Note on paranormal operators and operator equations $ABA = A^2$ and $BAB = B^2$

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Fredholm Operators

Definitions

Let $T \in B(\mathcal{H})$.

- T is called upper semi-Fredholm if $R(T)$ is closed and $\alpha(T) < \infty$,
- T is called lower semi-Fredholm if $\beta(T) < \infty$.
- T is called Fredholm if $\alpha(T) < \infty$ and $\beta(T) < \infty$, in this case, the index is defined by

$$i(T) := \alpha(T) - \beta(T).$$

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$$i(T) := \alpha(T) - \beta(T).$$

- T is called Weyl if it is Fredholm of index zero.
- T is called Browder if it is Fredholm of finite ascent and descent.

Notations

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- The smallest nonnegative integer q such that $R(T^q) = R(T^{q+1})$ is called the descent of T and denoted by $q(T)$. If no such integer exists, we set $q(T) = \infty$.

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$\sigma_w(T) := \{\lambda \in \mathbb{C} : T - \lambda$ is not Weyl$\}$

$\sigma_b(T) := \{\lambda \in \mathbb{C} : T - \lambda$ is not Browder$\}$

Local Spectrum

Given an arbitrary $T \in B(\mathcal{H})$ on a Hilbert space $\mathcal{H}$, the local resolvent set $\rho_T(x)$ of T at the point $x \in \mathcal{H}$ is defined as the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $f : U \to \mathcal{H}$ which satisfies $(T - \lambda)f(\lambda) = x$ for all $\lambda \in U$.

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We define the local spectral subspaces of T by

$$H_T(F) := \{x \in \mathcal{H} : \sigma_T(x) \subseteq F\} \text{ for all sets } F \subseteq \mathbb{C}.$$

Localized Single Valued Extension Property

Definitions [1952, N. Dunford]

$T \in B(\mathcal{X})$ has the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0) if for every open neighborhood U of λ_0 the only analytic function $f : U \longrightarrow \mathcal{X}$ which satisfies the equation

$$(T - \lambda)f(\lambda) = 0$$

is the constant function $f \equiv 0$ on U. The operator T is said to have SVEP if T has SVEP at every $\lambda \in \mathbb{C}$.

Well Known Facts

$$p(T-\lambda) < \infty \implies T \text{ has SVEP at } \lambda$$

$$q(T-\lambda) < \infty \implies T^* \text{ has SVEP at } \lambda$$

It is well known that if $T - \lambda$ is semi-Fredholm, then these implications are equivalent.

Operator equations $ABA = A^2$ and $BAB = B^2$

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[1964, Ivan Vidav]

A and B are self-adjoint operators satisfying the operator equations (1.1) if and only if $A = PP^*$ and $B = P^*P$ for some idempotent operator P.

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[2006, C. Schmoeger]

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The common spectral properties of the operators A and B satisfying the operator equations (1.1).

[2011, B.P. Duggal]

It is possible to relate the several spectrums, the single-valued extension property and Bishop's property (β) of A and B.

Operator equations $ABA = A^2$ and $BAB = B^2$

Note.

(1) If $\lambda \neq 0$, then

$$N(A - \lambda I) = N(AB - \lambda I) = A(N(B - \lambda I)),$$

$$N(B - \lambda I) = N(BA - \lambda I) = B(N(A - \lambda I)),$$

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Moreover, if $\lambda \neq 0$, then

$$p(A - \lambda I) = p(AB - \lambda I) = p(BA - \lambda) = p(B - \lambda I) \text{ and}$$

$$q(A - \lambda I) = q(AB - \lambda I) = q(BA - \lambda) = q(B - \lambda I)$$

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(2) $\sigma_x(A) = \sigma_x(AB) = \sigma_x(BA) = \sigma_x(B)$,

where $\sigma_x = \sigma,\ \sigma_p,\ \sigma_a,\ \sigma_{SF+},\ \sigma_{SF-},\ \sigma_e,\ \sigma_w,$ or σ_b.

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where $\sigma_x = \sigma,\ \sigma_p,\ \sigma_a,\ \sigma_{SF+},\ \sigma_{SF-},\ \sigma_e,\ \sigma_w,$ or σ_b.

(3) A has SVEP iff AB has SVEP iff BA has SVEP iff B has SVEP.

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$T \in B(\mathcal{H})$ is normal if $T^*T = TT^*$ and T is paranormal if

$$\|Tx\|^2 \leq \|T^2x\|\|x\| \text{ for all } x \in \mathcal{H}.$$

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$\subseteq$ {Polynomial roots of paranormal operators }

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A. No, it isn't.

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Example 1

let $P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} I & 0 \\ I & 0 \end{pmatrix}$ in $B(\mathcal{H} \oplus \mathcal{H})$. Then $P^2 = P$ and $Q^2 = Q$. If $A := PQ$ and $B := QP$, then (A, B) is a solution of the operator equations (1.1). Since $B^* = \begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix}$, a straightforward calculation shows that

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$$B^{2*}B^2 - 2\lambda B^*B + \lambda^2 I = \begin{pmatrix} (2 - 4\lambda + \lambda^2)I & 0 \\ 0 & \lambda^2 I \end{pmatrix},$$

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But, $(2 - 4\lambda + \lambda^2)I$ is not a positive operator for $\lambda = 1$, hence we obtain that for some $\lambda > 0$,

$$B^{2*}B^2 - 2\lambda B^*B + \lambda^2 \not\geq 0.$$

Therefore B is neither paranormal nor normal. On the other hand, A is normal, so that it is a paranormal operator.

Answer!

Example 2

If $P = \begin{pmatrix} I & 2I \\ 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ are in $B(\mathcal{H} \oplus \mathcal{H})$, then both P and Q are idempotent operators. Also, $A := PQ$ and $B := QP$ satisfy the operator equations (1.1). Since $B^*A^* = \begin{pmatrix} I & 0 \\ 2I & 0 \end{pmatrix}$, a straightforward calculation shows that

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$$(AB)^{2*}(AB)^2 - 2\lambda(AB)^*(AB) + \lambda^2 I = \begin{pmatrix} (1 - 2\lambda + \lambda^2)I & (2 - 4\lambda)I \\ (2 - 4\lambda)I & (4 - 8\lambda + \lambda^2)I \end{pmatrix}.$$

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If $P = \begin{pmatrix} I & 2I \\ 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ are in $B(\mathcal{H} \oplus \mathcal{H})$, then both P and Q are idempotent operators. Also, $A := PQ$ and $B := QP$ satisfy the operator equations (1.1). Since $B^*A^* = \begin{pmatrix} I & 0 \\ 2I & 0 \end{pmatrix}$, a straightforward calculation shows that

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However, $(4-8\lambda+\lambda^2)I$ is not a positive operator for $\lambda = 1$, hence AB is neither paranormal nor normal. On the other hand, A is normal, so that it is a paranormal operator.

Main Results 1

Let a pair (A, B) *denote the solution of the operator equations* (1.1) *throughout this talk.*

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Q. Suppose A is paranormal. How can the operators AB, BA, or B be paranormal or normal?

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Theorem

Let A be a paranormal operator on $\mathcal{H}$ and $N(A) = N(AB)$.

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(1) If $\dim \mathcal{H} < \infty$, then AB is a normal operator.

(2) If $\dim \mathcal{H} < \infty$ and $N(A - \lambda) = N(B - \lambda)$ for each $\lambda \in \mathbb{C}$, then all of A, AB, BA, and B are normal operators.

Main Result 1

Given $T \in B(\mathcal{H})$ and $S \in B(\mathcal{K})$ for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, the commutator $C(S, T) \in B(B(\mathcal{H}, \mathcal{K}))$ is the mapping defined by

$$C(S, T)(A) := SA - AT \text{ for all } A \in B(\mathcal{H}, \mathcal{K}).$$

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The iterates $C(S, T)^n$ of the commutator are defined by $C(S, T)^0(A) := A$ and

$$C(S, T)^n(A) := C(S, T)(C(S, T)^{n-1}(A))$$

for all $A \in B(\mathcal{H}, \mathcal{K})$ and $n \in \mathbb{N}$; they are often called the higher order commutators.

Main Results 1

There is the following binomial identity. It states that

$$C(S,T)^n(A) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k S^{n-k} A\, T^k,$$

which is valid for all $A \in B(\mathcal{H}, \mathcal{K})$ and all $n \in \mathbb{N} \cup \{0\}$.

Main Result 1

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$$C(A, X)^n(A^*) = 0 \text{ for all } n \in \mathbb{N}.$$

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(2) $\sigma_A(A^*x) \subseteq \sigma_{\alpha AB+(1-\alpha)A}(x)$ for all $x \in \mathcal{H}$.

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(2) $\sigma_A(A^*x) \subseteq \sigma_{\alpha AB+(1-\alpha)A}(x)$ for all $x \in \mathcal{H}$.

(3) $A^*\mathcal{H}_{\alpha AB+(1-\alpha)A}(F) \subseteq \mathcal{H}_A(F)$ for every set F in $\mathbb{C}$.

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Let A be a paranormal operator and $\sigma(A) = \{\lambda\}$. Then the following statements hold.

(1) If $\lambda = 0$, then $B^2 = 0$.

(2) If $\lambda \neq 0$, then $\lambda = 1$ and $A = B = I$.

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(1) If A is quasinilpotent, then AB, BA, and B are nilpotent.

(2) If $A - I$ is quasinilpotent, then B is the identity operator, that is, $AB - \lambda$, $BA - \lambda$, and $B - \lambda$ are invertible for all $\lambda \in \mathbb{C} \setminus \{1\}$.

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Corollary

If A is a paranormal operator, then iso $\sigma(T) \subseteq \{0, 1\}$ where $T \in \{A, AB, BA, B\}$.

Main Result 3

[2006, Uchiyama]

If T is a paranormal operator and λ_0 is an isolated point of $\sigma(T)$, then the Riesz idempotent $E_{\lambda_0}(T) := \frac{1}{2\pi i}\int_{\partial D}(\lambda - T)^{-1}d\lambda$, where D is the closed disk of center λ_0 which contains no other points of $\sigma(T)$, satisfies

$$R(E_{\lambda_0}(T)) = N(T - \lambda_0).$$

Here, if $\lambda_0 \neq 0$, then $E_{\lambda_0}(T)$ is self-adjoint and $N(T - \lambda_0)$ reduces T.

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$$\mathfrak{C} := \{(A, B) : A \text{ and } B \text{ are solutions of the operator equations (1.1) with } N(A-\lambda) = N(B-\lambda) \text{ for } \lambda \neq 0\}.$$

Lemma 2

Suppose that $(A, B) \in \mathfrak{C}$ and A is paranormal. If $\lambda_0 \in \text{iso}\, \sigma(BA) \setminus \{0\}$, then for the Riesz idempotent $E_{\lambda_0}(A)$ with respect to λ_0, we have that

$$R(E_{\lambda_0}(A)) = N(BA - \lambda_0) = N(A^*B^* - \overline{\lambda_0}).$$

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Theorem

Let $(A, B) \in \mathfrak{C}$ and A be a paranormal operator.

(1) If λ_0 is a nonzero isolated point of $\sigma(BA)$, then the range of $BA - \lambda_0$ is closed.

(2) If B^* is injective and $\lambda_0 \in \text{iso}\, \sigma(T) \setminus \{0\}$, then $N(T - \lambda_0)$ reduces T, where $T \in \{AB, B\}$.

Main Result 3

[Djor] S. Djordjević, I.H. Jeon and E. Ko, *Weyl's theoem throughout local spectral theory,* Glasgow Math. J. **44** (2002), 323–327.

It was shown by [Djor, Lemma 1] that for every $\lambda \in \pi_{00}(T)$, $\mathcal{H}_T(\{\lambda\})$ is finite dimensional if and only if $R(T - \lambda)$ is closed. Furthermore we can easily prove that

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$$\pi_{00}(A) \setminus \{0\} = \pi_{00}(AB) \setminus \{0\} = \pi_{00}(BA) \setminus \{0\} = \pi_{00}(B) \setminus \{0\}.$$

Corollary

Let $(A, B) \in \mathfrak{C}$ and A be a paranormal operator. If $\lambda_0 \in \pi_{00}(BA) \setminus \{0\}$, then $\mathcal{H}_{BA}(\{\lambda_0\})$ is finite dimensional.

Main Result 3

Remark

Let $(A, B) \in \mathfrak{C}$ and one of A, BA, AB, or B be paranormal. If λ_0 is a nonzero isolated point in the spectrum of one of them, then all of the ranges of $A - \lambda_0$, $BA - \lambda_0$, $AB - \lambda_0$, and $B - \lambda_0$ are closed. Moreover, if λ_0 is a nonzero isolated eigenvalue of the spectrum of one of them with finite multiplicity, then all of the spectral manifolds $\mathcal{H}_A(\{\lambda_0\})$, $\mathcal{H}_{AB}(\{\lambda_0\})$, $\mathcal{H}_{BA}(\{\lambda_0\})$, and $\mathcal{H}_B(\{\lambda_0\})$ are finite dimensional.

Main Result 4

R.E. Curto and Y.M. Han, *Generalized Browder's and Weyl's theorems for Banach space operators,* J. Math. Anal. Appl. **336** (2007), 1424–1442.

It is well known that every polynomial roots of paranormal operators satisfy generalized Weyl's theorem.

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Now, we would like to show that if A is paranormal, then Weyl's theorem holds for T, where $T \in \{AB, BA, B\}$. More generally, we prove that if A or A^* is a polynomial root of paranormal operators, then generalized Weyl's theorem holds for $f(T)$ for $f \in H(\sigma(T))$, where $T \in \{AB, BA, B\}$.

B-Fredholm Operators

Definitions [2001, M. Berkani]

Let $T \in B(\mathcal{H})$.

- For a nonnegative integer n define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_0 = T$).

B-Fredholm Operators

Definitions [2001, M. Berkani]

Let $T \in B(\mathcal{H})$.

- For a nonnegative integer n define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_0 = T$).
- T is called upper (resp., lower) semi-B-Fredholm if for some integer n the range $R(T^n)$ is closed and T_n is upper (resp., lower) semi-Fredholm.
- T is called semi-B-Fredholm if it is upper or lower semi-B-Fredholm.

B-Fredholm Operators

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- T is called semi-B-Fredholm if it is upper or lower semi-B-Fredholm.
- T is called B-Fredholm if T_n is Fredholm.
- T is called B-Weyl if it is B-Fredholm with index 0.

Well Known Facts

[Berk, Theorem 2.7] $T \in B(\mathcal{H})$ is B-Fredholm if and only if

$$T = T_1 \oplus T_2, \text{ where } T_1 \text{ is Fredholm and } T_2 \text{ is nilpotent.}$$

[Berk] M. Berkani and M. Sarih, *On semi B-Fredholm operators,* Glasgow Math. J. **43** (2001), no. 3, 457–465.

Notations

Let $T \in B(\mathcal{H})$.

$\sigma_{BF}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } B\text{-Fredholm}\}$

$\sigma_{BW}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } B\text{-Weyl}\}$

$\pi_0(T) := \{\lambda \in \text{iso}\,\sigma(T) : \alpha(T) > 0\}$

Concepts of Generalized Weyl type theorems

Definitions [2003, Berkani and Koliha]

Generalized Weyl's theorem holds for T, in symbol $(g\mathcal{W})$, if

$$\sigma(T) \setminus \sigma_{BW}(T) = \pi_0(T).$$

g-Weyl's theorem $\Longrightarrow$ Weyl's theorem

Main Result 4

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Theorem

Suppose that A or A^* is a polynomial root of paranormal operators. Then $f(T) \in g\mathcal{W}$ for each $f \in H(\sigma(T))$, where $T \in \{AB, BA, B\}$.

Main Result 4

Corollary

Suppose that $(A, B) \in \mathfrak{C}$ and A is a compact paranormal operator. Then we have that

$$BA = \begin{pmatrix} I & 0 \\ 0 & Q \end{pmatrix} \text{ on } N(BA - I) \oplus N(BA - I)^{\perp},$$

where Q is quasinilpotent.

Thank You !