff $x = e$;

for $M = L^\infty(\mathbb{R}^2)$ with $Sf(s,t) = f(s,-t)$, we have $\widehat{M} = C_0(\mathbb{R})^{**}$

Conjecture : if the canonical map $D_M : \widehat{\widehat{M}} \to M$ is surjective, then M carries a Haar weight

(satisfied for example if S is bounded and $\widehat{\widehat{\widehat{M}}} \neq 0$)

C^*-version

Theorem

There is a functor $A \mapsto \widehat{A}$ on the category of separable C^*-bialgebras with antipode defined as $\widehat{A} = \widehat{M}$, where $M = A^{**}$ is the enveloping VNA of A with induced structure.

$$A = \widehat{B} \text{ and } \left\{ \begin{array}{l} A \neq 0 \text{ is commutative} \Rightarrow A \simeq C_0(G) \\ \widehat{A} \neq 0 \text{ is commutative} \Rightarrow A \simeq C^*(G) \end{array} \right\} \text{ for some } G$$

(G is a locally compact group)

For every LCG G, the following diagram commutes :

$$\begin{tikzcd}
\widehat{C_*(G)} = C_0(G) \arrow[r, mapsto, "*"] & M(G) \arrow[d, "C^*_{un}-env"] \\
B(G) \arrow[u, "C^*_{un}-env"] & C^*(G) = \widehat{C_0(G)} \arrow[l, mapsto, "*"']
\end{tikzcd}$$